

THE DETERMINANT OF A ROW-FACTORIZATION MATRIX IN A NUMERICAL SEMIGROUP

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1. NOTATIONS AND DEFINITIONS

In this paper, we study the determinant of row-factorization matrix in a numerical semigroup. Row-factorization matrices, in short RF-matrices, for pseudo-Frobenius numbers f in a numerical semigroup S are defined by Moscariello in [5], to prove the type of almost symmetric semigroups generated by four elements is less than or equal to three. Their determinants, in general, is multiples of f . If its absolute value is f , then we get a basis of the kernel space defined by S , from the RF-matrix. Then we can also get a generating system of a defining ideal of the semigroup associated with S . Hence, it is important to investigate the determinants of RF-matrices.

First, we give notations and definitions. Let $\mathbb{Z}$ be the ring of integers, and $\mathbb{N}$ the set of non negative integers. Let S be a non empty subset in $\mathbb{N}$. We say that S is a *semigroup* in $\mathbb{N}$, if

- (1) $0 \in S$,
- (2) $a + b \in S$, if $a, b \in S$.

Let S be a semigroup in $\mathbb{N}$ and $n_1, \dots, n_s \in \mathbb{N}$. We say that S is *generated* by $n_1, \dots, n_s$ if

$$S = \{a_1 n_1 + \dots + a_s n_s : a_1, \dots, a_s \in \mathbb{N}\}.$$

We also say that S is *minimally* generated by $n_1, \dots, n_s$, if any proper subset of $\{n_1, \dots, n_s\}$ does not generate S . Then we denote S by $\langle n_1, \dots, n_s \rangle$ and call s the *embedding dimension* of S .

If $\mathbb{N} - S$ is finite, we say that S is *numerical*. We note that S is numerical if and only if the general common divisor of $n_1, \dots, n_s$ is one.

Example 1.

$$\langle 3, 5 \rangle = \{0, 3, 5, 6, 8, 9, 10, \dots\}$$

is a numerical semigroup generated by 3 and 5.

From now, all semigroups are assumed to be numerical semigroups in $\mathbb{N}$. Let S be a semigroup. The number

$$F(S) = \max\{a \in \mathbb{Z} : a \notin S\}$$

is called the *Frobenius number* of S . We also define

$$PF(S) = \{a \in \mathbb{Z} : a + x \in S \text{ if } x \in S \text{ and } x \neq 0\}$$

and an element in $PF(S)$ is called a *pseudo-Frobenius number*. Obviously, $F(S) \in PF(S)$. We say that the number of $PF(S)$ is the *type* of S , denoted by $t(S)$. For $d \in S$, We define the *Apery set* $Ap(S, d)$ as follows:

$$Ap(S, d) = \{x \in S : x - d \notin S\}.$$

Note $|Ap(S, d)| = d$ and, for any $a \in \{0, 1, \dots, d-1\}$, there is $x \in Ap(S, d)$ with $a \equiv x \pmod{d}$.

Next, we define RF-matrices. Let $S = \langle n_1, \dots, n_s \rangle$ be a semigroup and $f \in \mathbb{Z} - S$. For each i , there is $a_{ii} < 0$ with $f - a_{ii}n_i \in Ap(S, n_i)$. Then there are $a_{ij} \geq 0$ for $j \neq i$ satisfying $f - a_{ii}n_i = \sum_{j \neq i} a_{ij}n_j$. We say that the matrix $RF(f) = (a_{ij})$ is an RF-matrix (row-factorization matrix) for f in S . We denote it by $RF(f)$.

Example 2. (Examples of RF-matrices)

(1) Let $S = \langle 3, 4, 5 \rangle$ and $f = 2 \notin S$. Then

$$RF(2) = \begin{pmatrix} -1 & 0 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix}.$$

(2) Let $S = \langle 4, 5, 6 \rangle$ and $f = 7 \notin S$. Then

$$RF(7) = \begin{pmatrix} -1 & 1 & 1 \\ 3 & -1 & 0 \\ 2 & 1 & -1 \end{pmatrix} \text{ or } \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 2 & 1 & -1 \end{pmatrix}.$$

From above example, it follows that an RF-matrix for pseudo-Frobenius number is not unique in general.

2. THE DETERMINANTS OF RF-MATRICES

In this section, we consider the following question: Let S be a numerical semigroup with embedding dimension s and $f \in PF(S)$. Then, does the equation

$$(*) \quad \det RF(f) = (-1)^{s+1} f$$

hold?

Theorem 1. If $s = 2$ or 3 , then $(*)$ holds.

Proof. Assume $s = 2$ and let $S = \langle n_1, n_2 \rangle$. Then $F(S) = n_1 n_2 - n_1 - n_2$ is a unique pseudo-Frobenius number and

$$\text{RF}(F(S)) = \begin{pmatrix} -1 & n_1 - 1 \\ n_2 - 1 & -1 \end{pmatrix},$$

thus $\det \text{RF}(F(S)) = -F(S)$.

Assume $s = 3$ and let $S = \langle n_1, n_2, n_3 \rangle$. If $t(S) = 1$, then we may assume $dn_3 \in \langle n_1, n_2 \rangle$ where $d = \gcd(n_1, n_2)$. Then $F(S) = n_1 n_2 / d - n_1 - n_2 + (d - 1)n_3$ is a unique pseudo-Frobenius number and

$$\text{RF}(F(S)) = \begin{pmatrix} -1 & n_1/d - 1 & d - 1 \\ n_2/d - 1 & -1 & d - 1 \\ a_{31} & a_{32} & -1 \end{pmatrix},$$

where $dn_3 = (a_{31} + 1 - n_2/d)n_1 + (a_{32} + 1)n_2$ or $(a_{31} + 1)n_1 + (a_{32} + 1 - n_1/d)n_2$. Then $\det \text{RF}(F(S)) = F(S)$.

The rest case is that of $s = 3$ and $t(S) = 2$. Let $\text{PF}(S) = \{f_1, f_2\}$ and put $\text{RF}(f_1) = (a_{ij})$ and $\text{RF}(f_2) = (b_{ij})$. By classical result, they are unique and we may assume

$$\begin{aligned} a_{12} &= b_{32} = a_{32} + b_{12} + 1, \\ a_{23} &= b_{13} = a_{13} + b_{23} + 1, \\ a_{31} &= b_{21} = a_{21} + b_{31} + 1 \end{aligned}$$

Since

$$\begin{aligned} n_1 &= (a_{12} + 1)(a_{13} + 1) + (b_{12} + 1)(b_{23} + 1), \\ n_2 &= (a_{23} + 1)(a_{21} + 1) + (b_{23} + 1)(b_{31} + 1), \\ n_3 &= (a_{31} + 1)(a_{32} + 1) + (b_{31} + 1)(b_{12} + 1), \end{aligned}$$

we have $\det \text{RF}(f_i) = f_i$ for $i = 1, 2$. □

Definition. Let S_1, S_2 be numerical semigroups and $d_1 \in S_2$ and $d_2 \in S_1$. If d_1 and d_2 are coprime, then

$$S = d_1 S_1 + d_2 S_2 = \{d_1 x + d_2 y : x \in S_1, y \in S_2\}$$

is a numerical semigroup. We say that S is *glued* by S_1 and S_2 .

Definition. We say that S is *completely glued* if one of the following is satisfied:

- (1) $S = \langle 1 \rangle$,
- (2) S is glued by completely glued semigroups.

If the embedding dimension of S is 2, then S is completely glued. If S is completely glued, then its type is one.

Theorem 2. If S is completely glued, then there is an RF-matrix of $F(S)$ which satisfies (*).

Proof. If $S = \langle 1 \rangle$, then the assertion is clear. Assume $S = d_1 S_1 + d_2 S_2$ where $d_1 \in S_2$, $d_2 \in S_1$ and both S_1 and S_2 are completely glued. Then

$$F(S) = d_1 F(S_1) + d_2 F(S_2) + d_1 d_2$$

and there is an RF-matrix M_1 (resp. M_2) for $F(S_1)$ (resp. $F(S_2)$) in S_1 (resp. (S_2)) satisfying $\det M_1 = (-1)^{s_1} F(S_1)$ (resp. $\det M_2 = (-1)^{s_2} F(S_2)$) where s_1 (resp. s_2) is the embedding dimension of M_1 (resp. M_2). Since $F(S_1) + d_2 \in S_1$ (resp. $F(S_2) + d_1 \in S_2$), we may write $F(S_1) + d_2 = \sum_i a_i n_i$ (resp. $F(S_2) + d_1 = \sum_i a'_i n'_i$) where $S_1 = \langle n_1, \dots, n_{s_1} \rangle$ (resp. $S_2 = \langle n'_1, \dots, n'_{s_2} \rangle$) and $a_i \geq 0$ (resp. $a'_i \geq 0$) for each i . Let N_1 (resp. N_2) be an $s_2 \times s_1$ -matrix (resp. $s_1 \times s_2$ -matrix) whose ij -entry is a_i (resp. a'_i) for each i, j . And put

$$M = \begin{pmatrix} M_1 & N_2 \\ N_1 & M_2 \end{pmatrix}.$$

Then M is an RF-matrix for $F(S)$ in S and $\det M = (-1)^{s_1+s_2} F(S)$. $\square$

Theorem 3. Assume $s = 4$. If the type of S is one, or if S is pseudo-symmetric, then there is an RF-matrix of $F(S)$ which satisfies (*). We say that S is *pseudo-symmetric* if $\text{PF}(S) = \{F(S)/2, F(S)\}$.

Proof. Let $S = \langle n_1, n_2, n_3, n_4 \rangle$. Assume $t(S) = 1$. Further, we may assume that S is not completely glued. Then, by [1], after suitable renumbering, we have

$$\text{RF}(F(S)) = \begin{pmatrix} -1 & \alpha_2 - 1 & \alpha_3 - 1 & a_{14} \\ a_{21} & -1 & \alpha_3 - 1 & \alpha_4 - 1 \\ \alpha_1 - 1 & a_{32} & -1 & \alpha_4 - 1 \\ \alpha_1 - 1 & \alpha_2 - 1 & a_{43} & -1 \end{pmatrix},$$

where α_i is the minimal positive number satisfying that $(\alpha_i - 1)n_i$ has the unique factorization by $n_1, \dots, n_4$ for each i and $0 < a_{21} < \alpha_1$, $0 < a_{32} < \alpha_2$, $0 < a_{43} < \alpha_3$, and $0 < a_{14} < \alpha_4$. From this, we have $\det \text{RF}(F(S)) = -F(S)$.

Assume that S is pseudo-symmetric. By [3], after suitable renumbering, we also have

$$\begin{aligned} \text{RF}(F(S)) &= \begin{pmatrix} -1 & \alpha_2 - 2 & \alpha_3 - 1 & 0 \\ \alpha_1 - 1 & -1 & \alpha_3 - 2 & \alpha_4 - 1 \\ \alpha_1 - 2 & \alpha_2 - 1 & -1 & \alpha_4 - 1 \\ \alpha_1 - 1 & a_{42} - 1 & \alpha_3 - 1 & -1 \end{pmatrix}, \\ \text{RF}(F(S)/2) &= \begin{pmatrix} -1 & \alpha_2 - 1 & 0 & 0 \\ 0 & -1 & \alpha_3 - 1 & 0 \\ \alpha_1 - 1 & 0 & -1 & \alpha_4 - 1 \\ \alpha_1 - 1 & a_{42} & 0 & -1 \end{pmatrix}, \end{aligned}$$

where α_i is defined above and $0 < a_{42} < \alpha_2$. From this, we also have $\det \text{RF}(F(S)) = -F(S)$. $\square$

Finally, we give some examples of RF-matrices in an almost symmetric semigroup.

Definition. Let S be a semigroup. For any $f \in \text{PF}(S)$ with $f \neq F(S)$, if $F(S) - f \in \text{PF}(S)$, we say that S is *almost symmetric*.

Example 3 (Watanabe's example). Let $S = \langle 22, 46, 9, 57 \rangle$. Then S is almost symmetric of type 3 and $\text{PF}(S) = \{35, 70, 105\}$. We also have

$$\text{RF}(70) = \begin{pmatrix} -1 & 2 & 0 & 0 \\ 2 & -1 & 8 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 9 & -1 \end{pmatrix}, \text{RF}(35) = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 9 & 0 \\ 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{pmatrix}.$$

$$\det \text{RF}(70) = 0, \det \text{RF}(35) = -35.$$

Example 4. Let $S = \langle 22, 26, 79, 83 \rangle$. Then S is almost symmetric of type 3 and $\text{PF}(S) = \{57, 238, 295\}$. We also have

$$\text{RF}(57) = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 5 & 1 & -1 & 0 \\ 4 & 2 & 0 & -1 \end{pmatrix}, \text{RF}(238) = \begin{pmatrix} -1 & 10 & 0 & 0 \\ 12 & -1 & 0 & 0 \\ 0 & 9 & -1 & 1 \\ 11 & 0 & 1 & -1 \end{pmatrix}$$

and

$$\text{RF}(295) = \begin{pmatrix} -1 & 9 & 0 & 1 \\ 11 & -1 & 1 & 0 \\ 0 & 8 & -1 & 2 \\ 10 & 0 & 2 & -1 \end{pmatrix} \text{ or } \begin{pmatrix} -1 & 9 & 0 & 1 \\ 11 & -1 & 1 & 0 \\ 4 & 11 & -1 & 0 \\ 10 & 0 & 2 & -1 \end{pmatrix}.$$

$\det \text{RF}(57) = \det \text{RF}(238) = 0$. We note that the determinant of the former RF-matrix for 295 is zero, and that of the latter one is -295 .

From above examples, it follows that the condition $(*)$ does not hold for all RF-matrices for pseudo-Frobenius numbers. Hence we modify the question as follows:

Question. Let S be a semigroup with embedding dimension s . Then, does the equation

$$(*) \quad \det \operatorname{RF}(F(S)) = (-1)^{s+1} F(S)$$

hold for some RF-matrix for $F(S)$?

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