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STUDIES
OF
SPACE-DEPENDENT
NUCLEAR REACTOR CONTROLS

Ryuji KOGA

JANUARY 1975

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Abstract

Controlling problems on nuclear reactor cores are dealt with retaining their distributed-parameter characteristics. Approaches are made analytically and efforts are devoted to reducing concise system models in contrast to the practical approach which may be involved in complicated and large-sized system models.

In Chapter 1, a survey of the controlling problems of nuclear reactors is made. The studied problems are allocated to three chapters according to a measure, the controlling duration.

Chapter 2 is concerned with the steady state cores. A simple mathematical model to examine the criticality is reduced there. Also an optimization problem to distribute the controlling absorber is stated in connection with achieving the maximum flatness of the neutron flux.

Chapter 3 deals with the dynamical control problems to regulate the fluctuation of power distribution in the core. Arguments are made on the open loop control and the closed loop control separately.

Chapter 4 treats the burnup control problems. A concise neutronics model "*cumulus*" is devised by the author and the control rod programming problem to attain the maximum burnup is studied on the model. Also the fuel failure probability is taken into account in the latter part of the chapter in order to obtain the maximum expectation of the average burnup at the end of the core-life, where the neutronics is expressed by an ordinary modified one-group model.

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Chapter 1. Introduction

Since the invention, nuclear reactor has been brought up by many people including designers and the systems people. A nuclear reactor that intrinsically contains huge energy within it, prohibits a trial-and-error method in operation and requires precise study of the dynamics and the controlling schemes. The system engineers have participated in design and operation of reactors as well as the long-term power plant construction planning.

In early years when nuclear reactors were small in size, the mathematical model has been taken as a point or lumped model, which is written in an ordinary differential equation. Many studies on the stabilizing, minimum-time startup problems and so on, have been carried out by the point model (Weaver [We.68], Mohler [Mh.70]).

As the initial developing plan for fast reactors has fallen temporarily in fail, water reactors such as pressurized water reactors (PWR) or boiling water reactors (BWR) have been developed and are used as the power production plants. In these reactors, the physical values at separate positions in the core have weak interaction between them and a small parameter variation introduced partially into the core, results in skewed output-power-density profile than it does in a small-sized reactor. A number of control rods are installed to cope with this circumstance and further enable the operators to steer the core state for a desirable power distribution and dynamics.

The core structure which is composed of many fuel assemblies allows various allocation of the fuels which have different isotope composition and exposure histories. These space dependent freedom

can be utilized to improve the core performances.

There are many kinds of core performances to be improved. Some of the performances have weak interaction with others and there are many other coping performances. The controlling problem for nuclear reactors are characterized and classified by the duration for which a sequence of the controlling action runs.

A group of the problems is to allocate the materials in steady state core. In earlier days this type of problem has attracted much attention because of the easiness of problem formulation for a variety of criteria and restrictions. Many studies have been made from academic interests rather than from practical need for core designing or operation. Optimization of detailed core configuration is executed ordinarily at various phases of the core designing. The practical designing is, however, concerned with many numbers of variables and constraints and the optimization process is carried out by direct search methods using a large scale computers (Inoue [In.73]). The analytical approach rarely gives the physical solution.

The second group of problems is of dynamical control of power output from the core. This group contains the problem of startup, shutdown and regulation as well as power-level change. Undergoing reactors are subdued by very strict thermal stress constraints on the fuel-pin sheath and the regulation of output-power fluctuation is entrusted to the negative power feedback characteristics of the core. The participation of external regulating effort is intended for the regulation of xenon-induced spatial power oscillation whose controlling duration is measured by tens of hours. In a rated power operation, the total output power fluctuation can be absorbed into the intrinsic stability of the core that has been assigned by the design. The drastic dynamics of a core can well be expressed by a lumped model when the core is regarded as a component of the power generating plant. Therefore the space dependent controlling effort for the core is mainly intended to reduce the spatial instability inside the core.

The remaining group which requires the longest period is the problems concerning the economic consumption of fissile materials which are considered to be finite on a globe-wide scale. The power plant construction planning is necessary to maximally utilize the fissile materials and reduce the interests cost required to store the plutonium for tens of years (Yasukawa *et al.* [Ya.71]).

For a nuclear reactor, the fuel management affects the power generating cost and many studies have been carried out. These problems are concerned with the total life of the reactors, which are ranged from about 10 to 30 years. In these studies a fuel assembly corresponds to one state variable and the number of state variables that should be taken into account, amounts to very large. The mathematical device is required to deal with this situation. In these problems, the states are taken into account at each refueling time and the in-core poison management between the refuelings are usually not regarded under some assumptions.

The poison management problems have been attracted attentions as the subproblem of the refueling problems, which is concerned with how to bring the initial state to the desired end state within the prescribed constraints. Poison management problems are composed of the subproblems how to attain the temporarily desired output power profile under the limitations placed on the material density and the thermal, hydro-dynamical stress. They are the problems belonging to the first group. The second group problems are to attain these steady states within a minimum effort or to maintain them.

In regard to the tools to deal with these problems, various modern control theories and techniques are employed as well as the classical control theory in terms of frequency responses are. Making use of notions of state variables, the maximum principle (Pontryagin [Pnt.62] and the dynamic programming (Bellman [Be.57]) have helped to solve the optimal control problem associated with the inequality constraints. One of the most splendid results obtained through the modern control theory has been brought to the xenon-shutdown problem

(Ash [As.66])). The theory gives there the solution which cannot be obtained from the ordinary intuition.

The control theory for distributed systems have been developed as the extension of the theory for the lumped parameter system (Butkowskiy [Bu.69], Wang [Wa.66])). Mathematically complicated characteristics of distributed systems demand an abstract argument in the general treatment. General linear control systems, including distributed system as well as the lumped, can be described in terms of an abstract space such as Banach space or Hilbert space, associated with the specific norm which works as the criterion for the control (Balakrishnan [Ba.63])). Use of the abstract space lends the open prospect for the theory, but the objects that can be dealt with are limited to simply and analytically described systems.

For the practical intention, any final results ought to be obtained in real numbers and the distributed systems should be decomposed to a numbers of lumped systems. The spatial finite difference approximation is employed widely to obtain the lumped systems. This method is based on the direct substitution of the difference operator for the differential operator. The formulation is made straightforward and nonlinearities can be dealt with. The number of variables required to express the system within the satisfactory precision, however, often becomes tremendously large especially when a three-dimensional geometry is taken into account.

For systems with weak nonlinearities, the nodal approximation or the modal expansion approximation works well with benefits of the small amount of the memories and the computing time required. Much efforts have been devoted to the application of these methods to nuclear reactor analysis (Stacey [St.67], [St.69a])).

For controlling intention, the conciseness of the mathematical model obtained through the nodal or modal expansion approximation is much appreciated because of their conciseness of expression retaining the features of specific interests. Suitable choice of the geometrical partitioning for nodal approximation or the choice of the

base functions for modal expansions maximally exploits this benefits.

In theoretical works, the eigenfunctions of the system operators are often used as the base functions because of their simplicity of expressions and treatise. For an existing reactor, much effort is required to obtain the eigenfunctions, and then ought to be employed the "synthesis method", in the terminology of nuclear reactor science (Stacey [St.69a]).

This thesis is concerned with analytical treatments of the optimization of space dependent control of reactors. The neutronics equations, which characterize the mathematical model for the physical reactor, are taken to be the integral form except for the latter half of Chapter 2. The transformation from the original diffusion equations to the integral form is carried out by the synthesis method making use of Helmholtz mode expansion. Each chapter is devoted to the specific problems classified according to the controlling duration mentioned before.

Chapter 2 is concerned with two subjects dealing with the steady state core. One is to express the linear neutronics in an algebraic equation. Assuming a few numbers of discrete control rods in the core, the algebraic expression gives a criticality surface in the space of control values. The rest is devoted to the problem to allocate the controlling absorber in order to bring the flux distribution in the core to a desired distribution. The neutronics model is taken to be a one-group diffusion equation with slab geometry. The numerical examination has been carried out for the uniform desired distribution. Maximum principle is used to reformulate the original problem that is intended to minimize the integral criterion of squared error over the core, into a two-point boundary value problem. The shooting has been made to obtain the numerical solution. A numerical anomaly has been met there, and the reason is examined to give the validity of this formulation in comparison with the backward substitution which had been used in earlier literatures.

Chapter 3 is concerned with the regulating problems for the reactor core by using various mathematical models to express the neutronics. An optimum open loop control is obtained by the function space method (Balakrishnan [Ba.63]) for the reactor model of neutrons and precursors. An abstract description of the optimality condition on the control is translated by Helmholtz mode expansion into an algebraic equation. The quickly following property of neutronics enables us to reduce the order of the system and to represent the system only by the precursors, leading easiness of numerical solution.

Also the optimal feedback control problem is formulated for a thermo-hydrodynamics model, where the neutronics plays as the coupling subsystem. The result is obtained in terms of Riccati-type equation with respect to the kernel of feedback integral operator. The Helmholtz mode expansion is employed to decompose the Riccati-type integral equation into a set of algebraic equations.

Chapter 4 is of the burnup control problem which is concerned with the whole life of the core of the reactor. A new model that gives a concise description of steady state neutronics of the core is employed and named as "*cumulus*" by the author. The burnup maximization problem is formulated on the basis of this model and the numerical examination has been carried out for a two region core, and it is shown that the model can be described geometrically and easily gives the characteristic features of the burnup problem.

A practical aspect for the probabilistic fuel failure presents a problem to attain the maximum expectation of the burnup averaged over the core at the end of the core life.

More detailed surveys for literatures and the backgrounds of the problem are given in each introductory section of the chapters. The symbols which stand for the usual meanings such as diffusion coefficients, absorption cross section or the decay constant of precursors are used without explanation. As a standard textbook of neutron physics, the work by Weinberg & Wigner [W.58] can be referred to. The signature ω is used as the space variable common to all chapters.

Chapter 2 Analytical Approach to the Steady State Core

2.1 Introduction

The performance of a nuclear reactor core can be improved by re-distributing the space dependent parameters over the core. In early years, Goertzel [Goe.56] have showed through the importance consideration, that the fuel distribution which gives the minimum critical mass of the fissile material requires the flat flux over the entire core under a condition for the one neutron energy group model. This inspired many succeeding studies. The problems to realize the flat flux were studied by Bartosek & Zezula [Bar.66], Ravets & Lamarsh [Ra.60], Lelek [Le.65], Hara & Shibata [Ha.68], and Amano [Am.66], [Am.67]. All of these studies substitute the condition of the flatness of the flux into the neutron governing equation and then obtain the required parameter distribution of absorption cross section and moderator density.

In some case, however, this process, named backward substitution, results in a negative value of the material density, which should be physically positive, or give the practically unrealizable values. Introduction of the maximum principle by Pontryagin have enabled to find a solution within the specified closed class of controls.

Making use of the maximum principle, the original minimum critical mass problems have been studied by Zaritskaya & Rudik [Za.66] and by Kochurov [Koc.66] imposing a limiting condition on the uranium concentration without the assumption of the symmetricity of the slowing down kernel. Goldschmidt & Quenon [Go.70] studied the minimum

critical mass of a fast reactor under the power density constraint.

The interest has also arisen in the direct application for the core design to allow the maximum power output. Zaritzkaya & Rudik [Za.67] have studied the optimal fissile uranium distribution which gives the maximum power from a gas cooled reactor core, with the buckling as a function of the uranium concentration expressed in an experimental equation. Kochurov & Rudik [Koc.67] have considered a single channel reactor and have synthesized the optimum allocation of uranium concentration under the constraints of maximum fuel temperature and the uranium concentration in order to obtain the maximum power output. Zaritzkaya & Rudik [Za.74] also have considered a single channel reactor and have obtained the knowledge how to attain the maximally flat power release by adjusting the longitudinal distribution of fissile uranium concentration.

The maximum principle has been also used to obtain the flat flux distribution under the restriction on the concentration of the material distribution. Koga *et al.* [Ko.71] have taken the absorbing material concentration as the control and reformulated the original problem into a two-point boundary value problem through the maximum principle. Terney [Te.71] has obtained the information for the type of k_{∞} distribution through the maximum principle and synthesized the optimal solution to give the minimum integration of the squared deviation of the flux from its averaged value. Suda [Su.68] has obtained the optimal control to flatten the flux over the core in terms of the control absorber distribution in the radial direction.

The contrasting optimal problem to obtain the highest flux density in a thermal research reactor has been taken up by Strugar [Str. 70] with the aid of the maximum principle within the constraints of the total power, the power density and the fuel enrichment.

In many cases when we want to synthesize an optimum or some specific control system, the part to attain the criticality and/or to express the correspondence of the flux shape with the physical

value of the controlling absorber may occupy unreasonably large proportion compared with other facilities. Therefore a brief mathematical model in an algebraic form that expresses the steady state neutron distribution corresponding to the assigned material allocation will help the designers to construct their controlling system compactly.

The attempt to express the criticality condition in an algebraic form has been done by Hoshino [Ho.70], where the criticality factor is expressed in an empirical quadratic form of the cross sections after the experimental design technique.

The method presented in section 2 is the one to obtain the criticality condition for a reactor equipped with a few rods in terms of the values corresponding to the rod dislocation from its initial nominal position, which is similar to the ordinary perturbation method but allows larger extent of perturbations. The numerical example for the slab reactor with two rods shows good agreement with the strict solution.

The section 3 consists of the extension of the work by Koga *et al.* and includes the consideration of the singularity of the problem and an approximate method to avoid the difficulty is added.

In recent years, many large scale power reactors have been built succeedingly. The structural materials in these reactors are exposed to severe nuclear and thermal stress which restricts the output-power density or the temperature within some prescribed limit. On the other hand the huge initial constructing investment requires as much total power output as possible to reduce the energy cost. These situations require the operation to flatten the output-power density in the core.

This problem will be slightly modified if the fuel cost is taken into account, then a distorted flux distribution is required as will be shown in Chapter 4.

In section 3, the problem to bring the flux distribution into

the desired shape is formulated into a quadratic performance problem for a linear system including the control variable parametrically, that is called a bilinear system. The necessary condition is obtained for the control variable, which has both the upper and lower limits reflecting the physical condition, after the maximum principle by Pontryagin. The numerical example is tested for the flat desired shape and the numerical difficulties are pointed out. A substitute suboptimal method is proposed to avoid the difficulties and the numerical example is associated.

2.2 Criticality Surface

The steady state neutron flux distribution in a reactor core can be considered to be governed by an equation of the form

$$\mathcal{L}(\omega, u) \phi(\omega) = 0, \quad (2.2.1)$$

where $\mathcal{L}(\omega, u)$ is a spatial operator with u as a space dependent parameter corresponding to a controllable poison. Generally the equation is written in a partial differential equation, therefore it is natural that the boundary condition

$$\mathcal{A}(\omega) \phi(\omega) = 0, \quad \omega \in \partial\Omega \quad (2.2.2)$$

should be associated.

The mathematical condition translated from the physical critical condition is that Eq.(2.2.1) along with Eq.(2.2.2) has an existing non-trivial solution. More rigorously, the operator $\mathcal{L}$ should have a maximum zero eigenvalue.

The following discussion is devoted to the condition that the control function $u(\omega)$ should satisfy in order for the reactor core to

be critical.

Suppose that the operator $\mathcal{L}$ has a maximum simple zero eigenvalue when u vanishes, that is, for the existing nontrivial eigenfunction

$$\mathcal{L}_0(\omega) \phi_0(\omega) = \mathcal{L}(\omega, 0) \phi_0(\omega) = 0, \quad \omega \in \Omega \quad (2.2.3)$$

holds. Also let us suppose there exists the adjoint operator $\mathcal{L}_0^*$ of $\mathcal{L}_0$ and the adjoint eigenfunction $\phi_0^*(\omega)$.

These zero-subscripted signatures correspond to the reference or nominal critical state of the reactor core. The arbitrary function $\phi(\omega)$ can be expressed as

$$\phi(\omega) = \phi_0(\omega) + \delta \phi(\omega) \quad (2.2.4)$$

without loss of generality.

We consider the case where the control parameter $u(\omega)$ takes part in the operator $\mathcal{L}$ as

$$\mathcal{L}(\omega, u(\omega)) \equiv \mathcal{L}_0(\omega) + u(\omega). \quad (2.2.5)$$

The system (2.2.1) is then written in the form

$$\mathcal{L}_0 \phi = -u \phi. \quad (2.2.6)$$

If the linear homogeneous equation (2.2.6) has a nontrivial solution, Eq.(2.2.6) can be rewritten by Eq.(2.2.4), as

$$\mathcal{L}_0(\phi_0 + \delta \phi) = -u \phi. \quad (2.2.7)$$

Incorporating with Eq.(2.2.3), the lefthand-side of Eq.(2.2.7) should be written as

$$\mathcal{L}_0 \delta \phi = -u \phi. \quad (2.2.8)$$

The inner product of Eq.(2.2.8) by the adjoint eigenfunction ϕ_0^* of $\mathcal{L}_0$ suggests that

$$\langle \phi_0^*, \mathcal{L}_0 \delta \phi \rangle = \langle \mathcal{L}_0^* \phi_0^*, \delta \phi \rangle = -\langle \phi_0^*, u \phi \rangle = 0. \quad (2.2.9)$$

The first equality in Eq.(2.2.9) is approved by the definition of ϕ_0^* , ϕ_0^* .

On the other hand, the existing non-trivial solution $\phi(\omega)$ can be expressed by also existing modified Green's function $\tilde{G}(\omega|\omega')$ as

$$\phi(\omega) = \phi_0(\omega) + \int_{\Omega} \tilde{G}(\omega|\omega') u(\omega') \phi(\omega') d\omega' \quad (2.2.10)$$

Let us introduce a symbol "o" which means the integration with respect to the common variable between both sides of the symbol over the domain Ω . The repetitive substitution of Eq.(2.2.10) into Eq. (2.2.9) yields a functional on U as

$$\begin{aligned} P(u, \phi) = & \phi_0^*(\omega_1) \circ u(\omega_1) \phi_0(\omega_1) \\ & + \phi_0^*(\omega_1) \circ u(\omega_1) \tilde{G}(\omega_1|\omega_2) \circ u(\omega_2) \phi_0(\omega_2) \\ & + \phi_0^*(\omega_1) \circ u(\omega_1) \tilde{G}(\omega_1|\omega_2) \circ u(\omega_2) \tilde{G}(\omega_2|\omega_3) \circ u(\omega_3) \phi_0(\omega_3) \\ & \dots \dots \dots \\ & + \phi_0^*(\omega_1) \circ u(\omega_1) \tilde{G}(\omega_1|\omega_2) \circ \dots \dots \dots \\ & \dots \dots \dots \circ u(\omega_{m-1}) \tilde{G}(\omega_{m-1}|\omega_m) \circ u(\omega_m) \phi_0(\omega_m) \\ & + R_{m+1}(u(\omega), \phi(\omega)) , \end{aligned} \quad (2.2.11)$$

where the last term R_{m+1} is the residual term and is

$$\begin{aligned} R_{m+1}(u(\omega), \phi(\omega)) = & \phi_0^*(\omega_1) \circ u(\omega_1) \tilde{G}(\omega_1|\omega_2) \circ \dots \dots \dots \\ & \dots \dots \dots \circ u(\omega_m) \tilde{G}(\omega_m|\omega_{m+1}) \circ u(\omega_{m+1}) \phi(\omega_{m+1}) . \end{aligned} \quad (2.2.12)$$

When the norm $\|u(\omega)G(\omega|\omega')\|$ is small enough and m is large, the expression (2.2.11) will give a determinant functional independent of the unknown $\phi(\omega)$ as an alternate of Eq.(2.2.9).

The modified Green's function used above is a solution of the equation

$$L_0(\omega) G(\omega|\omega') = -\delta(\omega - \omega') + \frac{\phi_0(\omega) \cdot \phi_0^*(\omega')}{\|\phi_0(\omega)\| \cdot \|\phi_0^*(\omega')\|} . \quad (2.2.13)$$

When we consider a reactor which is equipped with a few control rods, the effect of control rods can be expressed by Dirac's distribution, that is

$$u(\omega) \sim \sum_{n=1}^N u_n \delta(\omega - \omega_n) . \quad (2.2.14)$$

This expression is substituted into Eq.(2.2.11) and generates

$$\begin{aligned} & \sum_{n_1=1}^N \phi_{0,n_1}^* u_{n_1} \phi_{0,n_1} \\ & + \sum_{n_1=1}^N \sum_{n_2=2}^N \phi_{0,n_1}^* \tilde{G}_{n_1,n_2} u_{n_2} \phi_{0,n_2} \\ & + \sum_{n_1=1}^N \cdots \sum_{n_m=1}^N \phi_{0,n_1}^* u_{n_1} \cdot \tilde{G}_{n_1,n_2} \cdot u_{n_2} \cdot \tilde{G}_{n_2,n_3} \cdots \\ & \quad \cdots \tilde{G}_{n_{m-1},n_m} \cdot u_{n_m} \cdot \phi_{0,n_m} \\ & = 0 , \end{aligned} \quad (2.2.15)$$

where the subscripts $n_1, n_2, \dots, n_m$ describe that the associated variable is evaluated at the point

Equation (2.2.15) prepares a very simple criteria for the criticality of a core equipped with a few rods. The points that satisfy the equation form a $(N-1)$ -dimensional hypersurface in the N -dimensional Euclidian space of u_i 's. This hypersurface may be called as the "criticality surface".

When the highest order m is taken to be 1, this expression coincides with the result given by the ordinary perturbation theory, where ϕ_0^* is considered as the importance. The validity of the perturbation theory is limited only when $\|u\|$ is very small, but Eq. (2.2.15) allows the wide value of $\|u\|$ when the order m of the series is taken enough large.

Example

Let us consider a bare homogeneous slab reactor, whose neutron flux distribution is governed by the equation

$$\frac{d}{dx} \left(D \frac{d}{dx} \phi \right) + (\nu \Sigma_f - \Sigma_a) \phi = \sum_{n=1}^2 \Sigma_c^n \delta(x - x_n) \phi \quad (2.2.16)$$

along with the boundary condition

$$\phi(0) = \phi(H) = 0, \quad (2.2.17)$$

where the points $x=0$ and $x=H$ are the extrapolated boundaries.

For the simplicity, the spatial variable x is transformed into y as

$$x = H y, \quad (2.2.18)$$

then Eqs.(2.2.16) and (2.2.17) become

$$\frac{d^2 \phi}{dy^2} + \pi^2 \phi = \sum_{n=1}^2 u_n \phi(y_n) \delta(y - y_n), \quad (2.2.19)$$

$$\phi(0) = \phi(1) = 0, \quad (2.2.20)$$

where

$$u_n = H^2 \cdot \Sigma_c^n / D, \quad (2.2.21)$$

and the criticality condition for $u=0$ has then been incorporated in the form

$$H^2 (\nu \Sigma_f - \Sigma_a) = D \pi^2. \quad (2.2.22)$$

Fortunately a rigorous criticality condition for the systems (2.2.19), (2.2.20) can be obtained analytically as the four-point boundary value problem. That is

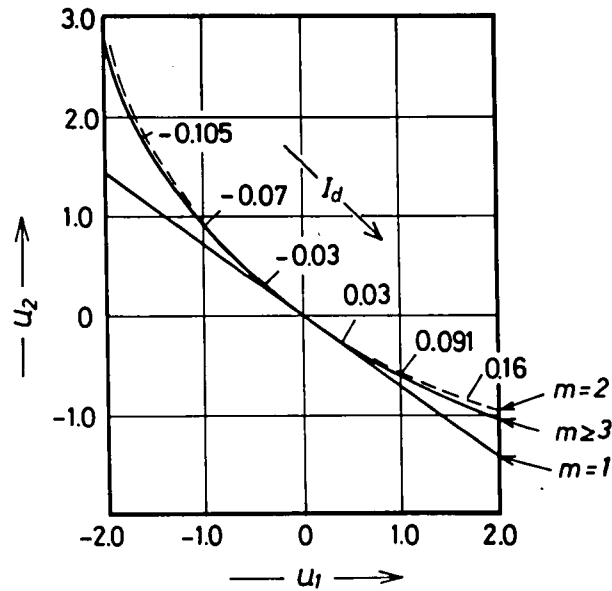
$$\begin{aligned} & \frac{1}{\pi} \sin \pi y_1 \cdot \sin \pi y_2 \cdot \sin \pi (y_2 - y_1) \cdot u_1 u_2 \\ & + \sin^2 \pi y_1 \cdot u_1 + \sin^2 \pi y_2 \cdot u_2 = 0. \end{aligned} \quad (2.2.23)$$

The approximate criticality condition obtained through Eq.(2.2.15) can be compared with the rigorous one in terms of the residual

reactivity error, $\delta\rho$, as

$$\begin{aligned}\delta\rho &= - \frac{\int_0^H \phi^*(x) \cdot \delta\Sigma_a \cdot \phi(x) dx}{\int_0^H \phi^*(x) \cdot \nu\Sigma_f \cdot \phi(x) dx} \\ &= - \frac{D}{H^2} \frac{1}{\nu\Sigma_f} \frac{\phi^2(y_2)(u_2 - \tilde{u}_2)}{\int_0^H \phi^2(x) dx},\end{aligned}\quad (2.2.24)$$

where u_2 is the approximate value by Eq.(2.2.15), and $\tilde{u}_2$ is the rigorous one after assigning u_1 . Figure 2.2.1 shows the criticality curve in u_1 - u_2 plane with the parameter m , the order of the polynomial (2.2.15). In this case, along with Table 2.2.1, it is seen that the order $m=2$ gives a good approximation in comparison with



No discrepancy is found between the strict curve $\tilde{u}_2$ and the approximate solution of order $m \geq 3$. $y_1 = 0.3$, $y_2 = 0.6$.

Fig. 2.2.1 Criticality curves for the core with two rods.

that of $m=1$ or the ordinary perturbation theory. Distortion is introduced into the flux shape by non-zero u_1 and u_2 , and is measured by an index I_d as

$$I_d = \frac{\int_0^1 \phi(y) \sin 2\pi y dy}{\int_0^1 \phi(y) \sin \pi y dy} . \quad (2.2.25)$$

The values of this index is associated with the criticality curves in Fig.2.2.1.

u_1	-2.0	-1.0	-0.5	0.5	1.0	2.0
$\tilde{u}_2$	2.57	0.927	0.406	-0.326	-0.594	-1.006
I_d	-0.125	-0.07	-0.037	0.042	0.091	0.216
$m=1$	-0.61E-3	-0.87E-4	-0.19E-4	-1.3 E-4	-0.4 E-4	-0.9 E-4
$m=2$	-0.2 E-4	-0.5 E-5	0.16E-5	-0.37E-6	0.2 E-5	0.9 E-5
$m=3$	-0.2 E-4	-0.35E-6	0.16E-6	-0.73E-6	-0.13E-5	-0.29E-5
$m=4$	0.11E-5	-0.35E-6	0.16E-6	-0.73E-6	-0.13E-5	-0.32E-6
$m=5$	0.11E-5	0.36E-6	0.16E-6	-0.73E-6	-0.13E-5	-0.32E-6

Table 2.2.1 Residual error in reactivity

2.3 Optimal Control of Steady-State Neutron Flux Distribution in Slab Reactors

2.3.1 Problem Statement

The neutron flux in a thermal reactor is described by the diffusion equation

$$\nabla D \nabla \phi + (\nu \Sigma_f - \Sigma_a) \phi = \frac{1}{v} \frac{\partial \phi}{\partial t} \quad (2.3.1)$$

with the boundary condition

$$\phi(r_B) = 0. \quad (2.3.2)$$

In steady state, the right-hand side of Eq.(2.3.1) must vanish.

Hence the steady state flux distribution satisfies

$$\nabla D \nabla \phi + (\nu \Sigma_f - \Sigma_a) \phi = 0. \quad (2.3.3)$$

The Helmholtz-type Eq.(2.3.3) has a non-trivial solution only when the parameters D , $\nu \Sigma_f$ and Σ_a , altogether, satisfy the criticality condition derived from the boundary conditions. Here the diffusion coefficient D and the fission cross section Σ_f are fixed and the absorption cross section Σ_a is available as "control variable".

The value Σ_a can be divided into two parts Σ_a^c and Σ_a^o , which are the controllable part and the remainder.

The part Σ_a^c has a maximum value $(\Sigma_a^c)_{\max}$ determined from the rod worth, as well as a minimum value, which is zero. In practice, the reactor may attain its criticality with half the length of each rod withdrawn where the incremental worth of the rod motion is largest. This means that Σ_a has upper and lower limits defined by

$$\Sigma_a^o \leq \Sigma_a \leq \Sigma_a^o + (\Sigma_a^c)_{\max}. \quad (2.3.4)$$

The performance criterion to be minimized is

$$J = \int_{\text{core}} \{ \phi(r) - \overline{\phi(r)} \} dr, \quad (2.3.5)$$

where $\overline{\phi(r)}$ is the desired flux distribution.

When a slab reactor is considered, only one spatial variable, t , is needed, which is the distance from the boundary as shown in Fig.2.3.1.

Thus the governing equation of steady state bare homogeneous slab reactor in one-group approximation becomes to;

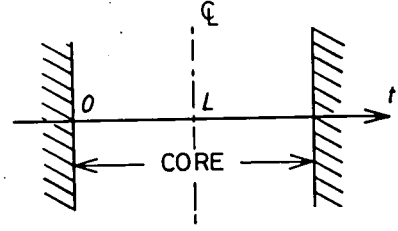


Fig.2.3.1 Bare slab reactor.

$$D \frac{d^2}{dt^2} \phi + (\nu \Sigma_f - \Sigma_a) \phi = 0 \quad (2.3.6)$$

along with the boundary conditions,

$$\phi(0) = \frac{d}{dt} \phi(L) = 0, \quad (2.3.7)$$

and the performance criterion to be minimized is

$$J = \int_0^L (\phi - \bar{\phi})^2 dt, \quad (2.3.8)$$

where D and $\nu \Sigma_f$ are assumed to be constant over the reactor.

Using the buckling notation, Eq.(2.3.6) is written in the more simple form

$$\frac{d^2}{dt^2} \phi + B^2(1 - u) \phi = 0. \quad (2.3.9)$$

In order that the reactor remains in steady state, or that Eq.(2.3.9) possesses a non-trivial solution for $u=0$, B^2 must be equal to the geometrical buckling:

$$B^2 = (\pi/2L)^2. \quad (2.3.10)$$

Assuming that the reactor is designed to attain its criticality at half value of $(\Sigma_a^c)_{max}$,

$$B^2 = \frac{1}{D} \left\{ \nu \Sigma_f - \Sigma_a^0 - \frac{1}{2} (\Sigma_a^c)_{max} \right\}. \quad (2.3.11)$$

Therefore

$$u = \frac{1}{D} \left(\frac{2L}{\pi} \right)^2 \left\{ \Sigma_a^c - \frac{1}{2} (\Sigma_a^c)_{max} \right\}. \quad (2.3.12)$$

The control variable u has the upper and lower limits

$$u_{\min} \leq u \leq u_{\max} , \quad (2.3.13)$$

where

$$u_{\min} = -\frac{2}{D} \left(\frac{L}{\pi}\right)^2 (\Sigma_a^c)_{\max} ,$$

$$u_{\max} = \frac{2}{D} \left(\frac{L}{\pi}\right)^2 (\Sigma_a^c)_{\max} .$$

Denoting ϕ and $d\phi/dt$ by x^1 and x^2 , respectively, the system equation (2.3.9) and the boundary conditions become

$$\left. \begin{aligned} \frac{d}{dt} x^1 &= x^2 \\ \frac{d}{dt} x^2 &= B^2(u-1)x^1 \end{aligned} \right\} , \quad (2.3.14)$$

$$x^1(0) = x^2(L) = 0 . \quad (2.3.15)$$

Then, the performance criterion becomes to

$$J = \int_0^L \{ x^1(t) - \bar{x}^1(t) \}^2 dt . \quad (2.3.16)$$

Pontryagin's maximum principle requires that the optimal control $u = u^*$ and the corresponding trajectory x must satisfy the four conditions P1 to P4 given below.

Consider auxiliary variables ψ^0, ψ^1, ψ^2 and the Hamiltonian H made up from ψ and x :

$$H = \psi^0(x^1 - \bar{x}^1)^2 + \psi^1 x^2 + B^2(u-1)\psi^2 x^1 . \quad (2.3.17)$$

Then there must exist $\psi^i(t)$ ($i = 0, 1, 2$) such that:

P1. ψ^0 is a negative constant.

P2. ψ^1, ψ^2 must be continuous and must satisfy the differential equations

$$\left. \begin{aligned} \frac{d}{dt} \psi^1 &= -\frac{\partial H}{\partial x^1} = -B^2(u-1)\psi^2 - 2\psi^0(x^1 - \bar{x}^1) \\ \frac{d}{dt} \psi^2 &= -\frac{\partial H}{\partial x^2} = -\psi^1 \end{aligned} \right\} \quad (2.3.18)$$

P3. (Transversality condition) The vector $(\psi^1, \psi^2)^t$ must be orthogonal to the initial and terminal manifolds composed of boundary values at $t=0$ and $t=L$, where $(\cdot)^t$ stands for the transpose of a vector.

P4. The function $H(x(t), \psi(t), u) = H(t, u)$ must attain its maximum with respect to $u \in [u_{min}, u_{max}]$ at $u = u^*(t)$ for almost all t .

Here the value ψ^0 can be taken arbitrarily as long as it is negative and is fixed to be $-1/2$ because the system (2.3.18) is homogeneous with respect to ψ 's.

Replacing ψ^1 and ψ^2 by x^3 and x^4 , respectively and linking the two systems (2.3.18) and (2.3.14), the optimal system can be written in the form of a vector differential equation associated with two-point homogeneous boundary conditions:

$$\frac{d}{dt} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \lambda^2 & 0 & 0 & 0 \\ 1 & 0 & 0 & -\lambda^2 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\bar{x}^1 \\ 0 \end{bmatrix}, \quad (2.3.19)$$

where $\lambda^2 = B^2(u-1)$, and

$$x^1(0) = x^4(0) = 0, \quad (2.3.20)$$

$$x^2(L) = x^3(0) = 0. \quad (2.3.21)$$

Employing a matrix $A(u)$, and a forcing vector g , Eq.(2.3.19) can be written in the form

$$\frac{d}{dt} x = A(u)x + g. \quad (2.3.22)$$

2.3.2 The Optimality Condition for the Uniform Desired Distribution

Here a uniform desired distribution $\bar{\chi}_1$ is considered. The magnitude of the desired value $\bar{\chi}_1$ can be taken as 1 without loss of generality because the system (2.3.19) is homogeneous.

As is often the case, the maximum principle may result in a singular solution where the conditions P1-P4 are not enough to decide a definite solution. In our problem, this occurs when $\psi^2=0$ on some closed interval. It is, however, guaranteed to be regular in case when $\bar{\chi}^1$ is uniform over the entire core after the following considerations.

The right-hand side of the Hamiltonian (2.3.17) contains solely u in the third term, and χ^1 --- which physically corresponds to the neutron flux density --- is positive for any t . Hence the condition P4 leads to the following control law:

$$\text{If } \chi^4 > 0 \text{ then } u^* = u_{\max}$$

$$\text{If } \chi^4 < 0 \text{ then } u^* = u_{\min}$$

Because both limits are given in such manner that $u_0 = u_{\max} = -u_{\min}$, the optimal control can be expressed as

$$u^*(t) = u_0 \operatorname{sgn}(\chi^4(t)). \quad (2.3.23)$$

This provides a control law for non-singular case.

The optimal trajectory in case of constant desired distribution, however, is non-singular in the interval $[0, t_1]$, where t_1 is a positive value. If not, the trajectory cannot satisfy the boundary condition (2.3.20) without violating the requirement of continuity.

Suppose that the trajectory is singular on $[t_1, t_2]$, where $\chi^4=0$ requires that $\chi^2 = \chi^3 = 0$ and $\chi^1 = u^1 = 1$ on this interval. Then this assumption introduces the following contradictions.

In the interval $[0, t_1]$, especially in the small subinterval $(t_1 - \varepsilon, t_1)$, $\chi^4(t)$ is definitely either positive or negative. From the requirement of continuity of the trajectory at $t = t_1$,

$x^4(t)$ for $t \in (t_1 - \varepsilon, t_1)$ should be either of the followings;

If $x^4 < 0$, then $u = u_{\min}$ and

$$x^4(t) = \frac{1}{\omega^2} (1 - \cos \bar{t} - \frac{\bar{t}}{2} \sin \bar{t}) \approx \frac{1}{\omega^2} \frac{1}{4!} \bar{t}^4 + O(\bar{t}^6) > 0, \quad (2.3.24)$$

where $\bar{t} = \omega(t - t_1)$,

and if $x^4 > 0$, then $u = u_{\max}$ and

$$\begin{aligned} x^4(t) &= -\frac{1}{\lambda^2} (1 - \cosh \bar{t} + \frac{\bar{t}}{2} \sinh \bar{t}) \\ &\approx -\frac{1}{\lambda^2} \frac{1}{4!} \bar{t}^4 + O(\bar{t}^6) < 0, \end{aligned} \quad (2.3.25)$$

where $\bar{t} = \lambda(t - t_1)$.

These contradictions prove that no optimal trajectory that satisfies the maximum principle has a singular arc.

The above discussion is valid also for reactors equipped with reflectors so long as the one group approximation is adopted.

While the control variable u remains constant, the system equation (2.3.19) is a linear autonomous system. Therefore an initial state x_j , whose fourth element x_j^4 is set to be zero at $t = t_j$ determines the state of the system at any arbitrary time thereafter, in particular at the next switching time t_{j+1} , at which the fourth element of x again vanishes for the first time. Both the time interval $\tau_{j+1} = t_{j+1} - t_j$ and the state in the subsequent time are the functions of x_j , namely

$$\tau_{j+1} = \tau_{j+1}(x_j) \quad (2.3.26)$$

$$x_{j+1} = x(\tau_{j+1}, x_j) = x_{j+1}(x_j) \quad (2.3.27)$$

Relation (2.3.27) can be rewritten using a mapping notation, that is,

$$x_{j+1} = F(x_j) \triangleq F \circ x_j, \quad (2.3.28)$$

where F is valid only for t_{j+1} within $[t_j, L]$. If the last

switching time $t_{\ell+1}$ exceeds L , another mapping F' which maps x_ℓ into $x(L)$ should be considered. Because we are interested only in $x^2(L)$ and $x^3(L)$,

$$\begin{aligned} v' &= (x^2(L), x^3(L))^t = F' \circ x_\ell \\ &= F' \circ F \circ \dots \circ F \circ x(0) \triangleq \hat{F} \circ x(0) \end{aligned} \quad (2.3.29)$$

should be considered. The τ_j 's ($j = 1, \dots, \ell$) and the rest, τ' , satisfy

$$\sum_{j=1}^{\ell} \tau_j + \tau' = L, \quad (2.3.30)$$

where τ_j 's are specified as the time when $x^4(\tau, x_{j-1})$ vanishes, so that

$$x^4(\tau_{j+1}) = x^4(\tau_{j+1}, x_j^1, x_j^2, x_j^3) = 0 \quad (2.3.31)$$

Recalling that x_j^4 is set at zero for any j below ℓ , the vector notation x_j and dx_j can be abbreviated to be (x_j^1, x_j^2, x_j^3) and (dx_j^1, dx_j^2, dx_j^3) without missing any information. Differentiating Eq.(2.3.31) and substituting the relation $\dot{x}_4 = -x^3$ into it, we obtain

$$d\tau_{j+1} = \sum_{i=1}^3 \frac{1}{x^3(\tau_{j+1})} \left. \frac{\partial x^4}{\partial x_j^i} \right|_{\tau=\tau_j} dx_j^i. \quad (2.3.32)$$

Also from Eqs.(2.3.27) and (2.3.32) we obtain

$$dx_{j+1} = \left. \frac{\partial x}{\partial x_j} \right|_{\tau=\tau_{j+1}} dx_j + \left. \frac{\partial x}{\partial \tau} \right|_{\tau=\tau_{j+1}} d\tau_{j+1} \quad (2.3.33)$$

$$= \left[\frac{\partial x}{\partial x_j} + \frac{1}{x^3} \frac{\partial x}{\partial \tau} \left(\frac{\partial x^4}{\partial x_j} \right)^t \right] \bigg|_{\tau=\tau_{j+1}} dx_j, \quad (2.3.34)$$

where $\partial x / \partial x_j$ is a 3×3 matrix whose (m, n) element is $\partial x^m / \partial x_j^n$, and $\partial x^4 / \partial x_j$ is the gradient of x^4 with respect to x_j . Then Eq.(2.3.34) is written in the more simple form

$$dx_{j+1} = D_{j+1} D_j \cdots D_1 dx(0) \quad (2.3.35)$$

$$\triangleq D_{j+1} dx(0), \quad (2.3.36)$$

where the (m, n) element of D_{j+1} is

$$\left(\frac{\partial x^m}{\partial x_j^n} + \frac{1}{x^3} \frac{\partial x^m}{\partial \tau} \frac{\partial x^4}{\partial x_j^n} \right) \Big|_{t=t_{j+1}}.$$

From Eqs. (2.3.20) and (2.3.32)

$$\begin{aligned} d\tau' &= - \sum_{j=1}^L d\tau_j \\ &= - \sum_{j=1}^L \left[\frac{1}{x_j^3} \left(\frac{\partial x^4}{\partial x_j} \right)^T \right] dx_j \\ &= - \left[\sum_{j=1}^L \frac{1}{x_j^3} \left(\frac{\partial x^4}{\partial x_{j-1}} \right)^T D_j \right] dx(0), \end{aligned} \quad (2.3.37)$$

$$\triangleq E^T dx(0). \quad (2.3.38)$$

Substituting $d\tau'$ into Eq. (2.3.33) instead of $d\tau_j$,

$$\begin{aligned} dx(L) &= \frac{\partial x}{\partial x_L} \Big|_{t=L} dx_L + \frac{\partial x}{\partial \tau} \Big|_{t=L} d\tau' \\ &= \left[\frac{\partial x}{\partial x_L} D_L + \frac{\partial x}{\partial \tau} E^T \right] \Big|_{t=L} dx(0) \end{aligned} \quad (2.3.39)$$

$$\triangleq P' dx(0), \quad (2.3.40)$$

where P' is a 4×3 matrix.

Thus a linear relationship has been obtained between the incremental variations of $x(0)$ and $x(L)$. In these formulas, $\partial x(\tau)/\partial x_j$ and $\partial x(\tau)/\partial \tau$ can be explicitly represented by making use of the transition matrix (Appendix A) of the linear state equation (2.3.23) associated with the constant u in two ways according to $u \gtrless 1$.

If the initial value $x(0)$ is given such as to satisfy the initial boundary condition, $x^1(0)$ and $x^4(0)$ are both set at zero and only the second and the third elements of $x(L)$ retain a part of P' :

Letting $v = (x^2(0), x^3(0))^T$, $v' = (x^2(L), x^3(L))^T$
and

$$P = \begin{bmatrix} P_{22} & P_{23} \\ P_{32} & P_{33} \end{bmatrix}$$

then

$$dv' = P dv . \quad (2.3.41)$$

As the system (2.3.29) is autonomous, there should exist an $\chi(L)$ uniquely for any $\chi(0)$, or a v' for any v . Among these v and the resulting v' , only $v'=0$ can be expected to correspond to the optimal control. Therefore if such a v as to give $v'=0$ should be found, this may be the optimal. Consequently the principle seeking the optimal control has been deduced to finding the zero of the function $v'(v)$.

The most classic method of solving an equation consisting of continuous and differentiable functions is the modified Newton's method. But this method requires the satisfaction of excessively demanding conditions in order to converge and lead to a solution, and in most cases it fails. The more complex but successful method is the steepest descent method, based on a dummy potential and a random number generator. This is the shooting method.

The optimal control must satisfy Pontryagin's maximum principle and the resulting equation is

$$v'(v) = 0 . \quad (2.3.42)$$

Now consider a dummy potential

$$R(v') = (v')^t Q v' , \quad (2.3.43)$$

where Q is a suitable symmetric positive definite 2×2 square matrix. The total derivative of R is the increment for a small change in v :

$$dR = 2(v')^t Q dv . \quad (2.3.44)$$

Substituting Eq.(2.3.41) into Eq.(2.3.44), we obtain

$$dR = 2(v')^t Q P dv . \quad (2.3.45)$$

Let the initial guess of v be v_i , and the resulting v' be v_i' ;

then R can take a positive value, and the slightly perturbed $\mathbf{v}_i$ with small variation $\Delta \mathbf{v}_i$ yields

$$R + \Delta R = (\mathbf{v}'_i)^t \mathbf{Q} \mathbf{v}'_i + 2(\mathbf{v}'_i)^t \mathbf{Q} \mathbf{P} \Delta \mathbf{v}_i. \quad (2.3.46)$$

If R were very small, the next guess $\mathbf{v}_{i+1} = \mathbf{v}_i + \Delta \mathbf{v}_i$ should result in $R + \Delta R = 0$, namely

$$R + \Delta R = (\mathbf{v}'_i)^t \mathbf{Q} (\mathbf{v}'_i + 2\mathbf{P} \Delta \mathbf{v}_i) = 0. \quad (2.3.47)$$

Therefore

$$\Delta \mathbf{v}_i = -\frac{1}{2} (\mathbf{P})^{-1} \mathbf{v}'_i. \quad (2.3.48)$$

Then the next guess $\mathbf{v}_{i+1}$ is

$$\mathbf{v}_{i+1} = \mathbf{v}_i + \Delta \mathbf{v}_i = \mathbf{v}_i - \frac{1}{2} (\mathbf{P})^{-1} \mathbf{v}'_i. \quad (2.3.49)$$

Equation (2.3.49) gives a recurrence formula applicable only when a very good initial guess is given, but in most cases, the initial guess cannot be expected to be so good, and Eq.(2.3.49) would then no longer apply: the algorithm would thus fail to converge. In such cases, the steepest descent method is more effective in obtaining an appropriate $\Delta \mathbf{v}_i$, especially when the method is modified by a random positive matrix though this requires more computation time. In Eq. (2.3.46), R decreases most rapidly when $\Delta \mathbf{v}$ runs inversely parallel to the vector $(\mathbf{Q} \mathbf{P})^t \mathbf{v}'_i$. But the complexity of R due to the involved nature of $\mathbf{v}'(\mathbf{v})$ leaves an ambiguity as to what value of $\Delta \mathbf{v}_i$ minimizes R . Some information on this point is obtained by examining the value of $\Delta \mathbf{v}_i$ given by Eq.(2.3.48). The correction that should be added to $\mathbf{v}_i$ becomes

$$\Delta \mathbf{v}_i = \frac{1}{2} \mu \frac{\|(\mathbf{P})^{-1} \mathbf{v}'_i\|}{\|(\mathbf{Q} \mathbf{P})^t \mathbf{v}'_i\|} (\mathbf{Q} \mathbf{P})^t \mathbf{v}'_i, \quad (2.3.50)$$

where μ is a positive number of the order of unity and is left undetermined for the convenience of numerical computation. A modification is brought by the matrix

$$T_n = k_n \begin{bmatrix} \cos \theta_n & \sin \theta_n \\ -\sin \theta_n & \cos \theta_n \end{bmatrix}, \quad (2.3.51)$$

where k_n and θ_n are random numbers bounded by $0 < k_n < 1$ and $-\pi/2 < \theta_n < \pi/2$. The modification of Δv_i by T_n is expressed by

$$\Delta v_{ni} = -T_n(QP)^t v'_i. \quad (2.3.52)$$

Among these Δv_{ni} , the one that minimizes R is taken as the next correction.

In order that the algorithm presented above should provide a good convergence, the smoothness of the mapping $v'(v)$ should be retained. But in our case neither this smoothness nor the homeomorphism can be expected especially when a large value is assigned to u_0 .

The nonlinearity in $\hat{F}$ arises mainly from the nonlinearity of the switching time τ_{j+1} with respect to the initial value $x(0)$, being the least positive root of a transcendental equation containing x_j as the parameter:

$$x^4(\tau_j, x_j) = 0 \quad (2.3.53)$$

When $u = u_{max} > 1$, the concrete form of $x^4(\tau, x_j)$ becomes

$$\begin{aligned} x^4(\tau, x_j) &= (x_j^3 + \lambda^{-2}) \cosh \lambda \tau \\ &+ \lambda^{-1}(x_j^2/2\lambda^2 - x_j^3) \sinh \lambda \tau \\ &- (\tau/2\lambda^2) x^2(\tau, x_j) - \lambda^{-2} \end{aligned} \quad (2.3.54)$$

$$x^2(\tau, x_j) = x_j^2 \cosh \lambda \tau + \lambda x_j' \sinh \lambda \tau,$$

and when $u = u_{min} < 1$,

$$\begin{aligned} x^4(\tau, x_j) &= (x_j^3 - \omega^{-2}) \cos \omega \tau \\ &- \omega^{-1}(x_j^2/2\omega^2 + x_j^3) \sin \omega \tau \end{aligned}$$

$$+ (\tau/2\omega^2) x^2(\tau, x_j) + \omega^{-2}, \quad (2.3.55)$$

$$x^2(\tau, x_j) = x_j^2 \cos \omega \tau - \omega x_j^1 \sin \omega \tau,$$

where $\lambda = B\sqrt{u_{\max}-1}$, $\omega = B\sqrt{1-u_{\min}}$.

Hereafter the case of $u=u_{\max}$ alone can be discussed without sacrificing any intrinsic property. The three independent initial values x_j^1 , x_j^2 and x_j^3 are transformed into new independent variables a , b and c by the transformation

$$a = x_j^1/x_j^2, \quad b = x_j^2/2 - x_j^3, \quad c = 2/x_j^2. \quad (2.3.56)$$

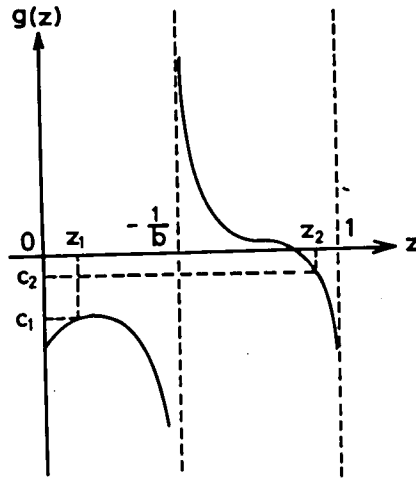
Further, we define a new variable z to replace τ :

$$\lambda \tau = \tanh^{-1} z.$$

Then the original equation (2.3.53) becomes

$$g(z) \triangleq \frac{z^2 + 2az + 1}{z + b} \cdot \frac{1}{z} \tanh^{-1} z = c. \quad (2.3.57)$$

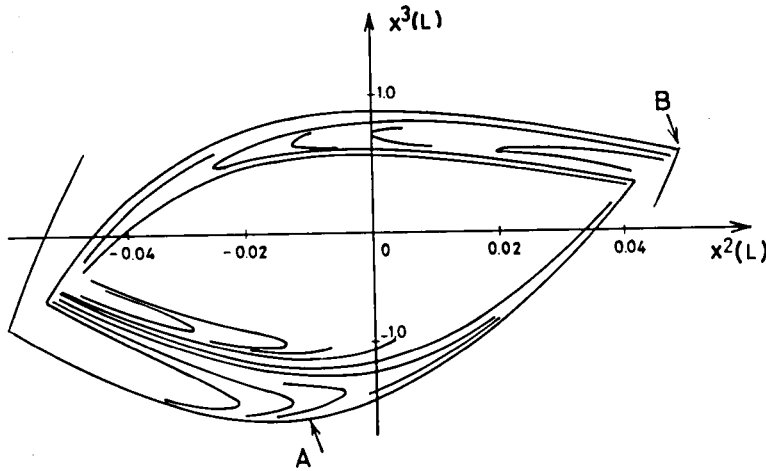
The conceptual curve of $g(z)$ for a , b and c in the near optimal



A root z_1 jumps to z_2 as c changes from c_1 to c_2 .

Fig.2.3.2 A conceptual curve of $g(z)$.

case is illustrated in Fig.2.3.2, which shows how small a change in variable C can cause a complicated change, such as jump, in the root of Eq.(2.3.57). When a jump occurs in τ_j , the mapping $\hat{F}$ acquires a nonlinearity which cannot be expressed analytically nor in Taylor's expansion. Such a case is shown in Fig.2.3.3, where the linear parallel lines in the $x^2(0) - x^3(0)$ plane are mapped into the $x^2(L) - x^3(L)$ plane, and the images are seen to be very complicated. In the neighbourhood of B , the number of switching times changes from 3 to 5. This is the case when the jump in τ_j occurs before the last switching time. On the other hand, the point A specifies a case where the last switching time was delayed and exceeded the end point $t=0$, and here the switching time changes in number from 3 to 4.



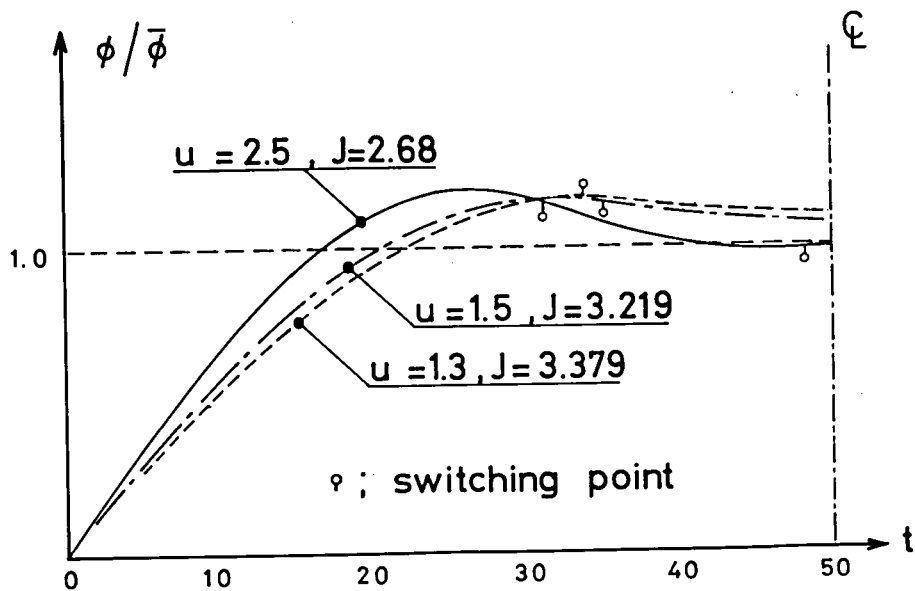
$L=50\text{cm}$, $u_0=5.0811$. The pre-images are located in the neighbourhood of $x^2(0)=0.0919$, $x^3(0)=5.4461$.

Fig.2.3.3 Images of parallel lines in $x^2(0) - x^3(0)$ plane
Mapped into $x^2(L) - x^3(L)$ plane.

2.3.3 Numerical Results and Discussions

The optimal neutron distribution is illustrated in Fig.2.3.4

for $u_0 = 1.3, 1.5$ and 2.5 . If u_0 is smaller than 1.5 , any initial guess results in rapid convergence. In such case, the number of switching times is only one. For u_0 greater than 1.6 , however, the convergence largely depends upon the initial guess. The mapping $\hat{F}$ near these junctures, such as at the points A and B in Fig.2.3.3, is very complicated. If a guess falls on one of these points, the next guess should deviate far from the optimal point. When $u_0=2.5$, a distinctly ill-conditioned case, the optimal solution has been obtained by almost random search after encircling the domain where the optimal solution is expected to lie making use of the ordinary steepest descent method.



The control variable u has its minimum value at the left end and the maximum that comes after the switching point. It is minimum also at the center for $u_0=2.5$.

Fig.2.3.4 Optimal Flux Distribution.

An examination of the solutions obtained indicates that the larger the bounds of u allowed, the smaller the value obtained of J_0 , which is the minimized J . However the peak value or the hot spot factor grows larger as J_0 is reduced. This undesirable phenomenon can be attributed to the fact that the desired flux distri-

bution is made constant all over the core including the boundary, despite the physical necessity of the flux to be continuously differentiable even at the boundary. Such a situation also occurs in the Fourier expansion of a square wave, which is known as Gibbs' phenomenon. However, if the governing equation is of a two group model provided with reflector, where the flux distribution is not required to be zero at the interface of the core and the reflector, such a difficulty would not appear. Figure 2.3.5 presents the trajectory of the optimal flux distribution in the phase plane for $u_0=2.5$.

The times of switchings increase as the control bound becomes larger and the calculational difficulties become serious. When u_0 is large, frequent switchings take place around the center region of the core as if the slipping motion occurs in a relay system.

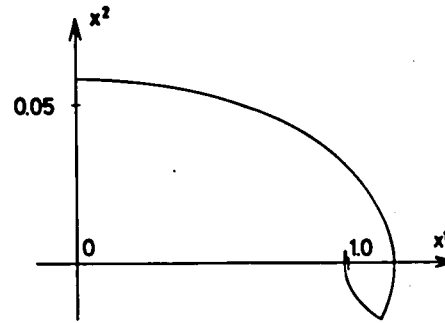


Fig.2.3.5 Optimal trajectory in phase plane for $u_0=2.5$.

Another approach to obtain this solution shall be presented after the slight modification of the control law (2.3.23) into the one using a saturation function instead of the stepwise *sgn* function. This method will provide a suboptimal but more realistic and realizable solution for $u(t)$.

In the preceding paragraph, the uniform distribution was adopted as the desired distribution and it was guaranteed that this desired distribution does not require a singular solution. However, the larger the bound u_0 for the control becomes, the more singular-like solution is required.

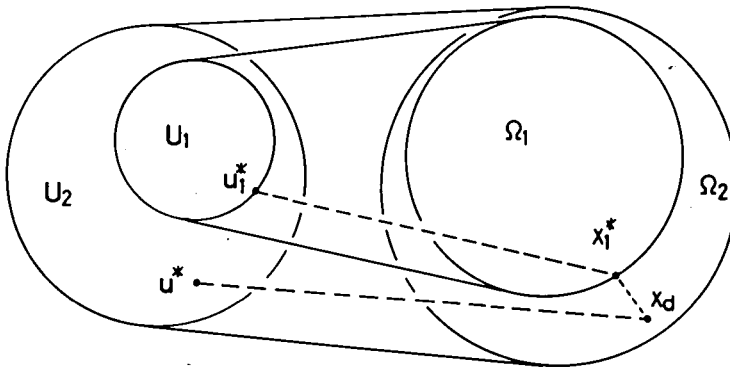
We can consider the attainable closed region Ω of x , the trajectory of the state variables, in X corresponding to the prescribed closed region of admissible control u . Let us consider two classes U_1 and U_2 of admissible controls with the relation

$$U_1 \subset U_2 \quad (2.3.58)$$

holds, then necessarily the corresponding Ω_1 and Ω_2 have the relation

$$\Omega_1 \subset \Omega_2 \quad (2.3.59)$$

as is depicted in Fig. 2.3.6.



When U is large, the desirable trajectory x_d is included in Ω .

Fig.2.3.6 Closed region of controls, U , and the corresponding region of the trajectories, Ω .

If the desired distribution is located outside of Ω , the bang-bang control (2.3.23) will be concluded. This is because the optimal bang-bang control is located on ∂U_1 of U_1 and the corresponding optimal distribution is also on the boundary $\partial \Omega$ of Ω which is the nearest from the desired point x_d . This is the case of the uniform desired distribution in this section. However, if the region becomes wider, then x_d may be contained inside of Ω , and the admissible variation δu for u will coincide with the full space: there may not be any suspending hyperplane and the condition P-2 will not provide any information about the optimal control. This situation requires that $\psi^2 = 0$ on some interval $[t_1, t_2] \in [0, L]$ and is generally named as the singular case.

As an example of the singular case, we can consider

$$\chi_d = \sin \pi t / 2L \quad (2.3.60)$$

as the desired distribution. Substituting this χ_d into Eq.(2.3.19), it is easily verified that the solution

$$u = 0 \quad (2.3.61)$$

$$\chi' = \sin \pi t / 2L \quad (2.3.62)$$

$$\psi^2 = 0, \quad t \in [0, L] \quad (2.3.63)$$

satisfies the necessary conditions P1-P4 and the performance index J attains the absolute minimum 0 for this solution. This solution cannot be determined from Eq.(2.3.23).

In the former works [Le.65], [Am.66] and [Am.67], the backward substitution was used in order to obtain the control to attain the desired flux distribution. That method yields the control after substituting the desired flux distribution into the neutron equation.

After the consideration above, it can be seen that the singular solution for our case may fall within the class of the solutions that are to be obtained by the backward substitution. Conversely the problem which should prepare the nonsingular solution for our method will not meet the backward substitution method.

2.3.4 Approximate Solution

It has been shown in the preceding section that the true optimal solution is apt to be difficult to be obtained and may fall into the singular case as u_0 tends to large. Our original aim was to distribute the neutron flux as we want and was not strictly concerned with minimizing the performance index J . Therefore some suboptimal control may be substituted instead of the intractable rigorous solution.

The method shown in the preceding subsection may fail in seeking the solution of the two-point boundary value problem. One of the reason that the mapping is very complicated to deal with is that the right side of Eq.(2.3.19) has the jump discontinuity because of the control law (2.3.23), and does not satisfy the Lipschitz's condition which assures the regularity of the mapping of the initial value to the terminal value.

Equation (2.3.23), composed of the step function, however, may be embedded in the class of the functions

$$u_{\Delta}^* = u_0 \text{sat}(\psi^2/\Delta) \quad (2.3.64)$$

which is associated by Δ as the parameter as shown in Fig.2.3.7.

For the fixed value of Δ , the Lipschitz's condition is satisfied and the continuous and differentiable mapping from one boundary to another is certified.

Because of this characteristic, the shooting method will make more success than in the original problem. The two-point boundary value problem to be solved here is

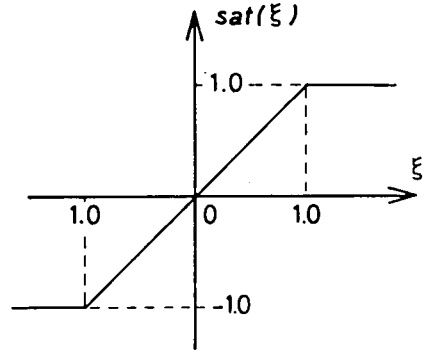


Fig.2.3.7 Graph of $\text{sat}(\xi)$.

$$\dot{x}^1 = x^2 \quad (2.3.65)$$

$$\dot{x}^2 = B^2 \{ u_0 \text{sat}(x^4/\Delta) - 1 \} x^1 \quad (2.3.66)$$

$$\dot{x}^3 = -B^2 \{ u_0 \text{sat}(x^4/\Delta) - 1 \} x^4 - x^1 + \bar{x}^1 \quad (2.3.67)$$

$$\dot{x}_4 = -x^5 \quad (2.3.68)$$

with the boundary conditions

$$x^1(0) = x^4(0) = 0 \quad (2.3.69)$$

$$x^2(L) = x^3(L) = 0 \quad (2.3.70)$$

The algorithm to solve this problem is similar to the exact one shown in the preceding paragraph but the neighbouring mapping matrix is substituted by the solution of the matrix differential equation

$$\dot{P}(t) = \left. \frac{\partial f(x)}{\partial x} \right|_K P(t) \quad (2.3.71)$$

with the initial conditions

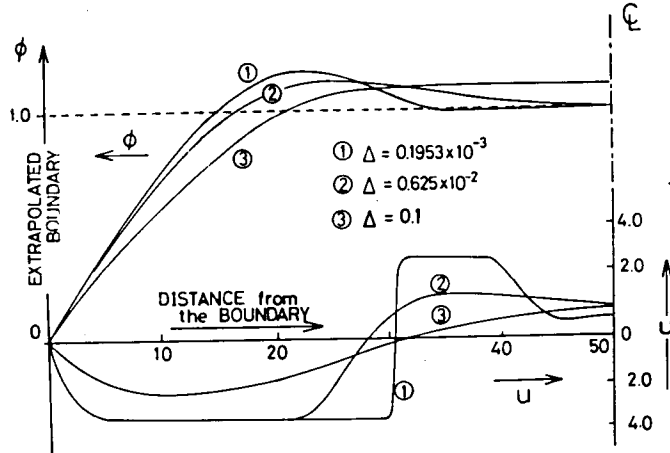
$$P(0) = \mathcal{I} \quad , \quad (2.3.72)$$

where $f(x)$ and $\left. \frac{\partial f(x)}{\partial x} \right|_K$ stand for the right-hand side of Eqs.(2.3.65-68) and the derivative evaluated along the trajectory corresponding to the previously assigned initial value, respectively, and $\mathcal{I}$, the identity matrix. As is easily verified, the right-hand side of Eq.(2.3.71) satisfies the Lipschitz's condition, and the continuity of the mapping is guaranteed.

The algorithm runs as follows: Initially, enough large value is assigned to Δ and the two-point boundary value problem is solved in the like manner as was shown in the preceding paragraph. The solution for this process can easily be obtained because the mapping of the initial value to the final value is rather plain when Δ is large. After the temporary solution has been obtained, it is used as the guess for the solution of the next step to which a half of the Δ is assigned. By this recursive process, each initial guess is always put near the right temporary solution, and each step can get to the solution after a few times of iterations.

This process has been applied for the case of $u_0 = 3.5$ as is shown in Fig. 2.3.8 that is difficult to be dealt with by the exact method. The shooting has been made from the center to the extrapolated boundary, contrary to the exact method. The figure shows that the shape of the neutron flux distribution has already arrived at its

optimal shape, though the control is not yet settled in its distribution near the center. This is because the contribution of u is much less in the center region, where the control is not bang-bang shaped, than it is around the boundary region and it is seen that the control is near singular.



The control u takes intermediate value about the center region.

Fig.2.3.8 Approximate solution by Eq.(2.3.64).

2.3.5 Two-Group Model

In this section the two-group model of a slab reactor equipped with reflectors is considered.

The governing equation in the core is

$$\nabla D_1 \nabla \phi_1 - \Sigma_R \phi_1 = -\nu \Sigma_f \phi_2$$

$$\nabla D_2 \nabla \phi_2 - \Sigma_a \phi_2 = -\rho \Sigma_R \phi_1, \quad (2.3.73)$$

where the subscripts 1 and 2 correspond to fast and thermal neutrons, respectively. In the reflectors, where no fuel is contained, Σ_f is zero, and to distinguish the variables and constants from those applying to the core, the notations relevant to the reflector are primed:

$$\nabla D_1' \nabla \phi_1' - \Sigma_R' \phi_1' = 0$$

$$\nabla D_2' \nabla \phi_2' - \Sigma_a' \phi_2' = -\rho' \Sigma_R' \phi_1' . \quad (2.3.74)$$

The performance index to be minimized is

$$J = \int_{\text{CORE}} (\phi_2 - \bar{\phi}_2)^2 dr . \quad (2.3.75)$$

The configuration of the slab reactor considered here is shown in Fig.2.3.9 and the symbol t is used as the space variable.

Replacing the differential operators in Eqs.(2.3.73) and (2.3.74) by ordinary differential operators and making a suitable variable transformation, the governing equation in the core becomes

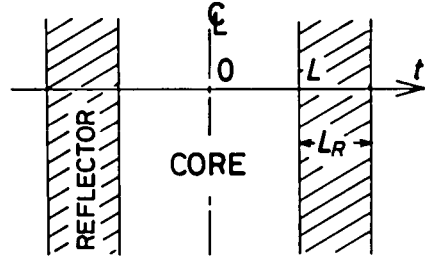


Fig.2.3.9 Reflected slab reactor.

$$\frac{d}{dt} x = A x , \quad 0 \leq t \leq L , \quad (2.3.76)$$

where t : distance from the center
of the core

L : half width of the core

$$x : (x^1, x^2, x^3, x^4)^t$$

$$x^1 = \phi_1, \quad x^2 = \phi_2$$

$$x^3 = \frac{d}{dt} \phi_1, \quad x^4 = \frac{d}{dt} \phi_2$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_{31} & a_{32} & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \end{bmatrix}$$

$$a_{31} = \Sigma_R / D_1 , \quad a_{32} = - \nu \Sigma_f / D_1$$

$$a_{41} = -p \Sigma_R / D_2 , \quad u = \Sigma_a / D_2 .$$

The absorption cross section Σ_a can be divided into two parts, Σ_a^c and Σ_a^0 , the former representing the part of controller and the latter, the remaining part. The variable u is then bounded by

$$0 \leq u_{\min} \leq u \leq u_{\max} , \quad (2.3.77)$$

where

$$u_{\max} = (\Sigma_a^0 + \Sigma_{a,\max}^c) / D_2$$

$$u_{\min} = \Sigma_a^0 / D_2 .$$

In the reflector

$$\frac{d}{dt} x' = A' x' , \quad L \leq t \leq L+L_R , \quad (2.3.78)$$

where

$$A' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ (a'_{31})^2 & 0 & 0 & 0 \\ -(a'_{41})^2 & (a'_{42})^2 & 0 & 0 \end{bmatrix}$$

$$(a'_{31})^2 = \Sigma_R' / D_1' , \quad (a'_{41})^2 = p \Sigma_R' / D_2'$$

$$(a'_{42})^2 = \Sigma_a' / D_2' , \quad L_R : \text{thickness of the reflector} .$$

The performance index (2.3.75) becomes

$$J = \int_0^L (x^2 - \bar{x}^2)^2 dt , \quad (2.3.79)$$

where

$$\bar{x}^2 = \bar{\phi}_2 .$$

The boundary conditions are that:

- (1) The neutron flux vanishes at the extrapolated boundary.
- (2) The net current of neutrons is continuous at the interface of the core and the reflector.
- (3) The symmetrical composition of the reactor requires that the net current is zero at the center plane.

These boundary conditions can be summed up into

$$\begin{bmatrix} k_1 & 0 & k_3 & 0 \\ 0 & k_1 k_2 & c' k_2 (k_3 - k_4) & k_1 k_4 \end{bmatrix} \chi(L) = 0, \quad (2.3.80)$$

where

$$k_1 = D'_1/D_1, \quad k_2 = D'_2/D_2$$

$$k_3 = (1/a'_{32}) \tanh a'_{31} L_R$$

$$k_4 = (1/a'_{42}) \tanh a'_{42} L_R$$

$$c' = (a'_{41})^2 / \{ (a'_{31})^2 - (a'_{42})^2 \}$$

and

$$\chi^3(0) = \chi^4(0) = 0. \quad (2.3.81)$$

The optimal control $u^* = u^*(t)$ must satisfy the maximum principle. We consider a Hamiltonian and the adjoint variables χ^i ($i = 5, 6, 7, 8$):

$$\begin{aligned} H = & -\frac{1}{2}(\chi^2 - \bar{\chi}^2)^2 + \chi^5 \chi^5 + \chi^4 \chi^6 + \chi^7(a_{31} \chi^1 + a_{32} \chi^2) \\ & + \chi^8(a_{41} \chi^1 + u \chi^2). \end{aligned} \quad (2.3.82)$$

Physically χ^2 corresponds to the thermal flux and must always be positive. Hence H acquires its maximum value when

$$u = u_{\max} \quad (\chi^8 > 0)$$

$$u = u_{\min} \quad (\chi^8 < 0). \quad (2.3.83)$$

The adjoint variables χ^i ($i = 5, 6, 7, 8$) must satisfy the adjoint equation

$$\frac{d}{dt} \hat{x} = -A^t \hat{x} + (0, x^2 - \bar{x}^2, 0, 0)^t \quad (2.3.84)$$

where

$$\hat{x} = (x^5, x^6, x^7, x^8).$$

Equations (2.3.80) and (2.3.81) define the two-folds S_0 and S_1 on which the initial and the terminal values respectively lie. Both

S_0 and S_1 are two-dimensional subspaces and therefore possess two linearly independent vectors:

$$S_0 : (1, 0, 0, 0)^t, (0, 1, 0, 0)^t$$

$$S_1 : (m_{13}/m_{11}, m_{23}/m_{22}, -1, 0)^t$$

$$(0, m_{24}, 0, -m_{22})^t.$$

where m_{ij} is the (i, j) element of the coefficient matrix M of Eq.(2.3.80). The transversality condition requires that $\hat{x}$ transverses these vectors at $t=0$ and $t=L$:

$$x^5(0) = x^6(0) = 0 \quad (2.3.85)$$

$$\begin{bmatrix} m_{13}/m_{11} & m_{23}/m_{22} & -1 & 0 \\ 0 & m_{24} & 0 & -m_{22} \end{bmatrix} x(L) = 0 \quad (2.3.86)$$

Denoting the matrix in Eq.(2.3.85) as $M^\perp$, and a new vector as χ , composed of $\hat{x}$ and the former χ , the boundary and transversality conditions at $t=0$ and $t=L$ are combined into the two equations

$$x^i(0) = 0, (i = 3, 4, 5, 6) \quad (2.3.87)$$

and

$$\begin{bmatrix} M & 0 \\ 0 & M^\perp \end{bmatrix} \chi(L) = 0. \quad (2.3.88)$$

Given an initial value $\chi(0)$ which satisfies Eq.(2.3.86), a

matrix that transfers an incremental change in $\chi(0)$ into $\chi(L)$ can be easily obtained analytically, as was done in the one-group case, and the same technique can be used in numerical computation. But the process is far more difficult to converge than in the previous case, because the sum of the order of the system equation (2.3.76) and the adjoint system (2.3.83) is as high as 8 in all, that is, the analytical solution for χ^8 is made up of 8 linearly independent exponential and sinusoidal functions. The equation

$$\chi^8(\tau) = 0 \quad (2.3.89)$$

is not only difficult to be solved but also the mapping $\hat{F}$ is too complicated to treat. Such circumstances have prevented our attempt on the machine computation of this case.

Chapter 3. Dynamical Control of In-Core Power Distribution

3.1. Introduction

Modern power reactors have their increasing size of cores in order to attain the less expensive energy cost. When the cores are small in size, the power distribution will hardly change the shape against the spatially biased parameter variation.

For the large sized cores, however, much distortion may be caused by the slight local change of parameters. This trend may inflict the xenon-induced spatial instability on the power reactors after a flux tilt has been introduced by startup, rod pattern alternation, or by load fluctuation. The regulating control is required not only from the standpoint of xenon instability but also because the distortion in power release profile will make the hot spot travel in the core and the thermal stress will be accumulated on the sheath of fuel pins that will increase the probability of the fuel failure. Therefore the flux tilt in the core is desired to be removed as fast as possible.

In the last decade, efforts have been devoted to synthesize the optimal control for the spatially distributed cores.

Weaver & Vanasse [We.67] have devised a technique to determine the optimal feedback coefficients for a system described in the frequency region, and have applied their method to a nodally represented core and also to a coupled core.

Stacey [St.70a] and Hsu [Hs.67] have used the partial differential form, and have applied a variational approach to obtain

the control. Stacey's method provides a method to calculate an optimal open loop control by the direct method of variational calculus, which yields an algebraic equation to be numerically solved. An artificial expression of the control function in the system equation permits to take account of the temperature feedback effect. Hsu have developed Pontryagin's maximum principle for distributed parameter systems, and applied the method to the regulator problem and to the minimum time problem. Also the Liapunov functional was constructed to verify the stability of the system, and was further utilized to find a suboptimal control law.

Wiberg [Wi.67] has treated the optimal control of xenon spatial oscillation, by using the method of expansion in Kaplan modes [Ka.61] in order to apply the modern control theory of lumped parameter systems, and the optimal feedback coefficients are given formally as a solution of the regulator problem. A numerical example is presented for a very simple case.

The modal expansion method has been used by Suda [Su.68] to examine the controllability and the maintainability of the flux distribution by control rods, and to solve the regulator problem.

Kyong [Ky.68] have applied the function space method to the regulator problem of the prompt neutron equation of the core, and has developed a method for solving numerically the integral equation, which was derived from the open loop optimal control problem for general linear system, initially formulated by Balakrishnan [Ba.63], [Ba.65] in the abstract space. The function space method used there has the advantage of giving the global uniqueness and the existency of the solution.

Kuroda & Makino [Ku.69] dealt by the principle of optimality with the terminal cost problem with a control energy constraint. Kuroda & Makino [Ku.71] also applied the results by Lions [Li.71] for the regulator problem of a linear one-group neutron model without the precursor.

A distinct approach has been tried by Sekimizu *et al.* [Se.72],

by means of the function space method, to obtain the control which minimizes the functional corresponding to the distance between the available trajectory and the set of desirable trajectories. This idea enabled them to avoid the excessive calculating labour worthless to the original aim.

A terminal cost problem which transfers the core state to another for a one group neutron equation associated with precursor density was studied by Iwazumi & Koga [Iw.73]. The feedback solution was obtained through the Kaplan mode expansion technique as well as by the function space method. An idea has there been devised to circumvent the space dependency of the core parameters that makes it difficult to obtain the Kaplan mode in an analytical manner.

The practical significance is further found in the regulation of the xenon-induced oscillation in pressurized water reactors. An elaborate survey paper was given by Stacey [St.70] covering the whole problems; history, physics, calculational problems as well as the control problems.

Stacey [St.68] obtained the optimal control by two regulating controller through dynamic programming method, and reformulated [St. 69] the problem as a problem in the calculus of variations for distributed parameter systems, both resulting in open loop controls. Christie & Poncelet [Chr.73] considered the problem, in contrast to Stacey's theoretical work, from the standpoint of view of practical operator manual control, based on control theory concepts and considerations of the physics of the control. El-Bassioni & Poncelet [ElB.74] obtained a simple optimal bang-bang control to suppress the xenon-induced oscillation knowing the switching time by observing the axial offset of the flux distribution.

This chapter contains the open loop regulator problem for the neutron-precursor dynamics in section 2; the stationary feedback control about the temperature-precursor dynamics coupled by steady state neutron diffusion equation, in section 3.

In section 2, the two group diffusion approximation is adopted as the neutron governing equation in order to suitably express the core which is large compared to the diffusion length of thermal neutrons. This neutron equation is reduced into a set of the eigenvalue problems for the matrices obtained by replacing the spatial operator, Laplacian, with the bucklings. The system equation, initially having been written in a partial differential equation, is rewritten in a Volterra type integral equation associated with the kernel readily synthesized in an infinite series. The integral expression of the system equation helps us to obtain the adjoint operator analytically, that is necessary in the course of synthesizing the ultimate integral equation which should be satisfied by the optimal control. The reduction of the equations are made in terms of a Hilbert space. The control equation in a Fredholm's integral equation of the 2nd kind is solved numerically by the successive approximation.

It is also shown that the introduction of steady state neutron approximation can successfully be made and reduces the calculational effort.

In section 3, the reactor core dynamics model is represented by the coolant temperature averaged along the channel, and by the precursor density. The neutrons are assumed to be governed by the one-group diffusion equation in steady state. This is because the discrepancy of the neutron distribution from the steady state which is determined by the temperature and the precursor density, dies out quickly after a change has been introduced in other affecting parameters. In order to obtain the stationary feedback control law for the pointwise control, the Hamilton-Jacobi theory is applied to the space dependent system. The result is given in a Riccati-type algebraic equation for the matrix kernel function. This equation is further transferred to a set of simultaneous equations with respect to the coupling matrices of the Helmholtz eigenfunction expansion of the kernel.

3.2 Optimal Open-Loop Regulator Problem

3.2.1 Integral Expression of Evolution Equation

The power distribution in a reactor core can be described by the equation

$$\frac{\partial}{\partial t} X(\omega, t) = S X(\omega, t) + \mathcal{F}(\omega, t), \quad \omega \in \Omega, \quad (3.2.1)$$

where ω and Ω stand for the place and the core respectively. The term $X(\omega, t)$ corresponds to K -dimensional state vector function and $\mathcal{F}$, to the control. The physical state should also be associated with the initial condition

$$X(\omega, 0) = X_0(\omega), \quad \omega \in \Omega, \quad (3.2.2)$$

and with the boundary condition

$$S_b X(\omega, t) = 0, \quad \omega \in \partial\Omega, \quad t \geq 0, \quad (3.2.3)$$

where $\partial\Omega$ stands for the extrapolated boundary.

One of the characteristics in the reactor core is that the state at the isolated place in the core is coupled by the diffusion and slowing down process, therefore the operator S in Eq.(3.2.1) is considered to operate on the spatial distribution of $X(\omega, t)$. Suppose that the operators S and S_b are time invariant, then the effort to solve Eq.(3.2.1) along with the initial condition and with the boundary condition is substituted by the eigenvalue problem

$$S \phi_k(\omega) = \mu_k \phi_k(\omega), \quad (3.2.4)$$

$$S^* \psi_k(\omega) = \mu_k \psi_k(\omega), \quad k=1, 2, 3, \dots, \quad (3.2.5)$$

where the operator S^* is the adjoint of S . The μ_k , $k=1, 2, 3, \dots$ are the eigenvalues and $\phi_k(\omega)$ and $\psi_k(\omega)$ are the K -dimensional vector eigenfunctions and the adjoints bi-normalized so that

$$\langle \phi_k(\omega), \psi_l(\omega) \rangle_{\Omega} = \delta_{kl} \text{ (Kronecker)} \quad (3.2.6)$$

holds, where the inner product is defined by

$$\langle x(\omega), y(\omega) \rangle_{\Omega} \triangleq \int_{\Omega} \{x(\omega)\}^t y(\omega) d\omega. \quad (3.2.7)$$

If every eigenvalue, corresponding vector eigenfunction and the adjoint is found, the solution of Eqs.(3.2.1), (3.2.2) and (3.2.3) can be synthesized as

$$\begin{aligned} X(\omega, t) = & \sum_{k=1}^{\infty} e^{\mu_k t} \phi_k(\omega) \int_{\Omega} \{\psi_k(\omega')\}^t X_0(\omega') d\omega' \\ & + \sum_{k=1}^{\infty} \phi_k(\omega) \int_0^t e^{\mu_k(t-t')} \int_{\Omega} \{\psi_k(\omega')\}^t \mathcal{F}(\omega', t') d\omega' dt'. \end{aligned} \quad (3.2.8)$$

Suppose that the infinite series converges for $\omega \in \Omega$ uniformly in wider sense, then the summation and the integration are commutable, to give

$$\begin{aligned} X(\omega, t) = & \int_{\Omega} \left[\sum_{k=1}^{\infty} e^{\mu_k t} \phi_k(\omega) \{\psi_k(\omega')\}^t \right] X_0(\omega') d\omega' \\ & + \int_0^t \int_{\Omega} \left[\sum_{k=1}^{\infty} e^{\mu_k(t-t')} \phi_k(\omega) \{\psi_k(\omega')\}^t \right] \mathcal{F}(\omega', t') d\omega' dt' \end{aligned} \quad (3.2.9)$$

which can be abbreviated into

$$X(\omega, t) = X_0(\omega, t) + \int_0^t \int_{\Omega} \tilde{G}(\omega, t | \omega', t') \mathcal{F}(\omega', t') d\omega' dt'. \quad (3.2.10)$$

Equation (3.2.10) gives, although is it formal, an integral expression of the solution for the original problem. The controlling input $\mathcal{F}(\omega, t)$ has been so far considered to be spatially distributed.

There are some reactors which are equipped with a few numbers of regulating rods whose tip size is small and that move finely compared to the whole reactor core geometry. Use of Dirac's distribution makes it able to express the effect of such controlling scheme. A control rod is made of deep absorber for the thermal neutrons. The effect of slightly moved control rod is therefore accounted as the yield or the extinction of thermal neutrons that are absorbed into the control rod surface, which is here considered black for the neutrons. After the monoenergetic neutron-physics consideration, the

number of the neutrons that are absorbed into the control rod can be roughly estimated as much as equal to the number of neutrons which are produced within the region of the distance L , the diffusion length of thermal neutrons, from the absorbing surface [Th.64].

If the partial differential equation is used for Eq.(3.2.1), use of Dirac's distribution is rationalized by interpreting the equation as, for any ψ , the relation

$$\langle \psi, \frac{\partial}{\partial t} X - SX \rangle_{\Omega} - \langle \psi, \mathcal{F} \rangle_{\Omega} = 0 \quad (3.2.11)$$

holds. On the other hand the integral expression (3.2.9) allows the direct use of the distribution.

Retaining the expression (3.2.11), we are able to carry out the formal manipulation with the forcing term $\mathcal{F}$ in Eq.(3.2.1) in the form

$$\mathcal{F}(\omega, t) = \sum_{n=1}^N d \cdot u_n \delta(\omega - \omega_n), \quad (3.2.12)$$

where d is a K -vector and δ stands for the Dirac's distribution on the Euclidean space concerned. The u_n coefficients for $n=1, \dots, N$ are taken to be the direct control values each of which are applied to the control rod locating at ω_n , and collected into a N -vector u in the following.

Making use of these terms, the original equation (3.2.1) is reduced to an integral expression

$$X(\omega, t) = X_0(\omega, t) + \mathcal{L}(\omega, t|t') u(t'), \quad (3.2.13)$$

where

$$\mathcal{L}(\omega, t|t') u(t') = \int_0^t G(\omega, t|t') u(t') dt'. \quad (3.2.14)$$

The closed description of $G(\omega, t|t')$ is given for each component by

$$\{ G(\omega, t|t') \}^{in} = \sum_{k=1}^{\infty} e^{\mu_k(t-t')} \{ \phi_k(\omega) \}^i e_k^n, \quad \begin{matrix} i=1, \dots, K \\ j=1, \dots, N \end{matrix}, \quad (3.2.15)$$

where

$$e_k^n = \{ \psi(\omega^n) \}^t d. \quad (3.2.16)$$

3.2.2 Regulator Problem in the Hilbert Space

In this section, the optimum regulator problem in the Hilbert space is considered in a general manner. This problem is to find the control $u(t)$ such as to minimize the criterion functional

$$J(u) = \int_0^T \left\{ \int_{\Omega} \|X(\omega, t) - X_d(\omega, t)\|^2 d\omega + r^2 u^t(t) u(t) \right\} dt, \quad (3.2.17)$$

where the term $X_d(\omega, t)$ represents the desirable behavior of the system. The system under consideration is written in the form

$$X(\omega, t) = X_0(\omega, t) + \mathcal{L}(\omega, t | t') u(t'), \quad X \in H_1, \quad u \in H_2, \quad (3.2.18)$$

where the Hilbert space H_1 stands for the space of the state vector functions and H_2 , the control vector function. The accompanying inner products are defined respectively as

$$\langle X_1, X_2 \rangle_{H_1} \triangleq \int_0^T \int_{\Omega} X_1^t(\omega, t) X_2(\omega, t) d\omega dt, \quad (3.2.19)$$

and

$$\langle u_1, u_2 \rangle_{H_2} \triangleq \int_0^T \sum_{n=1}^N u_1^n(t) u_2^n(t) dt. \quad (3.2.20)$$

The term $\mathcal{L}(\omega, t | t')$ stands for the operator which maps $u(t') \in H_2$ to $X(\omega, t) \in H_1$.

In this frame, we want to minimize

$$J(u) = \|\mathcal{L}u + X_0 - X_d\|_{H_1}^2 + r^2 \|u\|_{H_2}^2. \quad (3.2.21)$$

Let $\mathcal{L}^*(t' | \omega, t)$ be the adjoint operator of $\mathcal{L}$ which is defined as the existing operator that satisfies the following relation

$$\langle X, \mathcal{L}u \rangle_{H_1} = \langle \mathcal{L}^*X, u \rangle_{H_2}, \quad (3.2.22)$$

for any $X \in H_1$ and $u \in H_2$.

Following the definition of the norm in the Hilbert space, the criterion functional (3.2.21) can be rewritten as

$$\begin{aligned}
J(u) &= \langle Lu + X_0 - X_d, Lu + X_0 - X_d \rangle_{H_1} + r^2 \langle u, u \rangle_{H_2} \\
&= \langle (L^*L + r^2\mathcal{I})u - L^*(X_d - X_0), u \rangle_{H_2} - \langle L^*(X_d - X_0), u \rangle_{H_2} + \langle X_d - X_0, X_d - X_0 \rangle_{H_1}, \quad (3.2.23)
\end{aligned}$$

where the term $\mathcal{I}$ represents the identity operator which maps H_2 into another, but entirely similar Hilbert space H_2' . Here we define an operator as

$$\mathcal{R}(t', t) \triangleq L^*(t' | \omega'', t'') L(\omega'', t'' | t) + r^2 \mathcal{I}(t' | t) \quad (3.2.24)$$

which maps H_2 into H_2' .

When the operator L is completely continuous or compact, there exists the adjoint operator L^* and the reduced operator $\mathcal{R}$ is positive definite, self adjoint and has a unique, bounded inverse operator $\mathcal{R}^{-1}$, since L^*L is nonnegative and self adjoint [Ri.55].

These characteristics of $\mathcal{R}$'s admit further modification of the form (3.2.23) to give

$$\begin{aligned}
J(u) &= \langle u - \mathcal{R}^{-1}L^*(X_d - X_0), \mathcal{R}[u - \mathcal{R}^{-1}L^*(X_d - X_0)] \rangle_{H_2} \\
&\quad + \langle X_d - X_0, X_d - X_0 \rangle_{H_1} - \langle \mathcal{R}^{-1}L^*(X_d - X_0), L^*(X_d - X_0) \rangle_{H_2}. \quad (3.2.25)
\end{aligned}$$

Denoting $\mathcal{R}^{-1}L^*(X_d - X_0)$ by u^* , we get an expression

$$J(u) = \langle u - u^*, \mathcal{R}[u - u^*] \rangle_{H_2} + J^*, \quad (3.2.26)$$

where J^* is the fraction of J that is independent of the argument u .

The minimum of $J(u)$ is attained when and only when u coincides with u^* in a strong sense, that is,

$$\langle u - u^*, u - u^* \rangle_{H_2} = \|u - u^*\|^2 = 0 \quad (3.2.27)$$

holds.

This is asserted by the following consideration. Dividing u into u^* and the rest εz as

$$u = u^* + \varepsilon z, \quad (3.2.28)$$

the first term in the criterion functional in (3.2.26) can be evaluated as

$$\varepsilon^2 \langle z, \mathcal{R}z \rangle_{H_2} \leq \varepsilon^2 \|\mathcal{R}\| \cdot \|z\|^2, \quad (3.2.29)$$

where the equality holds when and only when $\|z\|=0$ because of positive definiteness of $\mathcal{R}$.

Thus the optimal control is found as u^* , whereas this is the solution of the equation

$$\mathcal{R}(t'|t) u(t) = \mathcal{L}^*(t'|\omega, t) (X_d(\omega, t) - X_0(\omega, t)). \quad (3.2.30)$$

From the property of the operator $\mathcal{R}$, this equation is certified to have a unique solution u^* . This result should be compared with the necessary condition derived from the calculus of variation or from the dynamic programming method. The latter two reduce to the two-point boundary value problem, which not only is difficult to solve numerically, but also is not assured of the existence of the solution.

The adjoint operator that appears in the above argument can be obtained analytically for our problem after the definition given by Eq.(3.2.14).

Making an inner product between $\mathcal{L}u$ and an arbitrary element ξ in the space H_1 , we obtain a bilinear concomitant defined by

$$\begin{aligned} \ell_1(\xi, u) &\triangleq \langle \xi(\omega, t), \mathcal{L}(\omega, t|t') u(t') \rangle_{H_1} \\ &= \int_{\Omega} \int_0^T \{ \xi(\omega, t) \}^t \mathcal{L}(\omega, t|t') u(t') dt d\omega. \end{aligned} \quad (3.2.31)$$

Substituting Eq.(3.2.14) into Eq.(3.2.31) and executing some manipulation, we have

$$\ell_1(\xi, u) = \int_0^T \{ u(t') \}^t \left[\int_{\Omega} \{ G(\omega, t|t') \}^t \xi(\omega, t) d\omega dt \right] dt'. \quad (3.2.32)$$

Regarding this equation as the linear functional of the $u \in H_2$ for an arbitrary $\xi \in H_1$, we have

$$\mathcal{L}^*(t'|\omega, t) \xi(\omega, t) = \int_{t'}^T \int_{\Omega} \{ G(\omega, t|t') \}^t \xi(\omega, t) d\omega dt. \quad (3.2.33)$$

Further the operator $\mathcal{L}^*(t'|\omega'',t'')\mathcal{L}(\omega'',t''|t)$, which appears in Eq. (3.2.24) can be directly calculated from Eqs.(3.2.14) and (3.2.33) in an integral form

$$\mathcal{L}^*(t'|\omega'',t'')\mathcal{L}(\omega'',t''|t)u(t) = \int_0^T K(t'|t)u(t)dt, \quad (3.2.34)$$

where

$$[K(t'|t)]^{m,n} = \sum_{k,\ell=1}^{\infty} a_{k\ell}^{mn} e^{(-\mu_k t' - \mu_\ell t)} \times \begin{cases} T - \tau & : \text{when } \mu_k + \mu_\ell = 0 \\ \frac{e^{(\mu_k + \mu_\ell)T} - e^{(\mu_k + \mu_\ell)\tau}}{\mu_k + \mu_\ell} & : \text{when } \mu_k + \mu_\ell \neq 0 \end{cases} \quad (3.2.35)$$

with

$$\tau = \max(t', t) \quad (3.2.36)$$

and

$$a_{k\ell}^{mn} = d^t \psi_k(\omega_m) \psi_\ell^t(\omega_n) d \int_{\Omega} \phi_k^t(\omega) \phi_\ell(\omega) d\omega. \quad (3.2.37)$$

3.2.3 Application for a Model with Two-Groups of Neutron and with One-Group Precursor

Here we attempt to obtain the optimal open-loop control for a two dimensional reactor associated with the N discrete control rods of the geometrical configuration depicted in Fig.3.2.1.

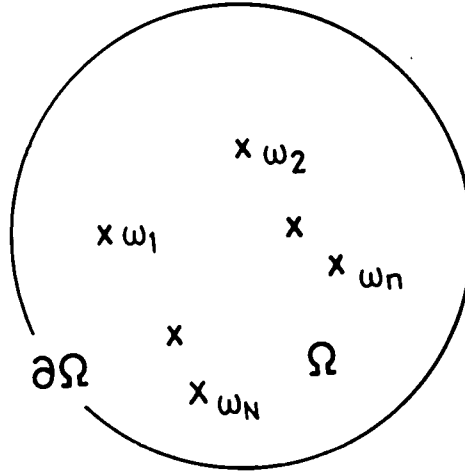


Fig.3.2.1 A reactor with discrete control rods at $\omega=\omega_n$, $n=1, \dots, N$.

The linearized dynamical model of the reactor core in the vicinity of a steady state can be written in terms of the fast and the thermal neutrons, ϕ_1 , ϕ_2 , and the one group of precursors C as

$$\frac{1}{v_1} \frac{\partial}{\partial t} \phi_1 = D_1 \nabla^2 \phi_1 - \Sigma_R \phi_1 + (1-\beta) \nu \Sigma_f \phi_2 + \lambda C \quad (3.2.38)$$

$$\frac{1}{v_2} \frac{\partial}{\partial t} \phi_2 = D_2 \nabla^2 \phi_2 - \Sigma_a \phi_2 + p \Sigma_R \phi_1 + f \quad (3.2.39)$$

$$\frac{\partial}{\partial t} C = \beta \nu \Sigma_f \phi_2 - \lambda C \quad (3.2.40)$$

associated with the boundary conditions at the extrapolated boundary

$$\phi_1(\omega) = \phi_2(\omega) = 0, \quad \omega \in \partial\Omega. \quad (3.2.41)$$

The effect of the slight movement of control rods are introduced into f in reference to Eq.(3.2.12) as

$$f = \sum_{n=1}^N u_n(t) \delta(\omega - \omega_n), \quad (3.2.42)$$

where δ stands for the Dirac's distribution introduced in order to represent the small geometry of the tips of the control rods located

at ω_n . The u_n coefficient is taken as the effect of slight motion of each control rod as has been stated in subsection 3.2.1, and therefore the null value of u_n corresponds to the steady state previously programmed.

Multiplying both sides of Eqs.(3.2.38) and (3.2.39) by $\mathcal{U}_1$ and $\mathcal{U}_2$ respectively, the set of Eqs.(3.2.38)-(3.2.40) can be shortened to

$$\frac{\partial}{\partial t} \Gamma(\omega, t) = S_\omega \Gamma(\omega, t) + d f, \quad (3.2.43)$$

where Γ and d stand for the vectors

$$\Gamma = (\phi_1, \phi_2, C)^t$$

$$d = (0, \mathcal{U}_2, 0)^t$$

and the term f is given in Eq.(3.2.42).

Let the material distribution be homogeneous over the core. Then it is well known that the eigenfunction of the operator S_ω in Eq.(3.2.43) can be factorized by the eigenfunction of the Helmholtz equation

$$\nabla^2 \varphi + B^2 \varphi = 0. \quad (3.2.44)$$

From the geometrical symmetry and the boundary condition, the eigenvalues are given by

$$B_i^2 = (i\pi/H)^2, \quad i=1, 2, \dots \quad (3.2.45)$$

and correspondingly, the normalized eigenfunctions are given by

$$\varphi_i(\omega) = \sqrt{2/H} \sin i\pi\omega/H. \quad (3.2.46)$$

Factorization of the eigenfunction φ_{ik} in the form

$$\varphi_{ik}(\omega) = \varphi_{ik} \varphi_i(\omega) \quad (3.2.47)$$

gives the eigenvalue problem with respect to the vector φ_{ik} ;

$$\begin{bmatrix} -v_1(D_1 B_i^2 + \Sigma_R) & v_1(1-\beta)\nu\Sigma_f & v_1\lambda \\ v_2 p \Sigma_a & -v_2(D_2 B_i^2 + \Sigma_a) & 0 \\ 0 & \beta\nu\Sigma_f & -\lambda \end{bmatrix} \varphi_{ik} = \mu_{ik} \varphi_{ik} \quad k = 1, 2, 3 \quad (3.2.48)$$

which has been reduced from the operator S_ω in Eq.(3.2.43) after substituting the Laplacian operator by $-B_i^2$ regarding Eq.(3.2.44).

In order to synthesize a physical control, we must truncate the infinite series appearing in the above reduction, to a finite number. Namely, we should pick up a finite number of eigenvalues out of the infinite set of μ_{ik} .

We can obtain the distribution of the eigenvalues and each contribution to the system dynamics as follows.

The eigenvalues are the solution of the algebraic equation

$$\det \begin{vmatrix} -B^2 D_1 - \Sigma_R - \frac{\mu}{v_1} & (1-\beta)\nu\Sigma_f & \lambda \\ p\Sigma_R & -B^2 D_2 - \Sigma_a - \frac{\mu}{v_2} & 0 \\ 0 & \beta\nu\Sigma_f & -\lambda - \mu \end{vmatrix} = 0 \quad (3.2.49)$$

For the B^2 large enough, the equation (3.2.49) can be separated into three cases according to the magnitude of the μ .

Assuming that $|\mu| \gg v_2$ holds, we obtain

$$\mu_1 = -v_1 \left\{ B^2 D_1 + \Sigma_R - \frac{p\Sigma_R(1-\beta)\nu\Sigma_f}{B^2 D_2 + \Sigma_a} \right\} \quad (3.2.50)$$

and if the condition $v_1 \gg |\mu| \gg \lambda$ is placed, the equation becomes to

$$\mu_2 = -v_2 \left\{ B^2 D_2 + \Sigma_a - \frac{p\Sigma_R(1-\beta)\nu\Sigma_f}{B^2 D_1 + \Sigma_R} \right\}. \quad (3.2.51)$$

In the smallest region in magnitude, or the assumption $v_2 \gg |\mu|$ holds, the μ is given by

$$\mu_3 = -\lambda \left\{ 1 - \frac{\beta \nu \Sigma_f p \Sigma_R}{(B^2 D_1 + \Sigma_R)(B^2 D_2 + \Sigma_a) - p \Sigma_R (1 - \beta) \nu \Sigma_f} \right\}. \quad (3.2.52)$$

These expressions for the eigenvalues show that μ_1 and μ_2 given by Eqs.(3.2.51) and (3.2.52) respectively, tend to infinity as B^2 increases. The other eigenvalue μ_3 , however, approaches to $-\lambda$, the value that comes from the decay constant of the precursors. In Table 3.2.2, the eigenvalues are listed according to the above classification for an ideal slab reactor with the parameters shown in Table 3.2.1, which is supposed to be a pressurized water reactor.

Geometry	:	Slab		
Core Width	:	120 cm		
Rod Position	:	30 cm, 70 cm		
		(from the one boundary)		
ν_1	1.969×10^9 cm/sec	Σ_a	6.5767×10^{-2} cm ⁻¹	
ν_2	2.2×10^5 cm/sec	$\nu \Sigma_f$	7.3639×10^{-2} cm ⁻¹	
D_1	1.5039 cm	p	9.5×10^{-1}	
D_2	0.4672 cm	β	7.5×10^{-3}	
Σ_R	1.8023×10^{-2} cm	λ	8.0×10^{-2} sec ⁻¹	

Table 3.2.1 Core parameters

i	B_i^2	μ_{i1}	μ_{i2}	μ_{i3}
1	6.85×10^{-4}	-2.81×10^5	-1.09×10^2	0.0
2	2.74×10^{-3}	-6.90×10^6	-2.33×10^3	-7.68×10^{-2}
3	6.17×10^{-3}	-1.79×10^7	-5.03×10^3	-7.88×10^{-2}
4	1.10×10^{-2}	-3.32×10^7	-7.63×10^3	-7.94×10^{-2}
5	1.71×10^{-2}	-5.29×10^7	-9.95×10^3	-7.96×10^{-2}

Table 3.2.2 Eigenvalues of the operator S_ω

The magnitude of μ_1 and μ_2 is far larger than that of μ_3 independent of the number i . This means that the μ_1 and μ_2 which are negative and large in magnitude, correspond to the mode which quickly dies out after the perturbation has been introduced. That is to say that the dynamics of the power distribution is governed solely by that of precursors. Therefore we can neglect the dynamics of the neutrons in fast group or in both fast and thermal groups from the viewpoint of ordinary regulating practice.

3.2.4 Steady Fast Neutron Approximation

The fast neutron flux can change quickly compared with thermal neutrons and precursors, and is regarded to be steady in the time scale of the other state variables.

Unless any stepwise motion of the control is expected, this assumption leads the equation

$$D_1 \nabla^2 \phi_1 - \Sigma_R \phi_1 + (1-\beta) \nu \Sigma_f \phi_2 + \lambda C = 0 \quad (3.2.53)$$

as the substitute of Eq.(3.2.38)

Eq.(3.2.53) along with the boundary condition (3.2.41) can be transferred to the explicit form in terms of an integral equation

$$\phi_1(\omega) = \int_{\Omega} G_1(\omega|\omega') \{ (1-\beta) \nu \Sigma_f \phi_2(\omega') + \lambda C(\omega') \} d\omega', \quad (3.2.54)$$

where Green's function $G_1(\omega|\omega')$ corresponds to Eq.(3.2.53) as

$$G_1(\omega|\omega') = \sum_{i=1}^{\infty} \frac{1}{D_1 B_i^2 + \Sigma_R} \varphi_i(\omega) \varphi_i(\omega'), \quad (3.2.55)$$

where $\varphi_i(\omega)$ is the eigenfunction given by Eq.(3.2.44) for the eigenvalue or the buckling B_i^2 .

Substituting Eq.(3.2.54) into Eq.(3.2.39) the governing equation reduced in order is given;

$$\frac{\partial}{\partial t} \phi_2 = \nu_2 D_2 \nabla^2 \phi_2 - \nu_2 \Sigma_a \phi_2 + \nu_2 \rho \Sigma_R \int_{\Omega} G_1(\omega|\omega') \{ (1-\beta) \nu \Sigma_f \phi_2(\omega') + \lambda C(\omega') \} d\omega' + \nu_2 f(\omega), \quad (3.2.56)$$

$$\frac{\partial}{\partial t} C = \beta \nu \Sigma_f \phi_2 - \lambda C. \quad (3.2.40)'$$

Again expanding the $\phi_2(\omega)$ and the $C(\omega)$ in terms of the Helmholtz's eigenfunctions, the Eqs.(3.2.56), (3.2.40) along with the boundary conditions are transferred to the integral equation

$$\Gamma(\omega, t) = \Gamma_0(\omega, t) + \int_0^t G(\omega, t|t') u(t') dt', \quad (3.2.57)$$

where

$$\Gamma \triangleq (\phi_2, C)^t.$$

The reduced algebraic eigenvalue problem arising from the factorization of the space dependency of the eigenfunctions for Eqs.

(3.2.56) and (3.2.40) corresponding to Eq.(3.2.48), is

$$\begin{bmatrix} -\nu_2(D_2 B_i^2 + \Sigma_a) + \frac{\nu_2 \rho \Sigma_R (1-\beta) \nu \Sigma_f}{D_1 B_i^2 + \Sigma_R} & \frac{\nu_2 \lambda \rho \Sigma_R}{D_1 B_i^2 + \Sigma_R} \\ -\beta \nu \Sigma_f & -\lambda \end{bmatrix} \varphi_{ik} = \mu_{ik} \varphi_{ik}, \quad k = 1, 2. \quad (3.2.58)$$

The asymptotic values of the eigenvalues are

$$\mu \sim -\lambda + \frac{\beta \nu \rho \Sigma_f \Sigma_R}{(D_1 B_i^2 + \Sigma_R)(D_2 B_i^2 + \Sigma_a)} \lambda \quad (3.2.59)$$

corresponding to the precursor decay constant, and

$$\mu \sim -\nu_2(D_2 B_i^2 + \Sigma_R) + \frac{\nu_2 \rho \Sigma_R \Sigma_f}{D_1 B_i^2 + \Sigma_R}, \quad (3.2.60)$$

for the thermal neutron dynamics.

3.2.5 Delayed Neutron Equation

As the extreme approximation, the total neutron flux may settle instantly after the control which is driven slowly enough. Then the Eq.(3.2.38) for the thermal neutrons as well as Eq.(3.2.39) may be substituted by the one for the steady state. That are

$$D_1 \nabla^2 \phi_1 - \Sigma_R \phi_1 + (1-\beta) \nu \Sigma_f \phi_2 = -\lambda C \quad (3.2.61)$$

$$D_2 \nabla^2 \phi_2 - \Sigma_a \phi_2 + \rho \Sigma_R \phi_1 = -f \quad (3.2.62)$$

associated with the boundary conditions

$$\phi_1(\omega) = \phi_2(\omega) = 0, \quad \omega \in \partial\Omega. \quad (3.2.41)'$$

By the modal expansion technique in terms of the Helmholtz's modes, $\varphi_i(\omega)$, the set of Eqs.(3.2.61) and (3.2.62) gives the solution for $\phi_2(\omega)$ as

$$\phi_2(\omega) = \sum_{i=1}^{\infty} \frac{\rho \Sigma_R \lambda c_i + (D_1 B_i^2 + \Sigma_R) \int_{\Omega} \varphi_i(\omega') f(\omega') d\omega'}{(D_1 B_i^2 + \Sigma_R)(D_2 B_i^2 + \Sigma_a) - \rho \Sigma_R (1-\beta) \nu \Sigma_f} \varphi_i(\omega), \quad (3.2.63)$$

where c_i is Helmholtz modal expansion coefficient of the precursor density C .

Substitution of Eq.(3.2.63) into Eq.(3.2.38) gives the equation for each modal expansion coefficient;

$$\begin{aligned} \dot{c}_i = & -\lambda \left\{ 1 - \frac{\beta \nu \Sigma_f \rho \Sigma_R}{(D_1 B_i^2 + \Sigma_R)(D_2 B_i^2 + \Sigma_a) - \rho \Sigma_R (1-\beta) \nu \Sigma_f} \right\} c_i \\ & + \frac{\beta \nu \Sigma_f (D_1 B_i^2 + \Sigma_R) \int_{\Omega} \varphi_i(\omega') f(\omega') d\omega'}{(D_1 B_i^2 + \Sigma_R)(D_2 B_i^2 + \Sigma_a) - \rho \Sigma_R (1-\beta) \nu \Sigma_f}. \end{aligned} \quad (3.2.64)$$

Here, in the first term of the right-hand side, the intrinsic decay constant λ is seen to be influenced by the neutron diffusion process.

The synthesized integral expression as the substitute of Eqs. (3.2.61) and (3.2.62) along with the boundary condition (3.2.41) is

$$c(\omega, t) = c_0(\omega, t) + \int_0^t G(\omega, t|t') u(t') dt', \quad (3.2.65)$$

where Green's function G is a row vector whose n -th element is given by

$$\{G(\omega, t|t')\}^n = \sum_{i=1}^{\infty} e^{\mu_i(t-t')} \varphi_i(\omega) m_i \varphi_i(\omega^n) \quad (3.2.66)$$

with the eigenvalues $\mu_i, i=1, 2, \dots$ and the coefficients m_i given by

$$\mu_i = -\lambda \left\{ 1 - \frac{\beta \nu \Sigma_f p \Sigma_R}{(D_1 B_i^2 + \Sigma_R)(D_2 B_i^2 + \Sigma_a) - p \Sigma_R (1-\beta) \nu \Sigma_f} \right\} \quad (3.2.67)$$

and

$$m_i = \frac{\beta \nu \Sigma_f (D_1 B_i^2 + \Sigma_R)}{(D_1 B_i^2 + \Sigma_R)(D_2 B_i^2 + \Sigma_a) - p \Sigma_R (1-\beta) \nu \Sigma_f} \quad (3.2.68)$$

Applying the result obtained in subsection 3.2.2, the integral equation for the optimal control is given by

$$\begin{aligned} & \int_0^T \sum_{i=1}^{\infty} \sum_{n=1}^N m_i^2 \{ \varphi_i(\omega^n) \varphi_i(\omega^m) \} K_i(t'|t'') u^n(t'') dt'' + r^2 \int_0^T \{ u^n(t') \}^2 dt' \\ & = - \sum_{i=1}^{\infty} e^{-\mu_i t'} m_i \varphi_i(\omega^m) g_i(t') b_i dt', \\ & m = 1, 2, \dots, N, \end{aligned} \quad (3.2.69)$$

where

$$K_i(t'|t'') = \begin{cases} T - \tau & : \text{when } \mu_i = 0 \\ e^{-\mu_i(t'+t'')} \frac{e^{2\mu_i T} - e^{2\mu_i \tau}}{2\mu_i} & : \text{when } \mu_i \neq 0 \end{cases} \quad (3.2.70)$$

with

$$\tau = \max(t', t'') \quad (3.2.71)$$

and

$$g_i(t') = \begin{cases} T - t' & : \text{when } \mu_i = 0 \\ \frac{e^{2\mu_i T} - e^{2\mu_i t'}}{2\mu_i} & : \text{when } \mu_i \neq 0 \end{cases} \quad (3.2.72)$$

The values $b_i, i=1, 2, \dots$ are the modal expansion coefficients of

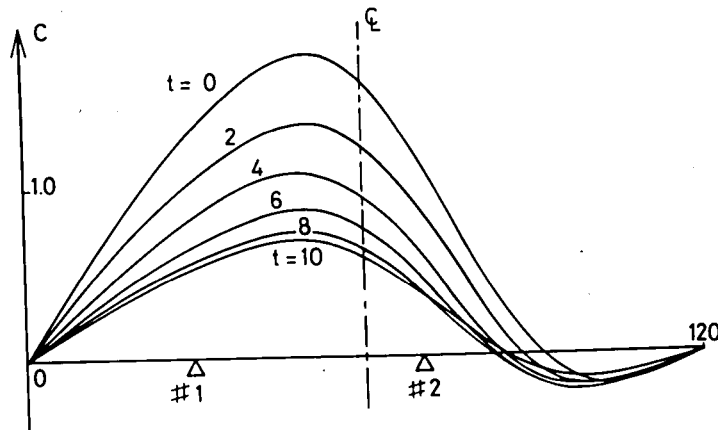
the initial perturbation of the precursor density, namely,

$$b_i = \int_{\Omega} \varphi_i(\omega) C(\omega, 0) d\omega . \quad (3.2.73)$$

The integral equation (3.2.69) can be approximately solved by discretizing the time [Tr.57] and making use of the successive substitution technique [Var.62].

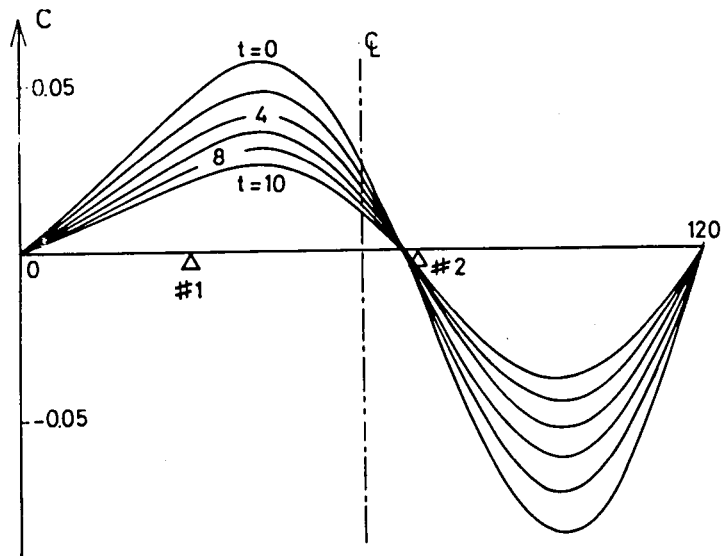
The numerical calculation has been carried out for the model of Eq.(3.2.63) with the core parameters listed in Table 3.2.1.

Figures 3.2.2 and 3.2.3 show the history of the perturbation of the precursor density C with the controlling period $T=10$ and with the weight $r^2=1.0$ for each initial perturbation. Figure 3.2.4 illustrates the transients of the amplitude of each mode and the control. Also the history of the precursor with the longer controlling period $T=30$, is given in Fig.3.2.5, where the higher modes are seen to die out faster than the fundamental mode. In Fig.3.2.6 it is shown that, as r^2 is lessened the control action becomes large and the perturbation is reduced quickly. If too large control action is undesirable, smaller value should be assigned to r^2 , though the discrepancy of the precursor density from the nominal value will be less achieved.

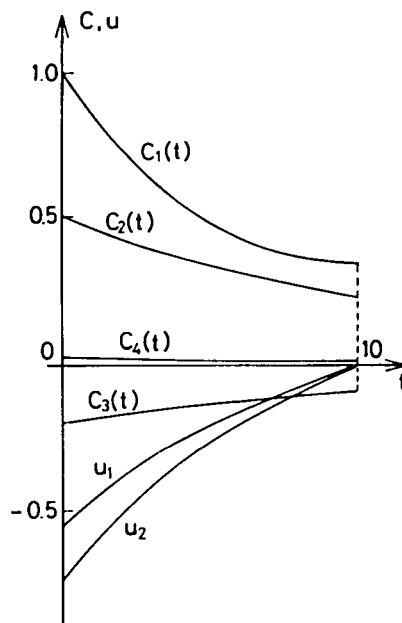


Control rods locate at the points marked by Δ . Core parameters are taken from Table 3.2.1 and $r^2=1.0$, $T=10(\text{sec})$.

Fig.3.2.2 History of the precursor distribution.

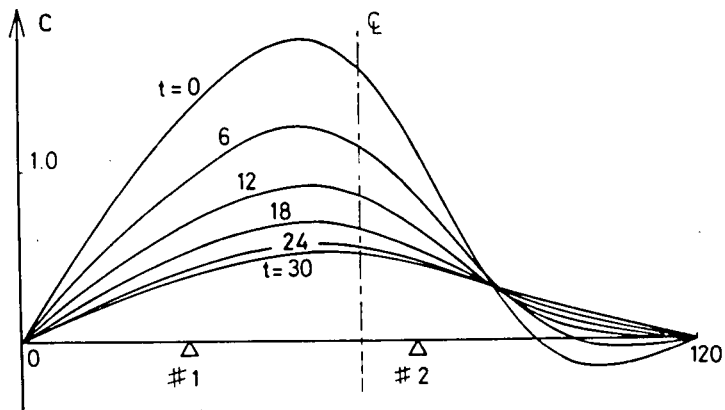


The initial state is different from that of Fig. 3.2.2
 Fig.3.2.3 History of the precursor distribution.



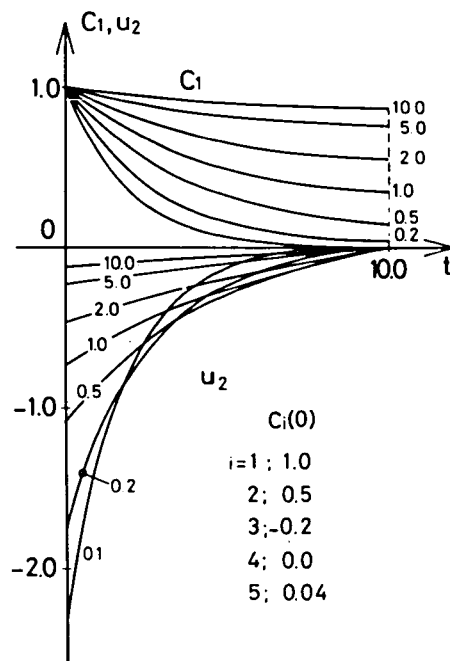
The same problem with that of Fig.3.2.2. $C_5(t)$ and the transverse axis overlap together.

Fig.3.2.4 History of the controls and the mode amplitude of the precursors.



Control period is taken to be $T=30(\text{sec})$. Initial distribution of C is same with that of Fig.3.2.2.

Fig.3.2.5 History of the precursor distribution.



Values of r^2 are associated with each curve. Initial value of C_i 's are also depicted in the figure.

Fig.3.2.6 Histories of C_1 and u_2 for various values of r^2 .

3.3 Optimal Feedback Control for the Regulator Problem of Coupled Nuclear Thermo-hydrodynamics Model

3.3.1 System Equation

Here the thermo-hydrodynamic equation coupled through the steady state neutron equation is chosen as the subject. The reactor at the rated power should be regulated in the vicinity of the steady state with respect to the coolant outlet temperature and the neutron flux level over the reactor core.

The governing equations with respect to the fluxtuated temperature θ and the precursor density c are written in the form

$$\dot{c} = -\lambda c + \beta \nu \Sigma_f \phi \quad (3.3.1)$$

$$\dot{\theta} = -\gamma \theta + \kappa \Sigma_f \phi, \quad (3.3.2)$$

where ϕ is the neutron flux density. The precursor density is considered to be of one group and the distributive neutron dynamics in a single channel is assumed to be far more quick compared to the thermo-hydrodynamics of the moderator and the control motion. The reduction of the parameter γ and κ is presented in Appendix B.

The two independent dynamics Eqs.(3.3.1) and (3.3.2) are coupled through the quickly following neutron system steered by the control absorber, namely,

$$\nu^2 \phi + \frac{1}{M^2} (k_\infty - 1) \phi = - \{ u + a \theta \} \Phi + \bar{\lambda} c \quad (3.3.3)$$

associated with the boundary condition,

$$\phi(\omega) = 0, \quad \omega \in \partial \Omega. \quad (3.3.4)$$

The variables are assigned by following meanings,

$$\begin{aligned} M^2 &= \Sigma_a / D_2 && \text{:migration area,} \\ k_\infty &= \nu \Sigma_f / \Sigma_a && \text{;multiplication factor for} \\ &&& \text{the infinite medium,} \end{aligned}$$

Φ	:nominal neutron flux density,
α	:moderator temperature coefficient,
$\bar{\lambda}$	: λ/Σ_a .

The differential equation (3.3.3) together with the boundary condition (3.3.4) can be transposed into the explicit form using Helmholtz mode expansion technique in the like manner used for the preceding section, as,

$$\phi(\omega) = \sum_{i=1}^{\infty} \varphi_i(\omega) \frac{1}{\eta_i} \int_{\Omega} \varphi_i(\omega') \{ (\alpha - \beta\theta)\Phi - \bar{\lambda}C \}_{(\omega')} d\omega' , \quad (3.3.5)$$

where $\varphi_i(\omega)$ is the Helmholtz eigenfunction corresponding to the eigenvalue

$$\eta_i = - \frac{M^2}{k_{\infty} - 1 - B_i^2 M^2} . \quad (3.3.6)$$

Substitution of this expression into the state equations (3.3.1) and (3.3.2) gives the system equation with respect to the state values $C(\omega, t)$ and $\theta(\omega, t)$ and the control value $u(\omega, t)$, in an evolution equation form with an integral operator as the spatial operator, instead of the differential operator, viz,

$$\begin{aligned} \frac{\partial}{\partial t} T(\omega, t) = & - \Lambda T(\omega, t) + b \int_{\Omega} m_r^t(\omega|\omega') T(\omega', t) d\omega' \\ & - b \int_{\Omega} m_u(\omega|\omega') u(\omega', t) d\omega' , \end{aligned} \quad (3.3.7)$$

where T is the state vector of C and θ , as

$$T = \text{col.}(C, \theta), \quad (3.3.8)$$

Λ , the diagonal matrix

$$\Lambda = \text{diag.}(\lambda, \gamma) \quad (3.3.9)$$

and b is given by

$$b = \text{col.}(\beta \nu \Sigma_f, \kappa \Sigma_f) . \quad (3.3.10)$$

The kernels of the integral operator $m_r(\omega|\omega')$ and $m_u(\omega|\omega')$ are given by

$$\{m_{\tau}(\omega|\omega')\}^1 = -\sum_{i=1}^{\infty} \frac{1}{\eta_i} \varphi_i(\omega) \varphi_i(\omega') a(\omega') \bar{\Phi}(\omega'), \quad (3.3.11)$$

$$\{m_{\tau}(\omega|\omega')\}^2 = -\sum_{i=1}^{\infty} \frac{1}{\eta_i} \varphi_i(\omega) \varphi_i(\omega') \bar{\lambda}(\omega'), \quad (3.3.12)$$

and

$$m_u(\omega|\omega') = \sum_{i=1}^{\infty} \frac{1}{\eta_i} \varphi_i(\omega) \varphi_i(\omega') \bar{\Phi}(\omega'), \quad (3.3.13)$$

respectively.

3.3.2 Optimal Regulator Problem; Fully Distributed Control

Let us consider a problem to minimize a functional of Γ and u which are regarded as the points of each space as

$$J = \int_t^{\infty} H(\Gamma(\omega, \tau), u(\omega, \tau)) d\tau, \quad (3.3.14)$$

choosing the control function $u(\omega, \tau)$ suitably under the constraint, Eq.(3.3.7). The H function in Eq.(3.3.14) is also a functional of the form

$$\begin{aligned} H(\Gamma, u) = & \frac{1}{2} \iint_Q \Gamma^t(\omega, t) Q(\omega, \omega') \Gamma(\omega', t) d\omega' d\omega \\ & + \frac{\gamma^2}{2} \int_{\Omega} \{u(\omega, t)\}^2 d\omega. \end{aligned} \quad (3.3.15)$$

The γ^2 variable is the parameter which should be adjusted according to the controlling aim. The Q in the first term is the symmetric weighting matrix function which is considered to be dependent both on ω and ω' also symmetrically, though we shall not consider the spatial dependency. This is because we want to make the reduction clearly.

The minimization problem as stated leads to a problem of the calculus of variations making use of the principle of optimality.

Let us introduce a functional π of Γ associated with a parameter t as,

$$\pi(\Gamma(t), t) = \min_{\substack{u(\tau) \\ \tau \in [t, \infty)}} \int_t^\infty H(\Gamma, u, \tau) d\tau. \quad (3.3.16)$$

Following the principle of optimality, Eq.(3.3.16) is to be re-written in a separated form;

$$\pi(\Gamma(t), t) = \min_{\substack{u(\tau) \\ \tau \in [t, t+\sigma]}} \left[\int_t^{t+\sigma} H(\Gamma, u, \tau) d\tau + \pi(\Gamma(t+\sigma), t+\sigma) \right]. \quad (3.3.17)$$

Gathering both sides into one and taking account of the minimization operation, we have

$$\min_{\substack{u(\tau) \\ \tau \in [t, t+\sigma]}} \left[\int_t^{t+\sigma} H d\tau + \pi\{\Gamma(t+\sigma), t+\sigma\} - \pi\{\Gamma(t), t\} \right] = 0. \quad (3.3.18)$$

Retaining the relation (3.3.18) to the limit $\sigma \rightarrow 0$, we obtain

$$\min_{u(t)} \left[H(\Gamma(t), u(t)) + \frac{d}{dt} \pi\{\Gamma(t), t\} \right] = 0, \quad (3.3.19)$$

for the optimal control u at an arbitrary calendar time t .

The total time derivative of π is written as

$$\frac{d\pi}{dt} = \frac{\partial \pi}{\partial t} + \frac{\delta \pi}{\delta \Gamma} \frac{\partial \Gamma}{\partial t}, \quad (3.3.20)$$

where the second term in the right side together means the Fréchet derivative of π with respect to the variable $\partial \Gamma / \partial t$.

Because only with the stationary case we are concerned, the *a priori* condition

$$\frac{\partial \pi}{\partial t} = 0 \quad (3.3.21)$$

should be retained. Then substitution of Eq.(3.3.20) together with Eq.(3.3.21) into Eq.(3.3.18) yields

$$\min_{u(t)} \left[H\{\Gamma(t), u(t)\} + \frac{\delta \pi}{\delta \Gamma} \frac{\partial \Gamma}{\partial t} \right] = 0. \quad (3.3.22)$$

This is the fundamental equation after which we are going to obtain the optimal feedback law analogously to the well known Hamilton-Jacobi theory for the lumped parameter system.

From the intuition, suggested from many stimulated former works on the various quadratic performance problems of the linear lumped parameter systems, let us put the form of the functional $\pi\{\Gamma(t)\}$ to be

$$\pi\{\Gamma(t)\} = \frac{1}{2} \iint_{\Omega} \left\{ \frac{\partial}{\partial t} \Gamma(\omega, t) \right\}^t P(\omega, \omega') \Gamma(\omega') d\omega' d\omega \quad (3.3.23)$$

by a square matrix function $P(\omega, \omega')$, which reduces to

$$\begin{aligned} \frac{\delta \pi}{\delta \Gamma} \frac{\partial \Gamma}{\partial t} &= \frac{1}{2} \iint_{\Omega} \left\{ \frac{\partial}{\partial t} \Gamma(\omega, t) \right\}^t P(\omega, \omega') \Gamma(\omega', t') d\omega' d\omega \\ &+ \frac{1}{2} \iint \Gamma^t(\omega, t) P(\omega, \omega') \cdot \frac{\partial}{\partial t} \Gamma(\omega', t) d\omega' d\omega. \end{aligned} \quad (3.3.24)$$

The substitution of Eq.(3.3.24) and the system equation(3.3.7) into Eq.(3.3.23) gives

$$\begin{aligned} \min_{u(t)} &\left[\iint_{\Omega} \Gamma^t(\omega, t) Q(\omega, \omega') \Gamma(\omega') d\omega' d\omega + r^2 \int_{\Omega} \{u(\omega, t)\}^2 d\omega \right. \\ &+ \iint_{\Omega} \left\{ \frac{\partial}{\partial t} \Gamma(\omega, t) \right\}^t P(\omega, \omega') \Gamma(\omega', t') d\omega' d\omega \\ &\left. + \iint_{\Omega} \Gamma^t(\omega, t) P(\omega, \omega') \left\{ \frac{\partial}{\partial t} \Gamma(\omega', t) \right\} d\omega' d\omega \right] \\ &= \min_{u(t)} \left[\iint_{\Omega} \Gamma^t(\omega, t) Q(\omega, \omega'') \Gamma(\omega'', t) d\omega'' d\omega + r^2 \int_{\Omega} \{u(\omega)\}^2 d\omega \right. \\ &- \iint_{\Omega} \Gamma^t(\omega, t) \{ \Lambda P(\omega, \omega'') + P(\omega, \omega'') \Lambda \} \Gamma(\omega'', t) d\omega'' d\omega \\ &+ \iint_{\Omega} \Gamma^t(\omega, t) \left[\int_{\Omega} m_T(\omega' | \omega) b^t P(\omega', \omega'') d\omega' \right. \\ &\left. + \int_{\Omega} P(\omega, \omega') b \{ m_T(\omega' | \omega'') \}^t d\omega' \right] \Gamma(\omega'', t) d\omega'' \\ &\left. - 2 \iint_{\Omega} u(\omega) \left[\int_{\Omega} b^t m_u(\omega' | \omega) P(\omega', \omega'') d\omega' \right] \Gamma(\omega'', t) d\omega'' d\omega \right] \\ &= 0. \end{aligned} \quad (3.3.25)$$

Because no restriction on the control $u(\omega, t)$ is posed, the minimum of Eq.(3.3.25) should be attained at $u = u^*(\omega, t)$ which makes the functional derivative of the operand in Eq.(3.3.25) be zero, the optimal control should be given by

$$u^*(\omega, t) = r^{-2} \int_{\Omega} \left[b^t m_u(\omega' | \omega) P(\omega', \omega'') \right] \Gamma(\omega'', t) d\omega'', \quad (3.3.26)$$

as a necessary condition for the optimality. Again substituting this intermediate result for $u^*(\omega, t)$ into the stationary Hamilton-Jacobi equation (3.3.25), the equation with respect to the P function which is as yet undetermined, is obtained as

$$\begin{aligned} & -r^2 \iint_{\Omega} \{P(\omega_2, \omega_1)\}^t \left[\int_{\Omega} m_u(\omega_2 | \omega_3) b b^t m_u(\omega_4 | \omega_3) d\omega_3 \right] P(\omega_4, \omega_3) d\omega_4 d\omega_3 \\ & + \int_{\Omega} m_r(\omega_2 | \omega_1) b^t P(\omega_2, \omega_3) d\omega_3 + \int_{\Omega} P(\omega_1, \omega_2) b \{m_r(\omega_2 | \omega_3)\}^t d\omega_3 \\ & - \Lambda P(\omega_1, \omega_3) - P(\omega_1, \omega_3) \Lambda + Q(\omega_1, \omega_3) = 0. \end{aligned} \quad (3.3.27)$$

Equation (3.3.27) is for a two dimensional square matrix and does not change in the form after transposing the whole equation or interchanging the variable ω_1 and ω_3 , and therefore the solution $P(\omega, \omega')$ should be symmetric not only in the form but also with respect to the two arguments. The equation is analogous to the Riccati-type differential equation which appears in the feedback control problem for the lumped system. The name "Riccati" is after the scalar Riccati differential equation which shares both the square nonlinearity and the linear part with ours [Red.72].

3.3.3 Optimal Regulator Problem; Rod Cluster Control

A large reactor may be operated by many control rods which are driven being grouped in a few clusters according to the prescribed partitioning pattern. This operation allows us to restrict the control function within the class that is expressed by

$$u(\omega, t) = \sum_{m=1}^M u_m(t) v_m(\omega), \quad (3.3.28)$$

where the $v_m(\omega)$, $m=1, \dots, M$ are the prescribed distribution chosen so that the control rod pattern may be well described.

Substitution of Eq.(3.3.28) into Eq.(3.3.7) gives a modification,

$$\begin{aligned} \frac{\partial}{\partial t} \Gamma(\omega, t) = & -\Lambda \Gamma(\omega, t) + b \int_{\Omega} \{m_T(\omega|\omega')\}^t \Gamma(\omega', t) d\omega' \\ & - b \{m_u(\omega)\}^t u(t), \end{aligned} \quad (3.3.29)$$

where the modified part $m_u(\omega)$ is an M -dimensional column vector function and is given by

$$\{m_u(\omega)\}^m = \int_{\Omega} m_u(\omega|\omega') v_m(\omega') d\omega' \quad (3.3.30)$$

and

$$u(t) = \text{col.} \{u_1(t), u_2(t), \dots, u_M(t)\}. \quad (3.3.31)$$

The integrand of the cost functional (3.3.15) should also be modified into

$$\begin{aligned} H = & \frac{1}{2} \iint_{\Omega} \{\Gamma(\omega, t)\}^t Q(\omega, \omega') \Gamma(\omega', t) d\omega' d\omega \\ & + \frac{1}{2} \{u(t)\}^t R u(t), \end{aligned} \quad (3.3.32)$$

where the positive definite square matrix R is introduced instead of r^2 in Eq.(3.3.15). The introduction of R allows us to evaluate the controlling effort on each controlling mode separately.

The parallel reduction along the previous subsection provides the stationary Riccati equation

$$\begin{aligned} & - \iint_{\Omega} \{P(\omega_2, \omega_1)\}^t b \{m_u(\omega_2)\}^t R^{-1} m_u(\omega_4) b^t P(\omega_4, \omega_3) d\omega_4 d\omega_2 \\ & + \int_{\Omega} [m_T(\omega_2|\omega_1) b^t P(\omega_2, \omega_3) + P(\omega_1, \omega_2) b \{m_T(\omega_2|\omega_3)\}^t] d\omega_2 \\ & - \Lambda P(\omega_1, \omega_3) - P(\omega_1, \omega_3) \Lambda + Q(\omega_1, \omega_3) = 0, \end{aligned} \quad (3.3.33)$$

and the optimum feedback law

$$u(t) = R^{-1} \iint_{\Omega} m_u(\omega_1) b^t P(\omega_1, \omega_2) T(\omega_2, t) d\omega_2 d\omega_1, \quad (3.3.34)$$

for the solution of (3.3.33).

3.3.4 Optimal Regulator Problem; A Few Regulating Rods

Let us consider a reactor equipped with a few regulating rods separately located in the core. The effect of the slight motion of the control rods are represented by

$$u(\omega, t) = \sum_{n=1}^N u_n(t) \delta(\omega - \omega^n) \quad (3.3.35)$$

as has been shown in section 2. Substituting Eq.(3.3.35) into Eq. (3.3.7) instead of Eq.(3.3.28), the same results with Eqs.(3.3.33) and (3.3.34) are obtained except for the definition of $m_u(\omega)$, which should be substituted by

$$\{m_u(\omega)\}^n = m_u(\omega | \omega^n), \quad n = 1, \dots, N. \quad (3.3.36)$$

One method to solve the Riccati-type equation (3.3.33) can be found in the literature Lion [Li.71] which has expanded the function $P(\omega, \omega')$ in terms of eigenfunctions of the system operator. These eigenfunctions coincide with the "natural mode" introduced by Kaplan [Ka.61] and are known to need much effort to obtain for the existing reactor.

We can expand the P function also in terms of the Helmholtz mode, which is given *a priori* for the specific core geometry, as

$$P(\omega, \omega') = \sum_{i,j=1}^{\infty} \varphi_i(\omega) P_{ij} \varphi_j(\omega'). \quad (3.3.37)$$

Substituting this form into Eq.(3.3.33) and multiplying it by $\varphi_i(\omega)$ and $\varphi_k(\omega')$ from the left and from the right respectively, we obtain a series of equations

$$\begin{aligned}
& - \sum_{j, \ell=1}^{\infty} \{P_{ji}\}^t K_{j\ell} P_{\ell k} + \sum_f \frac{1}{\eta_i} \sum_{\ell=1}^{\infty} \{B_{\ell i}\}^t P_{\ell k} + \sum_f \frac{1}{\eta_k} \sum_{\ell=1}^{\infty} P_{i\ell} B_{\ell k} \\
& - \Lambda P_{ik} - P_{ik} \Lambda + Q_{ik} = 0, \quad \begin{matrix} i=1, 2, \dots \\ k=1, 2, \dots \end{matrix}, \quad (3.3.38)
\end{aligned}$$

where

$$K_{j\ell} = b \sum_{n=1}^N \frac{w_j^n w_{\ell}^n}{r_n^2 \eta_j \eta_{\ell}} b^t \quad (3.3.39)$$

with

$$w_j^n = \varphi_j(\omega^n) \Phi(\omega^n), \quad (3.3.40)$$

and

$$B_{\ell k} = \begin{bmatrix} \beta \nu \tilde{b}^{\ell k} & \beta \nu \bar{\lambda} \\ \kappa \tilde{b}^{\ell k} & \kappa \bar{\lambda} \end{bmatrix} \quad (3.3.41)$$

with

$$\tilde{b}^{\ell k} = \int_{\Omega} \alpha(\omega) \Phi(\omega) \varphi_{\ell}(\omega) \varphi_k(\omega) d\omega \quad (3.3.42)$$

and

$$Q_{ik} = \iint_{\Omega} \varphi_i(\omega_1) Q(\omega_1, \omega_3) \varphi_k(\omega_3) d\omega_3 d\omega_1. \quad (3.3.43)$$

Truncating Eq.(3.3.38) at some finite number, we can obtain the **P** function.

The method by eigenfunction expansion may lend rather simple expression at the sacrifice of the effort for obtaining the eigenfunctions. Contrary to this, Helmholtz mode expansion retains crude parameters of the reactor core, and therefore prepares a tractable expression for further problems as sensitivity analyses, learning control problems and so forth.

Chapter 4. Burnup Control Problems

4.1 Introduction

As the nuclear energy has come to be competitive in cost with power production by conventional fossile fuels, effective use of nuclear fuel is becoming more and more important. Large portion of operational cost of nuclear power plant is occupied by the nuclear fuel cost. Many efforts have been devoted to the problem to reduce the fuel cost on each reactor type.

Wall & Fenech [Wal.65] have showed that an alternative to a dynamic programming algorithm can be applied to the refueling decision of a single-enrichment, three-zone, 1000-MWe pressurized water reactor (PWR) core for a minimum unit power cost.

Stover & Sesonske [Sv.69] and Stover [Sv.68] used this same technique, which is shown to be a computational acceleration method of an exhaustive search called "elimination of similar end states," for determining fresh fuel loading decisions which lead to a minimum fuel cost in a scatter-loaded three-zone 1000-MWe boiling water reactor (BWR) core.

Fagan & Sesonske [Fa.69] have used a direct search to determine the optimal loading patterns in a scatter-loaded PWR core with fuel shuffling between zones. A minimum fuel cycle cost is obtained by determining shuffle patterns that maximize the core life at each reload point in the life of the reactor, assuming a constant fraction of replacement in a 19-zone core which is in a quasi-equilibrium cycle.

Melice [Me.69] has presented a method for optimal core management

of PWR chemical shim reactor for finding the loading pattern of fresh fuels and exposed fuels in the light of the k profile deduced from the minimum critical mass problem.

Tabak [Tab.68] has used linear and quadratic programming in a simplified one-point reactor model to determine the optimum uranium mass loading which minimizes the usage of uranium mass charged or the plutonium mass removed from the core over the life of the reactor. The refueling system concerning a reactor core is regarded as a feedback control system in his study.

Motoda [Mo.70a] has showed that the variational method can be applied to the burnup optimization of continuously scattered refueling.

Suzuki & Kiyose [Suz.71a] and Sauer [Sau.71] have used the linear programming to minimize the consumption of fresh fuel throughout the plant life and the present-worth weighted total fuel cost, respectively.

Kawai & Kiguchi [Kw.71] and Naft & Sesonske [Na.72] have used a polynomial to express the local peaking factor in their work on the optimal in-core fuel management problem.

Hoshino [Ho.72] has studied the optimization of the multistage refueling decision process by the heuristic approach for four-region batch refueling, including shuffling. The use of heuristic approach lends a rule of thumb in the refueling policy.

No problems of poison management arise in PWR and heavy water reactor (HWR) in which the chemical shim control and on-power refueling are employed, respectively. In other studies, the poison management is considered to have weak interactions with fuel management and to be separable from that.

A number of control rods locating in the core can, however, bring the core to the end state which is most desirable with respect to discharge burnup, cycle cost or other criteria, consuming excess reactivity on the course.

For BWR, Haling's principle [Hal.63] is prevailing as the

standard and is presumably a good way of operation that suppresses the power peaking factor (PPF) throughout the core life.

Terney & Fenech [Te.70] have applied dynamic programming and direct flux synthesis to the space-time optimization problem of determining the optimum sequence of control rod motions in a representative PWR, which minimizes the maximum power peaking throughout the life of the core.

Motoda & Kawai [Mo.70] have studied the coupled effect of the control rod programming and the fuel burnup. The problem of optimizing the control-rod programming to maximize the fuel burnup with the constraint imposed on the power peaking factor, is formulated and solved for a typical BWR. The neutron governing equation there is taken to be a one-group equation. Introduction of the notion of burnup space and the maximum principle makes it possible to obtain an optimal solution for the two-region reactor.

Suzuki & Kiyose [Suz.71] have mathematically analyzed on the poison management problem for a maximum average burnup of multi-zone light-water reactor cores: developing an elaborate argument in the burnup space.

Wade & Terney [Wad.71] have represented a reactor core in a one-group, spatially nodalized model. The optimal problem is then formulated in terms of a generalized set of design objectives and a generalized control that influences the nodal material buckling. The problem is reduced to an algorithm based on gradient method through the maximum principle. A problem to minimize the residual fuel fraction at the specified terminal time is solved there numerically.

An attempt has been made by Motoda [Mo.71] to optimize control rod programming and fuel loading pattern simultaneously for a one-dimensional multiregion slab reactor by a nonlinear programming technique. Five-region reactor is tested as an example and the result is compared with that by Haling's principle.

Sekimizu & Monta [Se.74] have also dealt the poison management

problem for a three-zone reactor in attaining the largest fuel burnup. A geometrical study is made on the terminal manifold.

Common to these works, the neutron governing equation is taken to be a one-group model, which requires much computational effort in spite of its poor accuracy. The implicitness of the relation between the input control and the resulting power distribution, prevents the intuitive management of the controlling problem.

The resulting power distribution which is reduced to these optimizing algorithms cannot be far different from the prescribed shape. This leads to an assumption that the fast neutron flux density shape in the two-group model, does not change in its distribution despite of the small change in core parameters.

Section 2 of this chapter is concerned with the burnup maximization problem on this assumption. Different from the one group equation, this model always allows the existing solution of the flux distribution, and a new criticality condition is introduced. This model, named as "*cumulus*" in author's terminology, has its significance in giving a clear image of the system configuration. Reduction of the space dependent system to the lumped dynamical system is made by Walsh functions [Har.70], which are amenable to the core divided equally.

In section 3, the possibility that the small failures occur on the fuel sheath, is taken into consideration. The 10⁴ of fuel pins in the core are exposed to intensive irradiation of neutron flux and gamma ray, and left under severe thermal, mechanical conditions for about three years. It is, therefore, naturally expected that some of them carry pin-halls in their sheath. After the failure has been known to exist, the reactor may be still kept working under more strict thermal limitations than the ordinary, with the carefully prepared operating conditions.

These failures can be assumed to occur with a prescribed probability. If a control-rod programming is given, we can evaluate the expectation of the total average burnup. Adopting this expecta-

tion as the criterion for the control-rod programming, we can exploit the degenerate region that appears in the deterministic burnup optimization problem.

4.2 Neutron Governing Equation; *Cumulus Model*

Initially, assume that the neutron flux in a core is governed by the steady state two-group diffusion equation,

$$\nabla D_1 \nabla \phi_1 - \Sigma_R \phi_1 = -\nu \Sigma_f \phi_2 \quad (4.2.1)$$

$$\nabla D_2 \nabla \phi_2 - \Sigma_a \phi_2 + \rho \Sigma_R \phi_1 = \Sigma_c \phi_2 \quad (4.2.2)$$

Here, the elements of the equations are grouped into two parts. One is composed of the effect on the structural and the moderating materials, both of which are invariant on the course of burnup, and the other, of the fuel state and the manipulatable control absorber which may change on the course of burnup. A power reactor is necessarily equipped with reflectors in which the neutron flux is determined by the equations

$$\nabla D_1' \nabla \phi_1' - \Sigma_R' \phi_1' = 0 \quad (4.2.3)$$

$$\nabla D_2' \nabla \phi_2' - \Sigma_a' \phi_2' + \rho' \Sigma_R \phi_1' = 0 \quad (4.2.4)$$

along with the boundary conditions

$$\phi_1 = \phi_1' \quad (4.2.5)$$

$$D_1 \nabla \phi_1 = D_1' \nabla \phi_1' \quad (4.2.6)$$

at the core-reflector interface, and

$$\phi_1'(\omega) = \phi'(\omega) = 0, \quad \omega \in \partial\Omega \quad (4.2.7)$$

at the extrapolated boundary $\partial\Omega$ outside of the reflector with finite thickness. Existing water-moderated power reactors are equipped with neutronically enough thick reflectors, and the infinite thickness assumption for the reflectors provides a good approximation. Then Eq.(4.2.7) is substituted by

$$0 < \phi_1'(\omega), \phi_2'(\omega) < \infty, \quad |\omega| \rightarrow \infty. \quad (4.2.8)$$

In the following, the infinite thickness reflectors are supposed.

In dealing with the system (4.2.3-6) and (4.2.8), the criticality condition should always be satisfied. Much labour is, however, needed to examine the criticality or to adjust the control value within the prescribed conditions for the flux distribution or for other structural conditions. It is because the criticality condition is implicit rather than explicit, as has been pointed out in Chapter 2.

Undergoing power reactors cannot allow the extremely skewed power release profile over the core so that low energy cost may be attained. This circumstance leads us to an approximation assumption.

ASSUMPTION; *Fast neutron flux distribution is fixed at the prescribed shape $\varphi(\omega)$.*

It is natural to consider that the φ distribution is determined by

$$\nabla D_1 \nabla \varphi - \Sigma_R \varphi = -V^{-1} \int_{\Omega} (\nu \Sigma_f \phi_2) d\omega, \quad (4.2.9)$$

where the term $\overline{(\nu \Sigma_f \phi)}$ represents the power release profile which is uniform over the core of volume V .

Under this assumption the neutron governing equation becomes

$$\nabla D_2 \nabla \phi_2 - \Sigma_a \phi_2 = \Sigma_c \phi_2 - \rho \Sigma_R \varphi(\omega) \quad (4.2.10)$$

associated with the boundary conditions (4.2.5), (4.2.6) and (4.2.8).

Let us implement the mathematical model (4.2.9) into the physical reactor. Equation (4.2.9) is just the same as that for the

thermal neutron equation of the reactor with the source $p\Sigma_R\varphi$. If this reactor maintains the stationarity, the system without the source should have been subcritical. Therefore Eq.(4.2.9) always has a solution for an arbitrary positive Σ_c . This is in contrast with Eqs. (4.2.1) and (4.2.3), which do not allow the solution to exist unless the criticality condition is satisfied.

Making use of Green's function, Eq.(4.2.9) along with the boundary conditions is transformed into an integral expression

$$\phi_2(\omega) = \int_{\Omega} G(\omega|\omega') \{ p\Sigma_R\varphi(\omega') - \Sigma_c(\omega') \phi_2(\omega') \} d\omega'. \quad (4.2.11)$$

The lacking criticality condition can be complemented by defining an effective multiplication factor k_{eff}^* for the whole system as

$$k_{eff}^* \triangleq \frac{\int_{\Omega} \nu \Sigma_f \phi_2 d\omega}{\int_{\Omega} (\nu \Sigma_f \phi_2) d\omega}. \quad (4.2.12)$$

The criticality condition is defined by

$$k_{eff}^* = 1. \quad (4.2.13)$$

By this criterion, the solution of Eqs.(4.2.1), (4.2.2) along with the boundary conditions, which always exist for any Σ_c mathematically, can be examined for the criticality.

The concise neutron governing model stated above shall be called as "*cumulus*" hereafter for the convenience of explanation.

4.3 Burnup Maximization Problem

Let the burnup process in the core be described by

$$\frac{\partial}{\partial t} N_f = - \sigma_f N_f \phi_2, \quad (4.3.1)$$

where N_f and σ_f are, respectively, the number density and the microscopic cross section of the fissile material. Here, for the purpose of simplicity, we assume that the fissile material is composed of single species and the fission products do not have effect on the process.

Normalizing ϕ_2 by Φ as

$$\phi = \phi_2 / \Phi \quad (4.3.2)$$

and defining

$$\tau = \sigma_f \Phi t, \quad (4.3.3)$$

we obtain

$$\frac{\partial}{\partial \tau} \Sigma_f = - \Sigma_f \phi, \quad (4.3.4)$$

where the relation

$$\Sigma_f = \sigma_f N_f$$

is incorporated. Assigning the dimension of flux density to Φ and also normalizing Σ_f by the dimension itself, Eq.(4.3.4) is concerned only with dimensionless values.

There are some constraints on the operation of power reactors. One is that the control value Σ_c in Eq.(4.2.10) should be non-negative. The null value corresponds to the situation that the control absorber (rod) is fully withdrawn. The other is that the maximum power release density should be bounded by the prescribed value. A further constraint should be so posed that the reactor is operated at the rated power. Retaining this condition, the constraint on the maximum power release density is stated in terms of the power peaking factor, which shall be abbreviated to PPF in the following.

These conditions are written as

$$0 \leq \Sigma_c(\omega) \leq (\Sigma_c)_{max}, \quad (4.3.5)$$

$$V^{-1} \cdot \Sigma_f(\omega) \phi(\omega) \leq F, \quad (4.3.6)$$

$$\int_{\Omega} \Sigma_f(\omega) \phi(\omega) d\omega = 1. \quad (4.3.7)$$

Taking the normalization factor Φ so that

$$\int_{\Omega} (\overline{\Sigma_f \phi_2}) d\omega / \Phi = 1 \quad (4.3.8)$$

should hold, the rated power condition (4.3.7) coincide with the criticality condition (4.2.12).

The burnup process which has started from the initial state of Σ_f distribution, continues untill one of these conditions is violated.

In terms of the dynamical system theory, the state value is Σ_f which is governed by the state equation (4.3.4) associated with the control value ϕ , which is manipulated by Σ_c through the *cumulus model* (4.2.10). The space of Σ_f , the state space, is termed as the burnup space. The set of Σ_f for which there exist $\Sigma_c(\omega)$ and $\phi(\omega)$ that give the equality sign in conditions (4.3.5) and (4.3.6), forms a manifold, which shall be called as the terminal manifold.

The controlling object is taken as the maximization of the total output energy for the initial $\Sigma_f(\omega)$ given at $\tau=0$. Due to Eq. (4.3.7), this performance criterion is equivalent to the maximization of the time when the state arrives at the terminal manifold. Thus, the problem has been reduced to the "maximum time problem".

4.4 Walsh Function Expansion

The burnup maximization problem has been formulated as the maximum time problem of a distributed parameter system. In order to

deal with the distributed parameter system, we can make use of modal expansion technique. Suitable choice of the expansion functions helps us to express well the characteristics of the system with a small effort.

Walsh functions $Wal(i, \theta)$, $i = 0, 1, 2, \dots$, $\theta \in [-1/2, 1/2]$, are here employed because of the closedness with respect to multiplication and the adaptivity to the multi-region cores. A brief explanation about Walsh functions is given in Appendix C.

Assuming a slab geometry for the reactor, the core can be normalized to $\Omega = [0, 1]$. Then, shifting the region of definition of Walsh function, $[-1/2, 1/2]$ onto the normalized region, the modal expansion is carried out as follows.

Expanding $\phi(\omega)$, $\Sigma_c(\omega)$ and $G(\omega, \omega')$ as

$$\phi(\omega) = \sum_{i=0}^M p_i Wal(i, \omega) \quad (4.4.1)$$

$$\Sigma_c(\omega) = \sum_{i=0}^M r_i Wal(i, \omega) \quad (4.4.2)$$

$$G(\omega | \omega') = \sum_{i=0}^M \sum_{j=0}^M g_{ij} Wal(i, \omega) Wal(j, \omega') \quad (4.4.3)$$

and also expanding as

$$\Phi^{-1} \int_{\Omega} G(\omega | \omega') p \Sigma_R(\omega) \phi(\omega') d\omega' = \sum_{i=0}^M k_i Wal(i, \omega), \quad (4.4.4)$$

Eq.(4.2.11) becomes to

$$\sum_{k=0}^M \left\{ \sum_{j=0}^M g_{ij} p_{k \oplus j} \right\} r_k = k_i - p_i \quad (4.4.5)$$

where the symbol $\oplus$ comes of the multiplication law of Walsh functions (Appendix C).

Substituting the expanded form

$$\Sigma_f(\omega) = \sum_{i=0}^M s_i Wal(i, \omega) \quad (4.4.6)$$

into Eq.(4.3.4), we obtain

$$s_i = - \sum_{j=0}^M s_j p_{j \oplus i}, \quad i = 0, 1, 2, \dots \quad (4.4.7)$$

In case when $\dot{\lambda}=0$, the rated power condition or the criticality condition (4.3.6) gives

$$\dot{\lambda}_0 = -1 , \quad (4.4.8)$$

which is easily seen to give the definite solution

$$\lambda_0(\tau) = \lambda_0(0) - \tau . \quad (4.4.9)$$

In the following, expansion coefficients of each variables are collected into M-vectors p, r, λ and MxM matrix G .

Also the constraints (4.3.4-6) can be expanded as;
for power peaking,

$$\sum_{i=0}^M \left\{ \sum_{j=0}^M \lambda_i p_{i \oplus j} \right\} Wal(i, \omega) \leq F , \quad (4.4.10)$$

for criticality,

$$\sum_{i=0}^M p_i \lambda_i - 1 = 0 , \quad (4.4.11)$$

and for controlling absorber,

$$0 \leq \Sigma_c(\omega, p, h-p) \leq (\Sigma_c)_{max} , \quad (4.4.12)$$

where Σ_c is expressed in terms of r which has been obtained as the analytical solution of Eq.(4.4.5).

The number M of the utmost order of the expansion functions should preferably be taken as

$$M = 2^{N-1} - 1 , \quad (4.4.13)$$

where the number N is the number of regions of the core divided equally. This choice enables us to make the profit of the closedness of Walsh functions.

Confining the case to M=1, we can investigate the system more precisely. This number corresponds to the slab reactor core which is divided into two, the inner and the outer region as is illustrated in Fig.4.4.1.

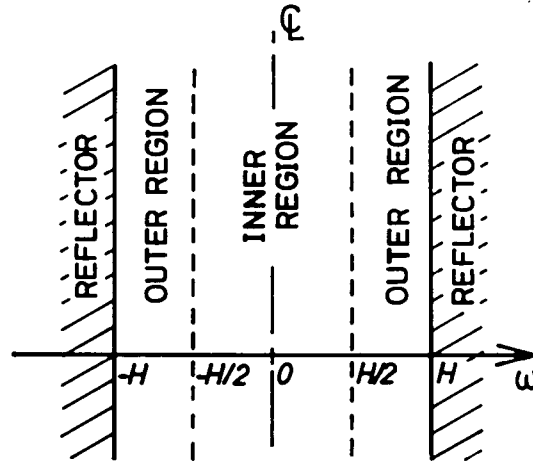


Fig.4.4.1 A slab reactor divided into two regions of the same volume.

From the geometrical symmetry, only the half is considered with $\omega=0$ and $\omega=H$ corresponding to the center and the core-reflector interface, respectively, and the width $H/2$ is shared by two regions in the half core.

In order to get correspondence between the domain of definition of Walsh functions and the physical configuration illustrated in Fig. 4.4.1, the space variable ω is normalized by H and meantime the domain of definition is shifted by $1/2$ to the right, different from the prevailing definition. In order that this new geometry is consistent, only the diffusion coefficients should be divided by H^2 . Thus, the half core is considered to lie on $[0,1]$ of ω in the following.

The burnup equation is written as

$$\dot{\lambda}_0 = \lambda_0(0) - \tau \quad (4.4.14)$$

$$\dot{\lambda}_1 = -(\rho_0 \lambda_1 + p_1 \lambda_0)$$

$$\triangleq -f \quad (4.4.15)$$

The value of f corresponds to the distortion of power density profile.

Making use of f , the PPF constraints become to

$$1 - F \leq f \leq F - 1 \quad (4.4.16)$$

where the left-side equality takes place when the power density in the inner region reaches to the bound, and the right-side, in the outer region.

The value of the control absorber in each region for the prescribed flux distribution is given by

$$\Sigma_c^{\text{IN}} = \frac{(g_{11} + g_{10})(k_0 - p_0) - (g_{00} + g_{01})(k_1 - p_1)}{\Delta (p_0 - p_1)} \quad (\text{INNER REGION}) \quad (4.4.17)$$

$$\Sigma_c^{\text{OUT}} = \frac{(g_{11} - g_{10})(k_0 - p_0) + (g_{00} - g_{01})(k_1 - p_1)}{\Delta (p_0 + p_1)} \quad (\text{OUTER REGION}) \quad (4.4.18)$$

where

$$\Delta = g_{00} g_{11} - g_{01} g_{10} . \quad (4.4.19)$$

The control absorber constraint given by Eq.(4.3.4) works on the values given by Eqs.(4.4.17) and (4.4.18) only requiring positiveness for the practical core parameters, as to give for the inner region

$$(g_{11} + g_{10})(k_0 - p_0) - (g_{00} + g_{01})(k_1 - p_1) \geq 0 \quad (4.4.20)$$

and

$$(g_{11} - g_{10})(k_0 - p_0) - (g_{00} - g_{01})(k_1 - p_1) \geq 0 \quad (4.4.21)$$

for the outer region.

The criticality condition, Eq.(4.3.6), becomes to

$$\Delta_0 p_0 + \Delta_1 p_1 = 1 . \quad (4.4.22)$$

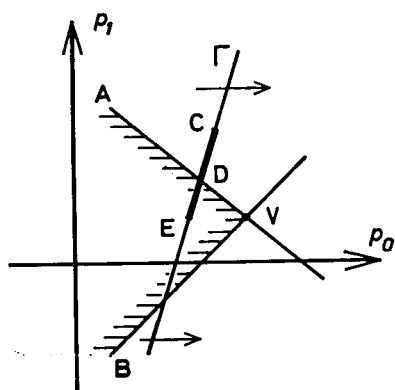
The conditions, Eqs.(4.4.6) and (4.4.20-22) can be geometrically interpreted in the $p_0 - p_1$ plane with Δ as the parameter.

In Fig.4.4.2, the line Γ corresponds to the criticality con-

dition, Eq.(4.4.22), on which the value of β can be marked. The bold faced section CE of the line stands for the part which satisfies Eq. (4.4.16).

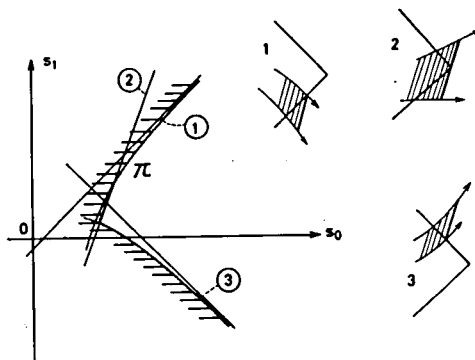
The constraint for the control absorber is satisfied if the reactor is operated at the point within the sector defined by AVB. The lines AV and BV are determined by the equality of Eqs.(4.4.20) and (4.4.21), respectively.

In the sequel, the reactor should be operated at a point on the common section of the part CE and the sector.



A feasible power shape ought to locate on the section DE for the illustrated situation.

Fig.4.4.2 Admissible power shape in p_0 - p_1 plane.



The small figures illustrate the movement of the CE section of Fig.4.4.2.

Fig.4.4.3 Terminal manifold.

Since $\lambda_0(\tau)$ decreases according to burnup as is shown by Eq.(4.4.4), the criticality line Γ moves to the right. The common section DE becomes smaller and at last disappears, that is, the life of the core ceases.

The ways how the core life ceases, are classified into three cases: 1. the PPF violates the constraint in the inner region, 2. there is no excess reactivity to maintain criticality or 3. the PPF violates the constraint in the outer region. Figure 4.4.3 shows the

process with the corresponding numbers. Movement of the fraction DE or CE in Fig.4.4.2 is also depicted in this figure in connection with the sector corresponding to the rod constraints. The set of Δ which gives the termination of the core life, named as the terminal manifold π , is shown also. The terminal manifold is composed of the parts of hyperbola (1,3) and the straight line (2). The detailed description is given in Appendix D.

Thus the constraints are seen to be abbreviated to

$$f_{min}(\Delta) \leq f \leq f_{max}(\Delta) , \quad (4.4.23)$$

where the upper and the lower bounds are the functions of the states, Δ .

4.5 Optimal Rod Programming and the Numerical Results

4.5.1 Optimality Condition

The necessary condition for the maximum time problem can be obtained through the variational calculus consideration and geometrical intuition.

Let T be the time when the trajectory arrives at the terminal manifold. Integrating the system Eq.(4.4.5) with respect to τ , we obtain

$$\Delta_1(T) = \Delta_1(0) - \int_0^T f(\Delta, p) d\tau . \quad (4.5.1)$$

Introducing the small variation δf to f , the deviation of $\Delta_1(T)$ on the terminal manifold is calculated to be

$$d\Delta_1(T) = -f^T dT - \int_0^T \delta f d\tau . \quad (4.5.2)$$

where f^T is the uniquely determined value of f at each point of the

terminal manifold.

Since the terminal point still lies on the manifold after the deviation δf has been introduced, the increment $d\mathcal{L}_1(T)$ is given by

$$d\mathcal{L}_1(T) = - \left[d\mathcal{L}_1/d\mathcal{L}_0 \right]_{\pi} dT, \quad (4.5.3)$$

where $[d\mathcal{L}_1/d\mathcal{L}_0]_{\pi}$ stands for the inclination of the manifold at the terminal point. Therefore Eq.(4.5.2) can be rewritten as

$$dT = - \frac{1}{f^T + [d\mathcal{L}_1/d\mathcal{L}_0]_{\pi}} \int_0^T \delta f \, d\tau. \quad (4.5.4)$$

In order that this trajectory be optimal, the inequality

$$dT \leq 0 \quad (4.5.5)$$

must hold for any variation δf which does not violate the restriction placed against the trajectory.

Due to the inequality (4.5.5), the necessary condition that

$$f = f_{\max}, \quad \tau \in [0, T] \quad (4.5.6)$$

should give the optimal trajectory, is given by

$$f^T + [d\mathcal{L}_1/d\mathcal{L}_0]_{\pi} > 0, \quad (4.5.7)$$

at the terminal point, because the inequality

$$\delta f \leq 0, \quad \tau \in [0, T] \quad (4.5.8)$$

holds for any admissible variation.

Similarly,

$$f^T + [d\mathcal{L}_1/d\mathcal{L}_0]_{\pi} < 0 \quad (4.5.9)$$

must hold in order that

$$f = f_{\min}, \quad \tau \in [0, T] \quad (4.5.10)$$

should be optimal.

Given the initial state, S^1 , in the burnup space, the region

where the trajectory can cover is restricted as is shown in Fig. 4.5.

1. The boundaries are the arcs $S^1-H_u^1$ and $S^1-H_l^1$ that respectively corresponds to $f=f_{max}$ and $f=f_{min}$.

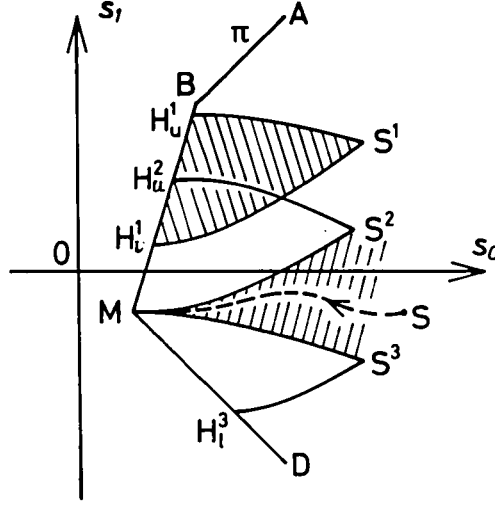


Fig.4.5.1 Reachable region and degenerate region.

Because of Eq.(4.4.4), the Δ_0 component of the terminal point is desirable to be as small as possible. The arc $S^1-H_l^1$ is, from the geometrical intuition, seen to be optimal when the initial state is located at S^1 .

The vertex M gives the fuel allocation corresponding to the solution of the minimum critical mass problem. Suppose two initial states: S^2 , which can arrive at M by $f=f_{max}$ through the pass S^2-M and S^3 , by $f=f_{min}$ through S^3-M . Any trajectory originating from the point located in the sector S^2-M-S^3 , can arrive at M, and the trajectory is not definite. This region is called as the degenerate region.

When an initial state is given, the optimal trajectory is obtained in the way as follows.

Calculate the two trajectories originating from the given initial state with the power shapes given by Eqs.(4.5.6) and (4.5.10), then examine the optimality conditions (4.5.7) and (4.5.9) at each

terminal point. The result falls into three cases.

When either of the criteria (4.5.6) or (4.5.10) holds, $f=f_{min}$ or $f=f_{max}$ is known to be optimal accordingly. If neither of the two criteria holds, the initial state must have been put in the degenerate region. The third is that both of the cases are approved. The last takes place when the part BM of the terminal manifold is heavily concaved as is illustrated in Fig.4.5.2, and then the optimality

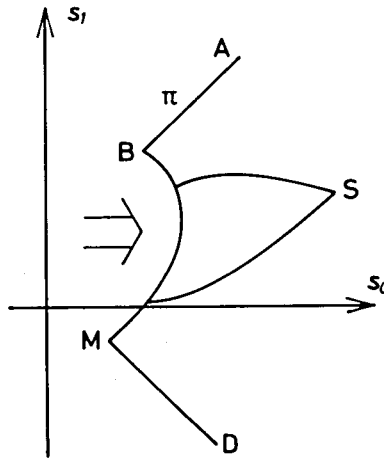


Fig.4.5.2 Terminal manifold with heavily concaved part.

should be examined by the direct comparison of the $\mathcal{A}_0$ -components of the two terminal points with each other. This case, however, does not arise for the neutron governing model, *cumulus*, used here, because the part BM is linear.

4.5.2 Numerical Results

Numerical calculation has been carried out for the reactor with the parameters listed in Table 4.5.1. The data are chosen to express a small sized Boiling Water Reactor (Hagen & Goin [Hag.65]).

CORE		REFLECTOR	
$D_1 = 1.5$	cm	$D_1 = 2.127$	cm
$D_2 = 0.4337$	cm	$D_2 = 0.2592$	cm
$\Sigma_r = 0.02471$	cm^{-1}	$\Sigma_r = 0.3946$	cm^{-1}
$\Sigma_a = 0.1012$	cm^{-1}	$\Sigma_a = 0.0115$	cm^{-1}
$p = 0.6705$		$p = 0.9840$	
Half			
Thickness	37.42	cm	
Reference			
Value	$\Sigma_f = 0.06336$	cm^{-1}	(Inner Region)
	0.12366	cm^{-1}	(Outer Region)
$(\Sigma_c)^{\max}$	$= 0.0153$	cm^{-1}	

Table 4.5.1 Core parameters

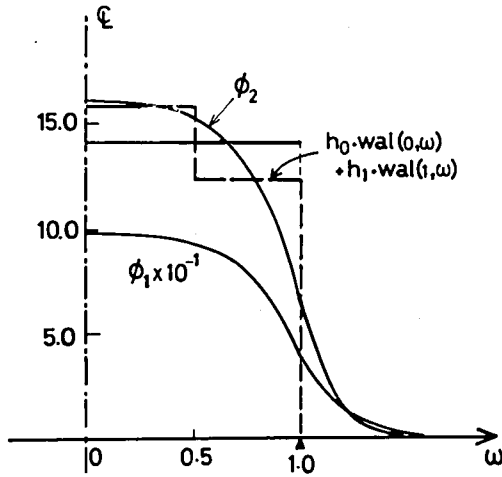
The half width H was adjusted so that the core had 1.15 of the effective multiplication factor defined by Eq.(4.2.13) with the reference Σ_f listed in the table. The maximum control absorber cross section $(\Sigma_c)^{\max}$ was determined to give the criticality for this Σ_f .

The reduced parameters are listed in Table 4.5.2 corresponding to Eqs.(4.4.3) and (4.4.4).

$g_{00} = 9.769$	$g_{01} = -0.1126$	$h_0 = 14.16$
$g_{10} = -0.1128$	$g_{11} = 9.223$	$h_1 = -1.742$

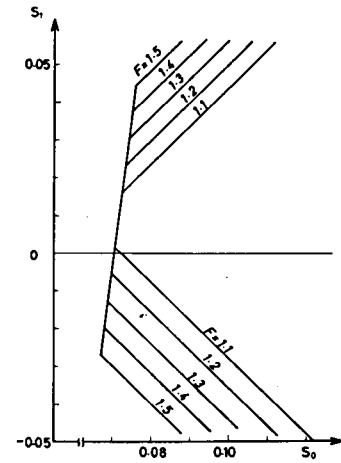
Table 4.5.2 Reduced system parameters

Figures 4.5.3 and 4.5.4 show the reference neutron flux distributions and the terminal manifolds for each maximum PPF.



The fast flux ϕ_1 is considered to be invariant in *cumulus* model. The reflector ranges to the infinite point.

Fig.4.5.3 Reference neutron flux distribution.



The PPF values are associated with each terminal manifold.

Fig.4.5.4 Terminal manifolds.

Figures 4.5.5-8 show the optimal trajectories and the degenerate regions for each value of PPF, F .

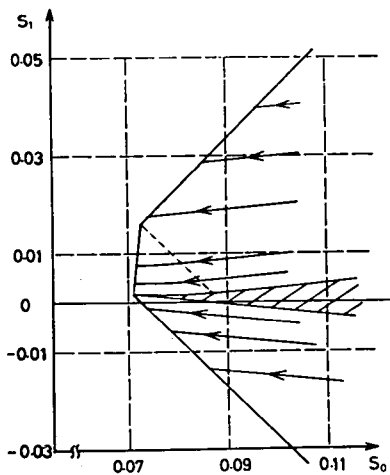


Fig.4.5.5 Optimal trajectories for PPF=1.1.

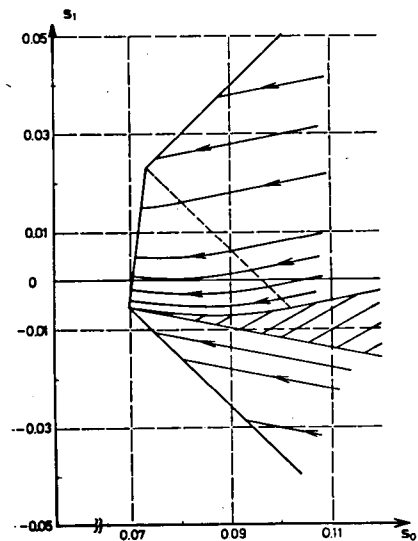


Fig.4.5.6 Optimal trajectories for PPF=1.2.

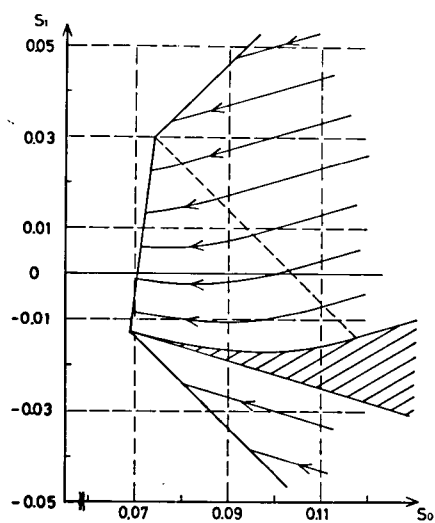


Fig.4.5.7 Optimal trajectories
for PPF=1.3.

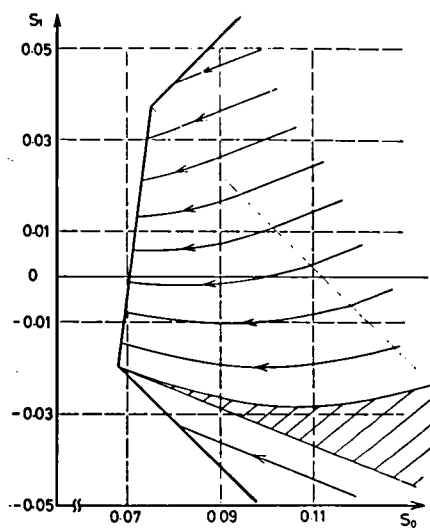


Fig.4.5.8 Optimal trajectories
for PPF=1.4.

4.5.3 Modification of Walsh Functions for Cylindrical Geometry

For reactors with cylindrical geometry, Walsh functions are not suitable because the equal division of radius does not provide the equal partition of the volume. The volume of each region is desirable to be equal from the viewpoint of fuel management.

Defining a new function as

$$Wa(i, r) \triangleq Wal(i, \sqrt{r}), \quad (4.5.11)$$

the equality consists between the volumes of the regions. The functions are illustrated in Fig.4.5.9.

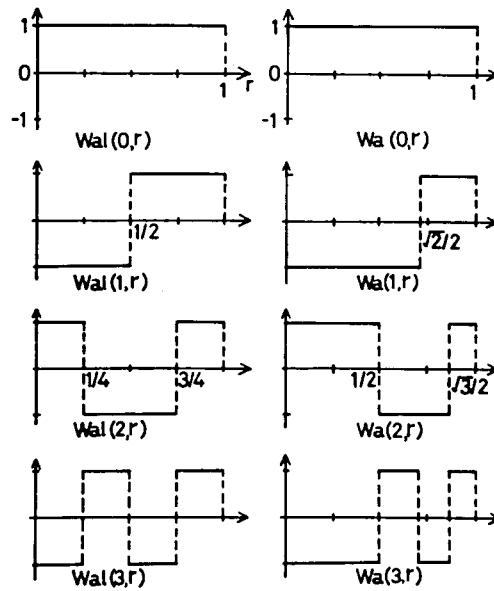


Fig.4.5.9 Binary functions for cylindrical geometry.

The orthogonarity of the functions is given by

$$\int_0^1 Wa(i, r) Wa(j, r) r dr = \delta_{ij} \text{ (Kronecker)}. \quad (4.5.12)$$

The burnup rate is measured by e , hereafter, defined by

$$e = \Sigma_f(\tau) - \Sigma_f(0) . \quad (4.6.1)$$

4.6.2 Expectation of Maximum Burnup with the Probabilistic Fuel Failure

The failure on the sheath of fuel pins is considered to take place in a prescribed probability. Three assumptions are put on the probability:

1. *The probability density of the occurrence of failure with respect to time is dependent only on the burnup e and not on the temporal output power density.*
2. *The probability is enough small and the failure takes place only in one region and once through the core life.*
3. *The control rod program after the failure has occurred, is prescribed independent of the degree and the position of the failure as far as it is not fatal.*

Subject to these hypotheses, the expectation of the total energy output can be determined.

Let us consider that the optimal control program $U(r, \tau)$ ($r \in \Omega$, $\tau \in [0, T]$) is given. The burnup of the fuel is given by

$$e(r, \tau, U) = \int_0^\tau P(r, t, U) dt \quad (4.6.2)$$

where

$$P(r, t) = \Sigma_f(r, t) \phi_2(r, t) . \quad (4.6.3)$$

Suppose that the terminal time has been changed to T_F after the fuel failure has occurred at the place r' and time $\tau = \tau_f$, then the resulting burnup distribution at $\tau = T$ is given by

$$e_T(r, e_f(r'), U, U_f) = \int_0^{T_f} P(r, t, U) dt + \int_{T_f}^{T_r} P(r, t, U_f) dt, \quad (4.6.4)$$

where U_f stands for the prescribed control rod program after the failure. Equation (4.6.4) directly gives the total energy output as

$$J(e_f(r'), U, U_f) = \int_{\Omega} e_T(r, e_f(r'), U, U_f) dr. \quad (4.6.5)$$

From the assumption 1, the expectation of J is given, incorporating with the prescribed probability density function $g(e_f)$, by

$$\begin{aligned} E[J(e_f(r'), U, U_f)] &= \int_K J(e_f(r'), U, U_f) g\{e_f(r')\} de_f(r') \\ &+ J^* \left[1 - \int_K g\{e_f(r')\} de_f(r') \right], \end{aligned} \quad (4.6.6)$$

where K and J^* stand for the optimal trajectory without the failure and the corresponding value of J , respectively.

Thus we have obtained the problem to maximize the performance criterion (4.6.6). The maximization should be performed with respect to the choice of the trajectory K .

In order to numerically evaluate Eq.(4.6.6), the core is divided into N regions and e_f is averaged in the region being defined as e_f^n , $n=1, \dots, N$. Then suppose that the failure at the region n takes place when the burnup arrives at e_f^{nm} , with the probability

$$P_r(e_f^{nm}) = \int_0^{e_f^{nm}} g(e) de, \quad m=1, \dots, M. \quad (4.6.7)$$

As is shown in Fig.4.6.2, the resulting performance J is determined uniquely because of the assumption 3. The approximate expectation of J is given by these $J(e_f^{nm})$ and the weights as

$$\begin{aligned} E[J(e_f, U, U_f)] &= \sum_{n=1}^N \sum_{m=1}^M J(e_f^{nm}, U, U_f) P_r(e_f^{nm}) \\ &+ J^* \left[1 - \sum_{n=1}^N \sum_{m=1}^M P_r(e_f^{nm}) \right]. \end{aligned} \quad (4.6.8)$$

If the initial state is put outside the degenerate region, the optimal solution that minimizes $E(J)$ coincides trivially with the

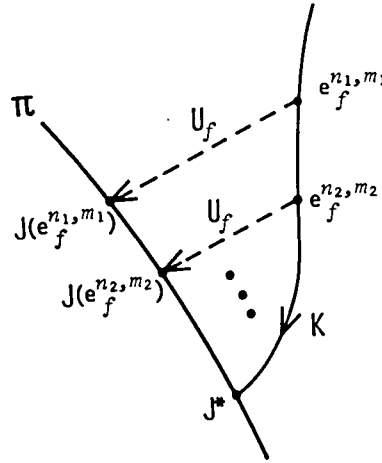


Fig.4.6.2 Discretization of the trajectory.

solution for the case without the probability of the failure.

Interest should be found in the case when the initial state is put in the degenerate region, where the freedom is left in the pass to arrive at the optimal terminal point. The optimal trajectory has been obtained numerically for two region reactor whose neutronics is governed by a modified one-group model. The burnup space is also described in terms of Walsh function expansion coefficients. The core parameters are taken to be of BWR with radius 1.5m. Figure 4.6.3 shows the region where the trajectory originated from the point can cover. The candidate of the solution has been chosen among the trajectories

$$\mathcal{A}_1(\mathcal{A}_0) = a \overline{\mathcal{A}}_1(\mathcal{A}_0) + (1-a) \underline{\mathcal{A}}_1(\mathcal{A}_0), \quad 0 \leq a \leq 1, \quad (4.6.9)$$

where the reference trajectories $\underline{\mathcal{A}}_1(\mathcal{A}_0)$ and $\overline{\mathcal{A}}_1(\mathcal{A}_0)$ are the uppermost and the lowermost boundary of the region, respectively, as is illustrated in Fig.4.6.4 for PPF=1.3. The broken lines stand for the terminal manifold and the degenerate region after PPF has suppressed to =1.2 because of the failure. Figure 4.6.5 shows $E(J)$ versus the parameter . The solid and the broken lines correspond to whether the control rod program after the failure, U_f , should be taken to be inner-high or outer-high, respectively. From this figure, the optimal

trajectory for inner-high U_f is given as $\alpha=0.68$.

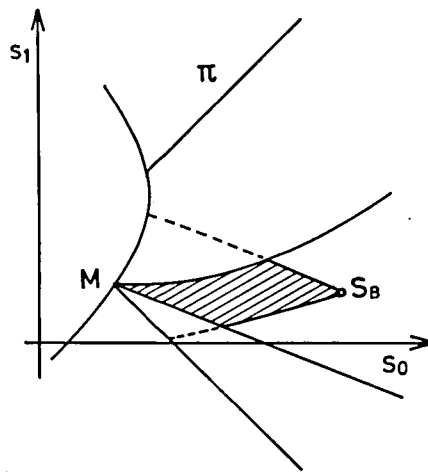


Fig.4.6.3 Reachable region from S_B in the degenerate region.

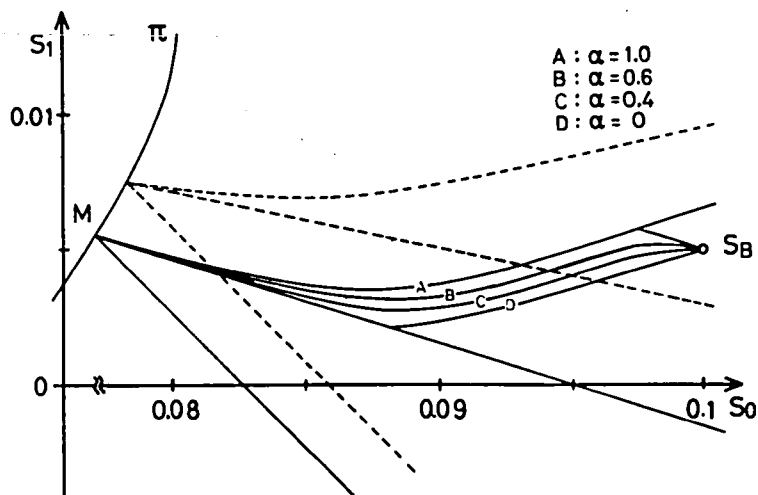
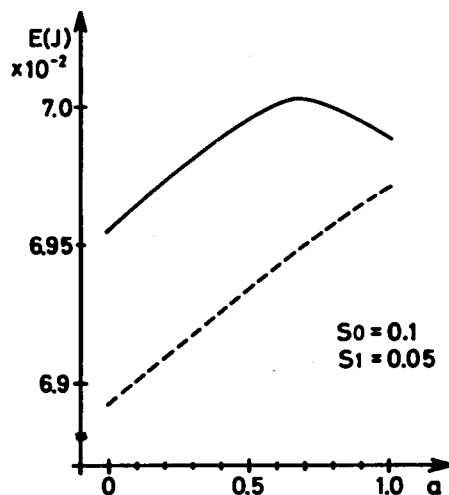


Fig.4.6.4 Candidate trajectories for the optimal.



Control rod programings after the failure are taken to be outer-high (solid line) and inner-high (broken line)

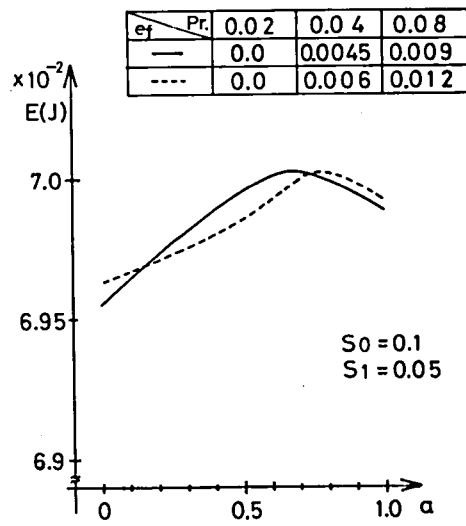
Fig.4.6.5 Evaluated expectation of J .

The probability of the fuel failure is hypothetically given in Table 4.6.1. These values are commonly used for both inner and outer regions

e_f	$P_f(e_f)$
0.0	0.02
0.0045	0.04
0.009	0.08

Table 4.6.1 Fuel failure probability

Figure 4.6.6 shows two curves of $E[J]$ corresponding to the different fuel failure probabilities depicted in the figure.



Rod programmings after the failure are commonly taken to be outer-high.

Fig.4.6.6 Two $E[J]$ for different failure probabilities.

Chapter 5. Conclusions

The optimal control problems of nuclear reactors have been solved with the recognition that the systems are distributive.

Efforts have been devoted to decompose the distributive and complex original core system to the proper small systems. At the sacrifice of accuracy and the practical applicability, the neutronics have been described in an integral equation using Green's functions, when the neutronics themselves are not of the prime importance. Reduced concise systems will help the intuitive approach of engineers on the site as well as of research people participating in the composition of the advanced controlling systems.

In chapter 2, the steady state core is dealt with. Employing a modified Green's function, the criticality condition for a core with a few rods located discretely in the core has been given in an algebraic form and it proves to give good accuracy in comparison with the strict solution. Also has been studied the optimal control problem to allocate the flux distribution in the core by adjusting the absorption cross section. The optimal solution has been obtained numerically for the moderate conditions. As the limiting condition for the control value has been widened, a singular feature of the problem has come out and has prevented the numerical calculation. The reason has been investigated and the property of the formulation employed there has been clarified in contrast with the backward substitution method. An alternative suboptimal formulation has been given by an intuitive consideration in the spaces of controls and the corresponding trajectories. Satisfactory solution has been obtained

numerically by the suboptimal method for an ill-conditioned case for which the strict method could not give a solution.

Chapter 3 has shown that the regulator problems for both open and closed loop controls are able to be reduced to the algebraic equations. The open loop control for the neutronics and precursor dynamics has been re-formulated to the dynamical system concerning with only the precursor dynamics through the adiabatic approximation for the neutronics. The analytical reductions on the course have been made in terms of function space method. Numerical examination has been made and the optimal controls and the responses have been obtained successfully. The closed loop control problem has been synthesized for dynamical system composed of the precursors and the temperature as the state values associated with the steady state neutronics representing the coupling scheme for the two state values. The result has been described in a Riccati-type integral equation. This equation has been further decomposed to a set of algebraic Riccati-type equations which are able to be solved numerically.

Chapter 4 is concerned with the burnup control problem of a core. The new neutronics model "*cowmulus*", introduced by the author, has made it possible to describe pictorially the burnup control problem of a two region reactor core. The burnup maximization problem has been solved for that model giving a numerical solution.

An attempt has been made to take account of the possibility that small failures occur on fuel pins. It has been shown that the formulation enables us to make use of the degenerate region which has not drawn much attention in the deterministic burnup problems. On the problem formulation, the probability of the fuel failures is evaluated with respect to the burnup to which the concerning fuel assembly has been attained. An optimization problem has been formulated so as to maximize the expectation of average burnup at the end of the core life, and the numerical results have been obtained.

References

- [Am.66] Amano, F., and K. Sunako, "An Analysis of Power-Flattening in a Multiregion Reactor (Part I)," *Bull. Electrotech. Lab.*, Vol.30, No.10, pp.839-848, 1966. (Japanese)
- [Am.67] Amano, F., "An Analysis of Power-Flattening in a Multiregion Reactor (Part II) — Discrete Model —," *Bull. of Electrotech. Lab.*, Vol.31, No.9, pp.49-58, 1967. (Japanese)
- [As.66] Ash, M, *Optimal Shutdown Control of Nuclear Reactors*, Academic Press, New York and London, 1966.
- [Ba.63] Balakrishnan, A. V., "An Operator Theoretic Formulation of Control Problems and a Steepest Descent Method of Solution," *SIAM. J. Control*, Vol.1, No.2, pp.109-127, 1963.
- [Ba.65] ———, "Optimal Control Problems in a Banach Spaces," *ibid.*, Vol.3, No.1, pp.152-180, 1965.
- [Bar.66] Bartošek, V., and R. Zezula, "Flat Flux in a Slab Reactor with Natural Uranium," *J. of Nuclear Energy, Parts A/B*, Vol.20, pp.129-134, 1966.
- [Be.57] Bellman, R., *Dynamic Programming*, Princeton Univ. Press, Princeton, N.J., 1957.
- [ElB.74] El-Bassioni A. A., and C. G. Poncelet, "Minimal Time Control of Spatial Xenon Oscillations in Nuclear Power Reactors," *Nucl. Sci. Eng.*, Vol.54, pp.164-176, 1974.
- [Bu.69] Butkovskiy, A. G., *Distributed Control Systems*, Am. Elsevier Pub. Co., New York, 1969.
- [Chr.73] Christie, A. M., and C. G. Poncelet, "On the Control of

- Spacial Xenon Oscillations," *Nucl. Sci. Eng.* Vol.33, pp.162-168, 1973.
- [Fa.69] Fagan, J. R., and A. Sesonske, "Optimal Fuel Replacement in Reactivity Limited System, *J. Nuclear Energy*, Vol.23, pp.683-696, 1969.
- [Go.70] Goldschmidt, P., and J. Quenon, "Minimum Critical Mass in Fast Reactors with Bounded Power Density," *Nucl. Sci. Eng.* Vol.39, pp.311-319, 1970.
- [Goe.56] Goertzel, G., "Minimal Critical Mass and Flat Flux," *J. Nuclear Energy*, Vol.2, pp.193-201, 1956.
- [Ha.68] Hara, F., and H. Shibata, "A Study of Core Optimization from the View Point of Neutron Flux Distribution," *J. Nucl. Sci. Tech.*, Vol.5, No.7, pp.322-332, 1968.
- [Hag.65] Hagan, J. I., and R. W. Goin, "Borax-V Neutronics," *USAEC Rept.*, Argonne National Laboratory, ANL-6964, 1965.
- [Hal.63] Haling, R. K., "Operating Strategy for Maintaining an Optimal Power Distribution Throughout Life," *Proc. Topical Meeting on Nuclear Performance Power Reactor Cores*, San Francisco, ANS, TID-7672, Sept., 1963.
- [Har.70] Harmuth, H., *Transmission of Information by Orthogonal Functions*, Springer-Verlag, Berlin/Heidelberg, 1970.
- [Ho.70] Hoshino, T., "Empirical Equation of Neutron Diffusion System, Application of the Experimental Design Technique," *J. Nucl. Sci. Tech.*, Vol.7, No.12, pp.610-614, 1970.
- [Ho.72] ———, "In-Core Fuel Management Optimization by Heuristic Learning Technique," *Nucl. Sci. Eng.* Vol.49, pp.59-71, 1972.
- [Hs.67] Hsu, C., "Control and Stability Analysis of Spatially Dependent Nuclear Reactor Systems," *USAEC Rept.*, Argonne National Laboratory, ANL-7322, 1967.
- [In.73] Inoue, K., "Fast Reactor Design Optimization System with a Method of Nonlinear Programming," *Nucl. Sci. Eng.*, Vol. 51, pp.1-9, 1973.
- [Iw.73] Iwazumi, T., and Ryuji Koga, "Optimal Feedback Control of

- a Nuclear Reactor as a Distributed Parameter System," *J. Nucl. Sci. Tech.*, Vol.10, No.11, pp.674-684, 1973.
- [Ka.61] Kaplan, S., "The Property of Finality and the Analysis of Problems in Reactor Space-Time Kinetics by Various Modal Expansions," *Nucl. Sci. Eng.*, Vol.9, pp.357-361, 1961.
- [Kw.71] Kawai, T., and T. Kiguchi, "Optimal Refueling Order for Minimum Local Power Peaking Factor," *ibid.*, Vol.43, pp. 342-345, 1971.
- [Ko.71] Koga, R., Y. Hattori, and T. Iwazumi, "Optimal Control of Steady State Neutron Flux Distribution in Slab Reactors," *J. Nucl. Sci. Tech.*, Vol.8, No.5, pp.267-276, 1971.
- [Koc.66] Kochurov, B. P., "Minimum Critical Mass at Limited Uranium Mass Concentration," *Sov. Atom. Energ.*, Vol.20, pp.267-276, 1966.
- [Koc.67] Kochurov, B. P., and A. P. Rudik, "The Problem of Maximum Reactor Power," *ibid.*, Vol.22, pp.44, 1967.
- [Ku.69] Kuroda, Y., and A. Makino, "Optimal Control of a Nuclear Reactor Systems with Distributed Parameters (I)," *Proceedings of the Faculty of Engineering, Tokai University*, No. 1, 1969.
- [Ku.71] ———, and ———, "Optimal Control of Nuclear Reactor Systems with Distributed Parameters," *IFAC Symp. on the Control of Distributed Parameter Systems, Banff, Canada*, 1971.
- [Ky.68] Kyong, S. H., "An Optimal Control of a Distributed Parameter Reactor," *Nucl. Sci. Eng.*, Vol.33, pp.146-149, 1968.
- [Le.65] Lelek, V., "Flattening of Flux in a Heterogeneous Slab Reactor," *IV. Reactor Conference, Budapest*, 1965.
- [Li.71] Lion, J. L., *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, Berlin/Heidelberg, 1971.
- [Me.69] Melice, M., "Pressurized Water Reactor Optimal Core Management and Reactivity Profiles," *Nucl. Sci. Eng.*, Vol.37,

pp.451-477, 1969.

- [Mh.70] Mohler, R. R., and C. N. Shen, *Optimal Control of Nuclear Reactors*, Academic press, New York and London, 1970.
- [Mo.70] Motoda, H., and T. Kawai, "A Theory of Control-Rod Programming Optimization in Two Region Reactors," *Nucl. Sci. Eng.*, Vol.39, pp.114-118, 1970.
- [Mo.70a] Motoda, H., "Burnup Optimization of Continuous Scattered Refueling," *ibid.*, Vol.41, pp.1-13, 1970.
- [Mo.72] ———, "Optimization of Control Rod Programming and Loading Pattern in Multiregion Reactor by the Method of Approximation Programming," *ibid.*, Vol.49, pp.515-524, 1972.
- [Na.72] Naft, B. N., and A. Sesonske, "Pressurized Water Reactor Optimal Fuel Management," *Nucl. Technol.*, Vol.14, pp.123-132, 1972.
- [Pnt.62] Pontryagin *et al.*, *The Mathematical Theory of Optimal Processes*, Interscience Publishers, John Wiley and Sons Inc., New York and London, 1962.
- [Ra.60] Ravets, J. M., and J. R. Lamarsh, "Flat Flux by Nonuniform Moderation," *Nucl. Sci. Eng.*, Vol.7, pp.496-501, 1960.
- [Red.72] Reid, W. T., *Riccati Differential Equations*, Academic Press, New York and London, 1972.
- [Ri.55] Riesz, F., and B. -Nagy, *Functional Analysis*, Frederick Ungar Publishing Co., New York, 1955.
- [Sau.71] Sauer, T. O., "Application of Linear Programming to In-Core Fuel Management Optimization in Light Water Reactors," *Nucl. Sci. Eng.*, Vol.46, pp.274-283, 1971.
- [Se.72] Sekimizu, K., *et al.*, "The Study on the Control of Neutron Flux Distribution in Nuclear Reactors," *Trans. Soc. Instrument and Control Eng.*, Vol.8, No.3, pp.321-330, 1972.
(Japanese)
- [Se.74] Sekimizu, K., and K. Monta, "Optimization of Control-Rod Programming in Multiregion Nuclear Reactors," *Nucl. Sci. Eng.*, Vol.54, pp.474-481, 1974.

- [St.67] Stacey, Jr., W. M., *Modal Approximations, Theory and an Application to Reactor Physics*, Research Monograph No.41, The M.I.T. Press, Cambridge, Mass., 1967.
- [St.68] ———, "Optimal Control of Xenon-Power Spatial Transients," *Nucl. Sci. Eng.*, Vol.33, pp.162-168, 1968.
- [St.69] ———, "Control of Xenon Spatial Oscillations," *ibid.*, Vol.33, pp.229-243, 1969.
- [St.69a] ———, *Space-Time Nuclear Reactor Kinetics*, Academic Press, New York and London, 1969.
- [St.70] ———, "Xenon-Induced Spatial Oscillations," *Reactor Technology*, Vol.13, No.3, pp.252-279.
- [St.70a] ———, "Application of Variational Synthesis to the Optimal Control of Spatially Dependent Reactor Models," *Nucl. Sci. Eng.*, Vol.39, pp.226-230, 1970.
- [Str.70] Strugar, P. V., "Maximum Neutron Flux in Thermal Research Reactors," *J. of Optimization Theory and Applications*, Vol.5, No.4, pp.301-312, 1970.
- [Sv.68] Stover, R. L., "Optimized Scatter-Loading of a Nuclear Reactor," Ph.D. Thesis, Purdue Univ., Aug., 1968.
- [Sv.69] Stover, R. L., and A. Sesonske, "Optimization of BWR Fuel Management using an Accelerated Exhaustive Search Algorithm," *J. Nuclear Energy*, Vol.23, pp.673-682, 1969.
- [Su.68] Suda, N., "On Controllability of Neutron Flux Distribution," *J. Nucl. Sci. Tech.*, Vol.5, pp.271-278, 1968.
- [Su.68a] Suda, N., *et al.*, "Optimization of Neutron Flux Distribution *ibid.*, Vol.5, No.9, pp.452-457, 1968.
- [Suz.71] Suzuki, A., and R. Kiyose, "Maximizing the Average Fuel Burnup Over the Entire Core: A Poison Management Optimization Problem for Multizone Light Water Reactor Cores," *Nucl. Sci. Eng.*, Vol.44, pp.121-134, 1971.
- [Suz.71a] ———, and ———, "Application of Linear Programming to Refueling Optimization for Light Water Moderated Power Reactors," *ibid.*, Vol.46, pp.112-130, 1971.

- [Tab.68] Tabak, D., "Optimization of Nuclear Reactor Fuel Recycle Via Linear and Quadratic Programming," *IEEE Tras. on Nuclear Science*, Vol.15, 1968.
- [Te.70] Terney, W. B., and H. Fench, "Control Rod Programming Optimization Using Dynamic Programming," *Nucl. Sci. Eng.*, Vol.39, pp.109-114, 1970.
- [Te.71] Terney, W. B., "Analytic Solution to the Flat Flux Problem," *ibid.*, Vol.45, pp.225-230, 1971.
- [Th.64] Thompson, T. J., and J. G. Beckerly (ed.), *The Technology of Nuclear Reactor Safety*, pp.48-52, M.I.T. Press, Mass., 1964.
- [Tr.57] Tricomi, F. G., *Integral Equations*, Interscience Pub., Inc., John Wiley & Sons, Inc., New York, 1957.
- [Ts.71] Tsai, T. Y., and L. L. Grigsby, "An Approach to Spacial Power Regulation in Nuclear Reactors," *Proceedings of 1971 JACC*, pp.255-260, 1971.
- [Var.62] Varga, R. S., *Matrix Iterative Analysis*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1962.
- [W.58] Weinberg, A. M., and E. P. Wigner, *The Physical Theory of Neutron Chain Reactors*. The Univ. of Chicago Press, Chicago, 1958.
- [Wa.66] Wang, P. K. C., "Control of Distributed Parameter Systems," *Adv. in Control Systems, Theory and Applications*, Vol.1, pp.75-172, C. T. Leondes (ed.), Academic Press, New York, 1966.
- [Wad.71] Wade, D. C., and W. B. Terney, "Optimal Control of Nuclear Reactor Depletion," *Nucl. Sci. Eng.*, Vol.45, pp.199-217, 1971.
- [Wal.65] Wall, Ian, and H. Fenech, "The Application of Dynamic Programming to Fuel Management Optimization," *ibid.*, Vol.22, pp.285-297, 1965.
- [We.67] Weaver, L. E., and R. E. Vanasse, "State Variable Feedback Control of Multiregion Reactors," *Nucl. Sci. Eng.*, Vol.29 pp.264-271, 1967.

- [We.68] Weaver, L. E., *Reactor Dynamics and Control*, Am. Elsevier Pub. Co., Inc., New York, 1968.
- [Wib.70] Wiberg, D. M., "Optimal Control of Nuclear Reactor Systems," *Adv. in Control Systems, Theory and Applications*, Vol.5, pp.301-388, C. T. Leondes (ed.), 1967.
- [Wit.70] Wittler, M., and C. N. Shen, "Time-Optimal Nuclear Rocket Startup with Thermal Stress Constraints based on Distributed Parameter Model," *Automatica*, Vol.6, pp.221-229, Pergamon Press, 1970.
- [Ya.67] Yasukawa, S., "An Analysis of Continuous Bi-Directional Reactor Refueling," *J. Nucl. Sci. Tech.*, Vol.4, pp.367-371, 1967.
- [Ya.71] Yasukawa, S., *et al.*, "Effect of Plutonium Recycled Reactors on Long-Term Fuel Cycle in Japan," *Rept. Japan Atomic Research Institute*, JAERI-1212, 1971.
- [Za.66] Zaritskaya, T. S., and A. P. Rudik, "Using L. S. Pontryagin's Maximum-Principle in Minimum Critical-Size and Maximum-Power Reactor Problems," *Sov. Atom. Energ.*, Vol.22, pp.5-9, 1966.
- [Za.67] ———, and ———, "Maximum Power from a Gas Cooled Reactor," *ibid.*, Vol.23, pp.940-944, 1967.
- [Za.74] ———, and ———, "Optimal Equalizing of the Heat Release over the Reactor Volume," *ibid.*, Vol.35, pp.843-846, 1974.

Appendix A. Transition Matrices of Eq.(2.3.22) with Constant u

The transition matrices are given as

$$\begin{pmatrix} \cosh \lambda t & , & \frac{1}{\lambda} \sinh \lambda t & , & 0 & , & 0 \\ \lambda \cdot \sinh \lambda t & , & \cosh \lambda t & , & 0 & , & 0 \\ \frac{1}{2}(t \cdot \cosh \lambda t + \frac{1}{\lambda} \sinh \lambda t), & -\frac{\lambda}{2} t \sinh \lambda t, & \cosh \lambda t & , & -\lambda \sinh \lambda t \\ -\frac{1}{2\lambda} t \cdot \sinh \lambda t, & \frac{1}{2\lambda^2}(\frac{1}{\lambda} \sinh \lambda t - t \cdot \cosh \lambda t), & -\frac{1}{\lambda} \sinh \lambda t, & \cosh \lambda t \end{pmatrix} \quad (\text{A.1})$$

when $u = u_{\max} > 1$, $\lambda = B\sqrt{u_{\max}-1}$, and

$$\begin{pmatrix} \cos \omega t & , & \frac{1}{\omega} \sin \omega t & , & 0 & , & 0 \\ -\omega \cdot \sin \omega t & , & \cos \omega t & , & 0 & , & 0 \\ \frac{1}{2}(t \cdot \cos \omega t + \frac{1}{\omega} \sin \omega t), & \frac{\omega t}{2} \sin \omega t, & \cos \omega t & , & \omega \sin \omega t \\ -\frac{1}{2\omega} t \cdot \sin \omega t, & \frac{1}{2\omega^2}(t \cdot \cos \omega t - \frac{1}{\omega} \sin \omega t), & -\frac{1}{\omega} \sin \omega t, & \cos \omega t \end{pmatrix} \quad (\text{A.2})$$

when $u = u_{\min} < 1$, $\omega = B\sqrt{1-u_{\min}}$.

Appendix B. A Model for a Single Channel Thermo-hydrodynamics

The argument are made on an assembly illustrated in Fig.B.

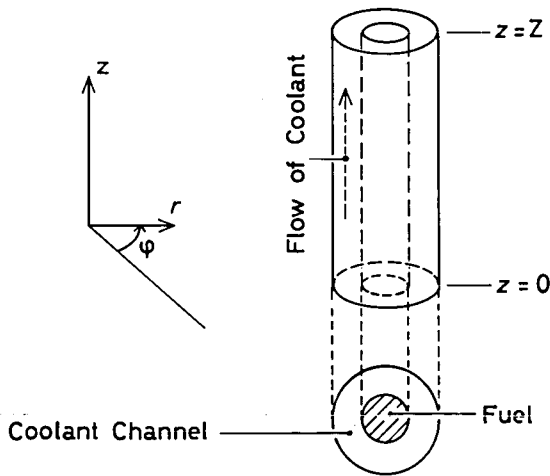


Fig.B Single fuel assembly.

The assembly is composed of the fuel pins and the coolant channels, and is installed in the reactor core at (r, φ) parallel to the axial z direction. Assuming the dynamics whose rate of change is within the range of controlling action, the thermo-hydrodynamics is written as

$$\rho c A \left\{ \frac{\partial}{\partial t} \theta_c(z, t) + V \frac{\partial}{\partial z} \theta_c(z, t) \right\} = q(z, t), \quad (B.1)$$

where the nomenclatures are given as

$\theta_c(z, t)$: fractional temperature of the coolant,

ρ : density of coolant,

C : specific heat of coolant,

A : cross sectional area of the coolant channel,
 V : speed of coolant flow,
 $q(z,t)$: total linear heat release from the fuel pins
 belonging to the assembly.

Assuming that the temperature-increase profile of the coolant along the longitudinal direction is invariant at $h(z)$ as

$$\theta_c(z,t) = \theta(t)h(z) + \theta_i, \quad (B.2)$$

where θ_i stands for the inlet temperature. Substitution of Eq. (B.2) into Eq. (B.1) yields

$$h(z)\dot{\theta}(t) + V\frac{\partial}{\partial z}h(z)\theta(t) = \frac{1}{\rho c A}q(z,t). \quad (B.3)$$

Also assuming that the thermal flux distribution along the axial direction at an assembly can be factorized by $\sin \frac{\pi z}{Z}$, the heat release density $q(z,t)$ is expressed as

$$q(z,t) = A_f E \Sigma_f \phi \frac{Z}{\pi} \sin \frac{\pi z}{Z}, \quad (B.4)$$

where the nomenclatures are given as

A_f : total cross sectional area of fuel pins,
 E : energy released per fission,
 Σ_f : macroscopic fission cross section,
 ϕ_2 : thermal flux density averaged over the full length
 of the assembly.

Substituting Eq. (B.4) into Eq. (B.3) and integrating both sides with weight $h^*(z)$ over the full length of the assembly, the system with respect to the representative coolant temperature θ is obtained as

$$\dot{\theta} = -\gamma\theta + \kappa \Sigma_f \phi_2, \quad (B.5)$$

where

$$\gamma = V \frac{\int_0^z h^*(z) \frac{\partial}{\partial z} h(z) dz}{\int_0^z h^*(z) h(z) dz} \quad (\text{B.6})$$

and

$$\kappa = \frac{2}{\pi} \frac{EA_f}{\rho c A} \frac{\int_0^z h^*(z) \sin \frac{\pi z}{Z} dz}{\int_0^z h^*(z) h(z) dz}, \quad (\text{B.7})$$

which have been required.

Appendix C. Walsh Functions [Har.70]

Walsh functions $Wal(i, j)$, $i=0,1,2,\dots$ are defined on the closed interval $-1/2 \leq \theta \leq 1/2$ and jump back and forth between +1 and -1 as are illustrated in Fig.C for the orders up to 15th. The orthogonality

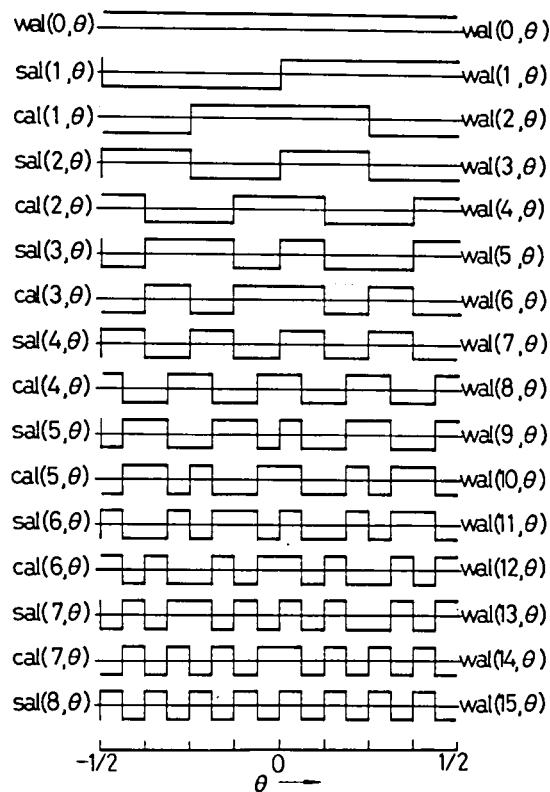


Fig.C Walsh functions.

relation holds as

$$\int_{-1/2}^{1/2} \text{Wal}(i, \theta) \text{Wal}(j, \theta) d\theta = \delta_{ij} \text{ (Kronecker)}. \quad (\text{C.1})$$

A distinct feature is that Walsh functions form a group with respect to multiplication and the group is isomorphic to a discrete dyadic group.

The multiplication law for two Walsh functions is given by

$$\text{Wal}(i, \theta) \text{Wal}(j, \theta) = \text{Wal}(i \oplus j, \theta), \quad (\text{C.2})$$

where the sign $\oplus$ stands for an operation between i and j briefed below.

Let i, j be nonnegative integers up to $2^{N-1}-1$ and be expressed in binary numbers as

$$i = k_{N-2} 2^{N-2} + k_{N-3} 2^{N-3} + \dots + k_1 2^1 + k_0 2^0 \leq 2^{N-1} - 1 \quad (\text{C.3})$$

and

$$j = l_{N-2} 2^{N-2} + l_{N-3} 2^{N-3} + \dots + l_1 2^1 + l_0 2^0 \leq 2^{N-1} - 1, \quad (\text{C.4})$$

where

$$k_0, k_1, \dots, k_{N-2}, l_0, l_1, \dots, l_{N-2} = 0 \text{ or } 1. \quad (\text{C.5})$$

The operation $\oplus$ is carried out according to the rules

$$1 \oplus 0 = 0 \oplus 1 = 1, \quad (\text{C.6})$$

$$0 \oplus 0 = 1 \oplus 1 = 0 \text{ (no carry)}. \quad (\text{C.7})$$

The addition is made for i and j as

$$i \oplus j = (k_{N-2} \oplus l_{N-2}) 2^{N-2} + \dots + (k_1 \oplus l_1) 2^1 + (k_0 \oplus l_0) 2^0. \quad (\text{C.8})$$

The minimum and the maximum numbers are seen from Eq.(C8), to be $2^{N-2} + \dots + 2 + 1 = 2^{N-1}$ and 0 respectively. From this fact the non-negative integers below 2^{N-1} are known to form a finite group with

respect to the operation $\oplus$. Table C shows the value of $i \oplus j$. In the table, the enclosing frames respectively stand for the subgroups.

$i \backslash j$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14
2	2	3	0	1	6	7	4	5	10	11	8	9	14	15	12	13
3	3	2	1	0	7	6	5	4	11	10	9	8	15	14	13	12
4	4	5	6	7	0	1	2	3	12	13	14	15	8	9	10	11
5	5	4	7	6	1	0	3	2	13	12	15	14	9	8	11	10
6	6	7	4	5	2	3	0	1	14	15	12	13	10	11	8	9
7	7	6	5	4	3	2	1	0	15	14	13	12	11	10	9	8
8	8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
9	9	8	11	10	13	12	15	14	1	0	3	2	5	4	7	6
10	10	11	8	9	14	15	12	13	2	3	0	1	6	7	4	5
11	11	10	9	8	15	14	13	12	3	2	1	0	7	6	5	4
12	12	13	14	15	8	9	10	11	4	5	6	7	0	1	2	3
13	13	12	15	14	9	8	11	10	5	4	7	6	1	0	3	2
14	14	15	12	13	10	11	8	9	6	7	4	5	2	3	0	1
15	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0

Table C Values of $i \oplus j$

Appendix D. The Terminal Manifold

The terminal manifold is composed of (Δ_0, Δ_1) which gives only one admissible power shape.

From Fig.4.4.2 the very conditions that determine the terminal power shape, are known. The analysis is made according to the numbers cited in the text.

1. In the inner region, the control rods are fully withdrawn and the power density arrives at the limit satisfying the criticality condition. Referring Eqs.(4.4.20), (4.4.16) and (4.4.22), this case is respectively expressed as

$$(g_{11} + g_{10})(k_0 - p_0) - (g_{00} + g_{01})(k_1 - p_1) = 0, \quad (D.1)$$

$$\Delta_1 p_0 + \Delta_0 p_1 = F - 1, \quad (D.2)$$

$$\Delta_0 p_0 + \Delta_1 p_1 = 1. \quad (D.3)$$

Regarding the set of Eqs.(D.1-3) as the linear simultaneous equations with respect to p_0 and p_1 , the necessary condition that an unique solution exist is that

$$\det \begin{vmatrix} g_{11} + g_{10} & -g_{00} - g_{01} & -(g_{11} + g_{10})k_0 + (g_{00} + g_{01})k_1 \\ \Delta_1 & \Delta_0 & 1 - F \\ \Delta_0 & \Delta_1 & -1 \end{vmatrix} = 0 \quad (D.4)$$

should hold. The result is reduced to

$$(\Delta_0 - U_1)^2 - (\Delta_1 - V_1)^2 = U_1^2 - V_1^2, \quad (\text{D.5})$$

where

$$U_1 = \frac{1}{2} \frac{(g_{11} + g_{10}) - (g_{00} + g_{01})(F-1)}{(g_{11} + g_{10})k_0 - (g_{00} + g_{11})k_1} \quad (\text{D.6})$$

$$V_1 = \frac{1}{2} \frac{(g_{11} + g_{10})(F-1) - (g_{00} + g_{01})}{(g_{11} + g_{10})k_0 - (g_{00} + g_{11})k_1}. \quad (\text{D.7})$$

As is trivial from the assumption, the value of f at the terminal point is given as

$$f = F - 1. \quad (\text{D.8})$$

2. All rods are fully withdraw in both regions. Then from Eqs. (4.4.20), (4.4.21) and the criticality condition (4.4.22), are obtained

$$(g_{11} + g_{10})(k_0 - p_0) - (g_{00} + g_{01})(k_1 - p_1) = 0 \quad (\text{D.9})$$

$$(g_{11} - g_{10})(k_0 - p_0) - (g_{00} - g_{01})(k_1 - p_1) = 0 \quad (\text{D.10})$$

and

$$\Delta_0 p_0 + \Delta_1 p_1 = 1 \quad (\text{D.11})$$

Assuming that

$$(g_{11} + g_{10})(g_{00} - g_{01}) - (g_{11} - g_{10})(g_{00} + g_{01}) = -2(g_{11}g_{01} - g_{10}g_{00}) \neq 0 \quad (\text{D.12})$$

holds, Eqs.(D.9) and (D.10) together give

$$(p_0, p_1) = (k_0, k_1) \quad (\text{D.13})$$

which corresponds to the point M in Fig.4.4.2. Substituting Eq. (D.13) into Eq.(D.3), the part of the terminal manifold is given by

$$\Delta_0 k_0 + \Delta_1 k_1 = 1. \quad (\text{D.14})$$

The corresponding value of f is given by

$$f^T = k_0 \lambda_1 + k_1 \lambda_0 . \quad (D.15)$$

3. In the outer region, the control rods are fully withdrawn and the power density arrives at the limit satisfying the criticality condition. From Eqs.(4.4.21), (4.4.16) and (4.4.22), we obtain

$$(g_{11}-g_{10})(k_0-p_0) - (g_{00}-g_{01})(k_1-p_1) = 0 \quad (D.16)$$

$$\lambda_1 p_0 + \lambda_0 p_1 = 1 - F \quad (D.17)$$

$$\lambda_0 p_0 + \lambda_1 p_1 = 1 . \quad (D.18)$$

The necessary condition for Eqs.(D16-18) to allow an unique solution (p_0, p_1) , is given by

$$(\lambda_0 - U_3)^2 - (\lambda_1 - V_3)^2 = U_3^2 - V_3^2 , \quad (D.19)$$

where

$$U_3 = \frac{1}{2} \frac{g_{11}-g_{10} + (g_{00}-g_{01})(F-1)}{(g_{11}-g_{10})k_0 - (g_{00}-g_{01})k_1} , \quad (D.20)$$

$$V_3 = -\frac{1}{2} \frac{(g_{11}-g_{10})(F-1) + g_{00}-g_{01}}{(g_{11}-g_{10})k_0 - (g_{00}-g_{01})k_1} . \quad (D.21)$$

The value of f is given as

$$f = 1 - F . \quad (D.22)$$