

Classification of Type II code over $\text{GF}(4)$ of some small lengths

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1 Introduction

Let $\text{GF}(2^r)$ be the finite field with 2^r elements. Let C be a code over $\text{GF}(2^r)$ of length n , which is a subspace of the vector space $\text{GF}(2^r)^n$. Let $B = \{b_1, b_2, \dots, b_r\}$ be a basis of $\text{GF}(2^r)$ over $\text{GF}(2)$. We denote by $\phi_B(a) = (a_1, a_2, \dots, a_r) \in \text{GF}(2)^r$ the representation of $a \in \text{GF}(2^r)$ over $\text{GF}(2)$ with respect to a basis B , that is, $a = \sum_{i=1}^r a_i b_i$. For $u = (u_1, u_2, \dots, u_n) \in \text{GF}(2^r)^n$, we also denote by $\phi_B(u)$ the vector in $\text{GF}(2)^{rn}$ obtained by concatenating $\phi_B(u_1), \dots, \phi_B(u_n)$. We call $\phi_B(C)$ the *binary image* of C with respect to B . B is called a *trace-orthogonal basis* (TOB) if $\text{Tr}(b_i b_j) = \delta_{ij}$ for $1 \leq i, j \leq r$ where Tr denotes the trace function of $\text{GF}(2^r)$ over $\text{GF}(2)$.

In 1980's, Pasquier and Wolfmann studied self-dual codes over $\text{GF}(2^r)$ whose binary images with respect to a TOB are binary Type II codes (that is, binary doubly-even self-dual codes) including the extended Hamming code and the extended Golay code (cf. [8, 10]). We say that such codes over $\text{GF}(2^r)$ are Type II codes with respect to a TOB [4] (see [6] for Type II codes over $\text{GF}(4)$). Recently, it has been proved by the author that the Type II property with respect to a TOB for self-dual codes over $\text{GF}(2^r)$ is independent of the choice of a TOB [1]. This allows us to call C a Type II code if C is a Type II code with respect to a TOB, without a reference to an explicit TOB. The *binary length* of a code over $\text{GF}(2^r)$ of length n is defined by rn , which is the length of its binary image. The Type II codes with binary length up to 24 have been classified (cf. [2, 4, 6]). We refer to [9] for the classification of binary Type II codes. The next problem would be to classify all Type II codes over $\text{GF}(2^r)$ for any r with binary length 32.

Theorem 1.1 (Munemasa [7]) *The total number of Type II $\text{GF}(2^r)$ -codes of length n is given by*

$$N_{II,r}(n) = \prod_{i=0}^{n/2-2} (2^{r^i} + 1), \quad (1)$$

if $rn \equiv 0 \pmod{8}$ and $n \equiv 0 \pmod{4}$, and 0 otherwise.

The formula (1) is called the mass formula for Type II codes over $\text{GF}(2^r)$. By Theorem 1.1, the possible cases for which there is a Type II code over $\text{GF}(2^r)$ of length n with binary length 32 are $(n, r) = (32, 1)$, $(16, 2)$, $(8, 4)$ and $(4, 8)$. For the cases $(n, r) = (32, 1)$ and $(n, r) = (8, 4)$, the complete classifications are given in [5] and [4], respectively. Furthermore, for the case $(n, r) = (4, 8)$, it is shown that there exists a unique Type II code up to permutation-equivalence [7].

In this paper, we give a classification for the case $(n, r) = (16, 2)$, that is, Type II codes over $\text{GF}(4)$ of length 16. It is the only remaining case to complete the classification of Type II codes with binary length 32.

2 Classification of Type II Codes

over $\text{GF}(4)$ of Length 16

In this section, we give a classification of Type II codes of length 16 over $\text{GF}(4) = \text{GF}(2)[\omega]/(\omega^2 + \omega + 1)$. A *bisorted* matrix is constructed by sorted vectors on a lexicographical order with both directions North to South and West to East. We check all the bisorted 8×8 $\text{GF}(4)$ -matrices A 's such that $(I \ A)$ generates a Type II code, where I is the 8×8 identity matrix. It is sufficient to consider such matrices to complete the classification [3]. Indeed, we obtain 82588 distinct Type II codes by the method above. Let C be a code over $\text{GF}(4)$ of length 16, and let

$$\tilde{C} = \{(\hat{c}_1, \hat{c}_2, \dots, \hat{c}_{16}) \mid \text{wt}(c) = 6, c = (c_1, c_2, \dots, c_{16}) \in C\},$$

where $\text{wt}(c)$ is the Hamming weight of c and $\hat{c}_i = 0$ if $c_i = 0$ and $\hat{c}_i = 1$ otherwise. Then $\tilde{C}$ is a non-linear binary code of length 16. We now consider three invariants of codes under the permutation-equivalence:

1. the Hamming weight enumerator $W(C)$,
2. the order $|\text{Aut}(C)|$ of the permutation automorphism group of C ,
3. the order $|\text{Aut}(\tilde{C})|$ of the permutation automorphism group of $\tilde{C}$.

Calculating the invariants for every code using MAGMA, we find 48 codes $D_1, \dots, D_{48}$ with distinct sets of invariants. The minimum Hamming weight $d(D_i)$ of D_i , the Hamming weight enumerator $W(D_i)$ of D_i and the values $|\text{Aut}(D_i)|$ are listed in Table 1. The weight enumerators are listed in Table 2 where only the coefficients of the monomials $x^{16-i}y^i$ for $i = 3, 4, 6, 7, \dots, 16$ are given. For all weight enumerators, the coefficients of the monomials x^{16} , $x^{15}y$, $x^{14}y^2$ and $x^{11}y^5$ are 1, 0, 0 and 0, respectively. We verified that

Table 1: Properties of the Type II codes over GF(4) of length 16

	$d(D_i)$	$W(D_i)$	$ \text{Aut}(D_i) $		$d(D_i)$	$W(D_i)$	$ \text{Aut}(D_i) $
D_1	3	W_1	497664	D_{25}	4	W_{24}	5160960
D_2	3	W_2	387072	D_{26}	4	W_{25}	2304
D_3	3	W_3	11664	D_{27}	4	W_{26}	73728
D_4	3	W_4	5184	D_{28}	4	W_{27}	6144
D_5	3	W_5	648	D_{29}	4	W_{28}	576
D_6	3	W_6	7776	D_{30}	4	W_{28}	6144
D_7	3	W_7	324	D_{31}	4	W_{28}	3072
D_8	3	W_8	13608	D_{32}	4	W_{29}	18432
D_9	3	W_9	576	D_{33}	4	W_{29}	18432
D_{10}	3	W_{10}	90	D_{34}	4	W_{30}	128
D_{11}	3	W_{11}	216	D_{35}	4	W_{30}	64
D_{12}	3	W_{12}	432	D_{36}	4	W_{31}	240
D_{13}	3	W_{13}	3456	D_{37}	4	W_{32}	24
D_{14}	3	W_{14}	34560	D_{38}	4	W_{32}	144
D_{15}	3	W_{15}	276480	D_{39}	4	W_{32}	48
D_{16}	3	W_{16}	13824	D_{40}	4	W_{33}	48
D_{17}	3	W_{17}	12096	D_{41}	4	W_{33}	96
D_{18}	3	W_{18}	288	D_{42}	6	W_{34}	96
D_{19}	3	W_{19}	216	D_{43}	6	W_{34}	384
D_{20}	3	W_{20}	36	D_{44}	6	W_{35}	16
D_{21}	3	W_{21}	192	D_{45}	6	W_{35}	336
D_{22}	3	W_{22}	7920	D_{46}	6	W_{35}	8
D_{23}	3	W_{23}	18	D_{47}	6	W_{35}	14
D_{24}	4	W_{24}	3612672	D_{48}	6	W_{35}	24

Table 2: The weight enumerators

i	3	4	6	7	8	9	10	11	12	13	14	15	16
W_1	48	12	864	432	54	6912	5184	1296	20844	20736	7776	1296	81
W_2	24	48	312	1080	306	4032	7056	4104	24840	12096	7992	3240	405
W_3	24	3	522	360	351	5952	6156	5544	18165	17856	7722	2520	360
W_4	18	12	384	522	414	5232	6624	6246	19164	15696	7776	3006	441
W_5	12	0	360	300	534	5376	6816	7428	17136	16128	7896	3012	537
W_6	12	3	378	252	603	5184	7164	6948	17757	15552	8298	2772	612
W_7	9	3	282	405	531	5112	6876	8019	17325	15336	7722	3375	540
W_8	9	21	150	837	405	4440	6804	8019	19731	13320	7254	4095	450
W_9	6	6	252	366	636	4752	7368	8034	17778	14256	8124	3306	651
W_{10}	6	0	264	318	606	5040	7104	8418	16800	15120	7896	3354	609
W_{11}	6	3	234	414	567	4944	7020	8514	17157	14832	7722	3546	576
W_{12}	6	9	222	462	597	4656	7284	8130	18135	13968	7950	3498	618
W_{13}	6	18	132	750	480	4368	7032	8418	19206	13104	7428	4074	519
W_{14}	6	30	204	558	756	3600	8424	6498	21690	10800	9036	3114	819
W_{15}	12	48	120	1116	450	3360	7632	6084	24168	10080	7992	3924	549
W_{16}	12	12	288	540	486	4896	6912	7236	18828	14688	7776	3348	513
W_{17}	3	30	156	567	792	3432	8568	6993	21522	10296	9036	3285	855
W_{18}	3	12	144	567	594	4392	7344	8721	18324	13176	7776	3861	621
W_{19}	3	3	234	279	711	4680	7596	8433	17253	14040	8298	3285	720
W_{20}	3	3	186	423	603	4776	7164	9009	16989	14328	7722	3717	612
W_{21}	3	6	204	375	672	4584	7512	8529	17610	13752	8124	3477	687
W_{22}	12	3	330	396	495	5280	6732	7524	17493	15840	7722	3204	504
W_{23}	3	0	216	327	642	4872	7248	8913	16632	14616	7896	3525	645
W_{24}	0	84	336	0	1854	0	14112	0	32004	0	15120	0	2025
W_{25}	0	18	228	192	984	3648	9048	7104	19926	10944	9732	2688	1023
W_{26}	0	36	48	768	750	3072	8544	7680	22068	9216	8688	3840	825
W_{27}	0	24	168	384	906	3456	8880	7296	20640	10368	9384	3072	957
W_{28}	0	12	96	576	630	4224	7488	9216	18156	12672	7776	4032	657
W_{29}	0	12	288	0	1062	3840	9216	6912	19212	11520	10080	2304	1089
W_{30}	0	6	156	384	708	4416	7656	9024	17442	13248	8124	3648	723
W_{31}	0	9	126	480	669	4320	7572	9120	17799	12960	7950	3840	690
W_{32}	0	3	138	432	639	4608	7308	9504	16821	13824	7722	3888	648
W_{33}	0	3	186	288	747	4512	7740	8928	17085	13536	8298	3456	756
W_{34}	0	0	216	192	786	4608	7824	8832	16728	13824	8472	3264	789
W_{35}	0	0	168	336	678	4704	7392	9408	16464	14112	7896	3696	681

$|\text{Aut}(\tilde{D}_{32})|$ and $|\text{Aut}(\tilde{D}_{33})|$ are 7962624 and 36864, respectively. By Table 1, $D_1, \dots, D_{48}$ are not permutation-equivalent. By Theorem 1.1, we have that

$$\begin{aligned} \sum_{i=1}^{48} \frac{16!}{|\text{Aut}(D_i)|} &= 11925737086250 \\ &= N_{II,2}(16). \end{aligned}$$

Thus, we obtain the following:

Theorem 2.1 *There exist 48 Type II codes over $\text{GF}(4)$ of length 16, up to permutation-equivalence.*

Generator matrices of the codes are given in Section 3. The Frobenius automorphism is the field automorphism on $\text{GF}(4)$ defined by $a \mapsto a^2$. Each of these 48 codes is permutation-equivalent to its Frobenius image. Finally, we calculate the binary images of the codes using MAGMA. In Table 3, the binary codes $\phi(C)$ are given and the minimum Hamming weights $d(\phi(C))$ of $\phi(C)$ are also given. We use the notation in [9] for the binary Type II codes of length 32. There exist 7 Type II codes whose binary images are of minimum Hamming weight 8 up to permutation-equivalence. Only 4 codes among the 5 extremal binary Type II codes of length 32 are the binary images of Type II codes over $\text{GF}(4)$.

As a consequence, we obtain the complete classification of Type II codes over $\text{GF}(2^r)$ with binary length 32. The numbers of Type II codes over $\text{GF}(2^r)$ with binary length 32 up to permutation-equivalence are in Table 4. In the table, #1, #2 and #3 denote the number of codes up to permutation-equivalence, the number of codes whose binary images are extremal, and the number of extremal binary codes obtained as the binary images, respectively.

3 Generator Matrices

In order to save space, we list generator matrices $(I \{a_{i,j}\})$ as

$$a_{1,1}a_{1,2} \cdots a_{1,8}, a_{2,1} \cdots a_{2,8}, \dots, a_{8,1} \cdots a_{8,8},$$

where $\bar{\omega}$ denotes ω^2 .

$$\begin{aligned} D_1: & 000000\omega\bar{\omega}, 000000\bar{\omega}\omega, 0000\omega\bar{\omega}00, 0000\bar{\omega}\omega00, 00\omega\bar{\omega}0000, 00\bar{\omega}\omega0000, \omega\bar{\omega}000000, \bar{\omega}\omega000000 \\ D_2: & 000000\omega\bar{\omega}, 000000\bar{\omega}\omega, 00011100, 00101100, 00110100, 00111000, \omega\bar{\omega}000000, \bar{\omega}\omega000000 \\ D_3: & 000000\omega\bar{\omega}, 000000\bar{\omega}\omega, 0000\omega\bar{\omega}00, 00\omega\bar{\omega}0000, 0\omega\bar{\omega}\omega1\bar{\omega}00, 0\bar{\omega}\omega1\bar{\omega}\omega00, \omega01\bar{\omega}\bar{\omega}\omega00, \bar{\omega}0\bar{\omega}\omega\omega100 \\ D_4: & 000000\omega\bar{\omega}, 000000\bar{\omega}\omega, 00011100, 00101100, 0\omega\bar{\omega}\bar{\omega}\omega100, 0\bar{\omega}\omega\omega1\bar{\omega}00, \omega0\bar{\omega}\bar{\omega}1\omega00, \bar{\omega}0\omega\omega\bar{\omega}100 \\ D_5: & 000000\omega\bar{\omega}, 0000\omega\bar{\omega}00, 000\omega\bar{\omega}\omega1\bar{\omega}, 00\omega01\bar{\omega}\bar{\omega}\omega, 0\omega\bar{\omega}100\bar{\omega}\omega, 0\bar{\omega}\omega\bar{\omega}00\omega1, \omega01\bar{\omega}\bar{\omega}\omega00, \bar{\omega}0\bar{\omega}\omega\omega100 \end{aligned}$$

Table 3: Binary images of the Type II code over GF(4) of length 16

$\phi(D_i)$	$d(\phi(D_i))$	Codes over GF(4)	$\phi(D_i)$	$d(\phi(D_i))$	Codes over GF(4)
C5	4	D_{25}	C59	4	D_{11}, D_{29}
C10	4	D_{15}	C60	4	D_{10}, D_{36}
C17	4	D_{14}	C62	4	D_{21}
C24	4	D_1, D_2, D_{24}	C64	4	D_{19}, D_{20}
C26	4	D_3, D_4	C66	4	D_{23}
C27	4	D_{16}	C67	4	D_{30}, D_{32}, D_{33}
C28	4	D_{22}	C69	4	D_{31}
C30	4	D_{27}	C75	4	D_{34}, D_{35}
C33	4	D_{13}	C77	4	D_{41}
C34	4	D_{28}	C78	4	$D_{37}, D_{38}, D_{39}, D_{40}$
C45	4	D_8, D_{17}	C81	8	D_{46}
C51	4	D_6	C83	8	D_{43}, D_{45}
C53	4	D_5, D_{12}, D_{26}	C84	8	D_{47}
C55	4	D_7, D_{18}	C85	8	D_{42}, D_{44}, D_{48}
C57	4	D_9			

Table 4: The number of Type II codes with binary length 32

r	#1	#2	#3	reference
1	85	5	5	[5]
2	48	7	4	Section 2
4	6	1	1	[4]
8	1	0	0	[7]

D_6 : 000000 $\bar{x}$, 0000 $\bar{x}$ 00, 0001 $\bar{x}$ $\bar{x}$ $\bar{x}$, 0010 $\bar{x}$ $\bar{x}$ $\bar{x}$, 0 $\bar{x}$ 000 $\bar{x}$ 1, 0 $\bar{x}$ $\bar{x}$ $\bar{x}$ 001 $\bar{x}$, ω 0 $\bar{x}$ $\bar{x}$ $\bar{x}$ 100, $\bar{\omega}$ 0 $\bar{x}$ $\bar{x}$ 1 $\bar{\omega}$ 00
 D_7 : 000000 $\bar{x}$, 0000 $\bar{x}$ 00, 000 $\bar{x}$ $\bar{x}$ 1 $\bar{x}$, 0 $\bar{x}$ $\bar{x}$ 1 ω 1 ω 1, 0 $\bar{x}$ 0100 $\bar{x}$, 0 $\bar{x}$ $\bar{x}$ 100 $\bar{x}$, ω 1111 $\bar{\omega}$ 00, $\bar{\omega}$ $\bar{x}$ $\bar{x}$ $\bar{x}$ $\bar{x}$ 00
 D_8 : 000000 $\bar{x}$, 0000 $\bar{x}$ 00, 0011 ω 11 $\bar{x}$, 0101 ω 11 $\bar{x}$, 0110 ω 11 $\bar{x}$, 01110000, ω 000 $\bar{x}$ 1 $\bar{x}$, $\bar{\omega}$ 000 ω 1 $\bar{x}$ $\bar{x}$
 D_9 : 000000 $\bar{x}$, 000 $\bar{x}$ $\bar{x}$ 1 $\bar{x}$, 000 $\bar{x}$ $\bar{x}$ 1 $\bar{x}$, 0 $\bar{x}$ $\bar{x}$ 0 $\bar{x}$ $\bar{x}$ $\bar{x}$, 0 $\bar{x}$ $\bar{x}$ $\bar{x}$ 001 $\bar{x}$, 0 $\bar{x}$ $\bar{x}$ $\bar{x}$ 001 $\bar{x}$, ω 11 $\bar{\omega}$ 1100, $\bar{\omega}$ $\bar{x}$ $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 00
 D_{10} : 000000 $\bar{x}$, 000 $\bar{x}$ $\bar{x}$ 1 $\bar{x}$, 00 ω 0 $\bar{\omega}$ 1 $\bar{x}$ $\bar{x}$, 0 ω 0 $\bar{\omega}$ 01 $\bar{x}$ $\bar{x}$, 0 $\bar{x}$ $\bar{\omega}$ 0 $\bar{\omega}$ 0 ω 1, 0 $\bar{\omega}$ 1101 ω 1, ω 1 $\bar{\omega}$ $\bar{x}$ $\bar{x}$ 1 $\bar{x}$, $\bar{\omega}$ $\bar{x}$ $\bar{x}$ $\bar{x}$ 11 $\bar{x}$ $\bar{x}$
 D_{11} : 000000 $\bar{x}$, 000 $\bar{x}$ $\bar{x}$ 1 $\bar{x}$, 00 ω 1111 $\bar{x}$, 0 ω 1 $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 00, 0 ω 1 $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 00, 0 $\bar{\omega}$ 1 $\bar{\omega}$ $\bar{x}$ 11 $\bar{x}$, ω 110011 $\bar{x}$, $\bar{\omega}$ $\bar{x}$ $\bar{\omega}$ 00 $\bar{x}$ $\bar{x}$
 D_{12} : 000000 $\bar{x}$, 000 $\bar{x}$ $\bar{x}$ 1 $\bar{x}$, 00111000, 0 ω 1 $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 00, 0 ω 1 $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 00, 0 $\bar{\omega}$ 0 $\bar{\omega}$ $\bar{x}$ $\bar{x}$ $\bar{\omega}$ $\bar{x}$, ω 1000 $\bar{x}$ $\bar{x}$, $\bar{\omega}$ $\bar{x}$ 000 $\bar{x}$ $\bar{x}$ 1
 D_{13} : 000000 $\bar{x}$, 000 $\bar{x}$ $\bar{x}$ 1 $\bar{x}$, 00111000, 01011000, 01101 ω 1 $\bar{x}$, 01110 ω 1 $\bar{x}$, ω 001111 $\bar{x}$, $\bar{\omega}$ 00 $\bar{x}$ $\bar{x}$ $\bar{x}$ $\bar{x}$
 D_{14} : 000000 $\bar{x}$, 00011100, 00101100, 01001100, 011101 $\bar{x}$ $\bar{x}$, 011110 $\bar{x}$ $\bar{x}$, ω 000 $\bar{x}$ $\bar{x}$ 1, $\bar{\omega}$ 000 $\bar{x}$ $\bar{x}$ 1 $\bar{x}$
 D_{15} : 000000 $\bar{x}$, 000000 $\bar{x}$, 00011100, 00101100, 01001100, 10001100, 1111 $\bar{x}$ $\bar{\omega}$ 00, 1111 $\bar{x}$ $\bar{\omega}$ 00
 D_{16} : 000000 $\bar{x}$, 000000 $\bar{x}$, 00011100, 0 $\bar{x}$ $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 100, 0 $\bar{x}$ $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 100, 1 $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 0 $\bar{\omega}$ 00, 1 $\bar{\omega}$ $\bar{x}$ 0 $\bar{x}$ $\bar{x}$ 00, 111 $\bar{x}$ $\bar{\omega}$ 100
 D_{17} : 000000 $\bar{x}$, 000 $\bar{x}$ $\bar{x}$ 1 $\bar{x}$, 000 $\bar{x}$ $\bar{x}$ 1 $\bar{x}$, 000 $\bar{x}$ $\bar{x}$ 1 $\bar{x}$, 011 $\bar{x}$ $\bar{x}$ $\bar{x}$ 1, 101 $\bar{x}$ $\bar{x}$ $\bar{x}$ 1, 110 $\bar{x}$ $\bar{x}$ $\bar{x}$ 1, 11100000
 D_{18} : 000000 $\bar{x}$, 000 $\bar{x}$ $\bar{x}$ 1 $\bar{x}$, 000 $\bar{x}$ $\bar{x}$ 1 $\bar{x}$, 00 ω 1111 $\bar{x}$, 01 $\bar{\omega}$ 011 ω 1, 10 $\bar{\omega}$ 011 ω 1, 11 $\bar{x}$ $\bar{\omega}$ 1100, 111 $\bar{\omega}$ 00 ω 1
 D_{19} : 000000 $\bar{x}$, 000 $\bar{x}$ $\bar{x}$ 1 $\bar{x}$, 00 ω 0 $\bar{\omega}$ 1 $\bar{x}$ $\bar{x}$, 0 ω 0 $\bar{\omega}$ 01 $\bar{x}$ $\bar{x}$, 0 $\bar{\omega}$ $\bar{x}$ 00 $\bar{x}$ $\bar{x}$ 1, 1 ω 1 $\bar{\omega}$ 1100, 1 $\bar{\omega}$ $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 00, 11 ω 1 $\bar{\omega}$ 100
 D_{20} : 000000 $\bar{x}$, 000 $\bar{x}$ $\bar{x}$ 1 $\bar{x}$, 00 ω 0 $\bar{\omega}$ 1 $\bar{x}$ $\bar{x}$, 0 ω 01111 $\bar{x}$, 0 $\bar{\omega}$ $\bar{x}$ 1 ω 00, 1 ω 1 $\bar{\omega}$ 1100, 1 $\bar{\omega}$ $\bar{x}$ $\bar{\omega}$ 101 $\bar{x}$, 11 ω 0011 $\bar{x}$
 D_{21} : 000000 $\bar{x}$, 000 $\bar{x}$ $\bar{x}$ 1 $\bar{x}$, 00111000, 0 $\bar{x}$ $\bar{x}$ 0 $\bar{x}$ $\bar{x}$ $\bar{x}$, 0 $\bar{x}$ $\bar{x}$ $\bar{x}$ 001 $\bar{x}$, 1 $\bar{x}$ $\bar{\omega}$ 0 $\bar{x}$ $\bar{x}$ 00, 1 $\bar{\omega}$ 0 $\bar{\omega}$ $\bar{x}$ 11 $\bar{x}$, 11 $\bar{\omega}$ $\bar{x}$ 0 $\bar{\omega}$ 1 $\bar{x}$
 D_{22} : 000000 $\bar{x}$, 000000 $\bar{x}$, 0 $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 100, ω 0 $\bar{x}$ $\bar{x}$ 1 $\bar{\omega}$ 00, $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 0 ω 100, $\bar{x}$ $\bar{x}$ 0 $\bar{\omega}$ 1 ω 00, $\bar{\omega}$ 1 ω 11100, 1 $\bar{\omega}$ 1 ω 1100
 D_{23} : 000000 $\bar{x}$, 000 $\bar{x}$ $\bar{x}$ 1 $\bar{x}$, 0 $\bar{x}$ $\bar{x}$ 0 $\bar{x}$ $\bar{x}$ $\bar{x}$, ω 0 $\bar{\omega}$ 0 ω 01 $\bar{x}$, $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 1 ω 1 $\bar{x}$, $\bar{x}$ $\bar{x}$ 0 $\bar{\omega}$ 00 ω 1, $\bar{\omega}$ 1 $\bar{x}$ $\bar{\omega}$ $\bar{x}$ $\bar{x}$ 1, 1 $\bar{\omega}$ 1 $\bar{x}$ $\bar{x}$ $\bar{x}$ $\bar{x}$
 D_{24} : 00000111, 00001011, 00001101, 00001110, 01110000, 10110000, 11010000, 11100000
 D_{25} : 00000111, 00001011, 00010011, 00100011, 01000011, 10000011, 11111101, 11111110
 D_{26} : 00000111, 00001011, 000 $\bar{x}$ $\bar{\omega}$ $\bar{x}$ 1, 00 ω 0 $\bar{\omega}$ $\bar{x}$ 1 ω , 01 $\bar{\omega}$ $\bar{x}$ 00 $\bar{x}$, 10 $\bar{\omega}$ $\bar{x}$ 00 $\bar{x}$, 11 ω 1 $\bar{x}$ $\bar{\omega}$ 0 ω , 111 $\bar{x}$ $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 0
 D_{27} : 00000111, 00001011, 00010011, 001111 $\bar{x}$ $\bar{x}$, 010111 $\bar{x}$ $\bar{x}$, 100111 $\bar{x}$ $\bar{x}$, 111 $\bar{x}$ $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 0, 111 $\bar{\omega}$ $\bar{x}$ $\bar{\omega}$ 0 $\bar{\omega}$
 D_{28} : 00000111, 00001011, 00010011, 0 $\bar{x}$ $\bar{x}$ $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 0 ω , 0 $\bar{x}$ $\bar{x}$ $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 0 ω , 1 $\bar{x}$ $\bar{x}$ $\bar{\omega}$ $\bar{x}$ 1 ω , 1 $\bar{\omega}$ $\bar{x}$ 000 $\bar{x}$ $\bar{x}$, 111 $\bar{\omega}$ $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 0
 D_{29} : 00000111, 00001011, 001111 $\bar{x}$ $\bar{x}$, 0 $\bar{x}$ $\bar{x}$ $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 0 ω , 0 $\bar{x}$ $\bar{x}$ $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 0, 1 $\bar{x}$ $\bar{\omega}$ 01101, 1 $\bar{\omega}$ 0 ω 1110, 11 $\bar{x}$ $\bar{\omega}$ 0011
 D_{30} : 00000111, 000 $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 1, 000 $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 1, 0 $\bar{x}$ $\bar{x}$ $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 0, 0 $\bar{x}$ $\bar{x}$ $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 0, 1 $\bar{x}$ $\bar{x}$ $\bar{x}$ 011, 1 $\bar{\omega}$ $\bar{x}$ $\bar{x}$ 1 $\bar{x}$ $\bar{x}$, 111001 $\bar{x}$ $\bar{x}$
 D_{31} : 00000111, 00 $\bar{x}$ $\bar{x}$ 10 $\bar{\omega}$ $\bar{x}$, 010111 $\bar{x}$ $\bar{x}$, 01 $\bar{x}$ $\bar{x}$ 00 $\bar{\omega}$ $\bar{x}$, 011011 $\bar{x}$ $\bar{x}$, 1 $\bar{x}$ $\bar{\omega}$ $\bar{x}$ $\bar{x}$ 1, 1 $\bar{\omega}$ $\bar{x}$ $\bar{\omega}$ $\bar{x}$ 110, 11001 $\bar{x}$ $\bar{\omega}$ 1
 D_{32} : 00000111, 000 $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 1, 000 $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 1, 0 $\bar{x}$ $\bar{\omega}$ 00 $\bar{x}$ $\bar{\omega}$ 1, 0 $\bar{x}$ $\bar{\omega}$ 00 $\bar{x}$ $\bar{\omega}$ 1, 1 $\bar{x}$ $\bar{x}$ $\bar{x}$ 011, 1 $\bar{\omega}$ $\bar{x}$ $\bar{\omega}$ $\bar{x}$ 101, 11111110
 D_{33} : 00000111, 00 $\bar{x}$ $\bar{x}$ 10 $\bar{\omega}$ $\bar{x}$, 0 ω 0 ω 1 $\bar{\omega}$ 0 $\bar{x}$, 0 $\bar{x}$ $\bar{\omega}$ 01 $\bar{\omega}$ $\bar{x}$ 0, 01110000, 10 $\bar{\omega}$ $\bar{x}$ 00 $\bar{x}$, 1 $\bar{\omega}$ 0 $\bar{\omega}$ 0 ω 0 ω , 1 $\bar{\omega}$ $\bar{x}$ 00 $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 0
 D_{34} : 00000111, 000 $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 1, 00 ω 0 $\bar{\omega}$ 1 $\bar{x}$ $\bar{x}$, 0 ω 0 $\bar{\omega}$ 0 $\bar{x}$ $\bar{x}$ 1, 0 $\bar{\omega}$ $\bar{x}$ 010 $\bar{x}$ $\bar{x}$, 1 ω 1 $\bar{\omega}$ 0110, 1 $\bar{\omega}$ $\bar{x}$ $\bar{x}$ 1 $\bar{x}$ $\bar{x}$, 11 ω 1 ω 0 $\bar{x}$ $\bar{x}$
 D_{35} : 00000111, 000 $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 1, 001 $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 0 $\bar{x}$, 0 $\bar{x}$ $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 0 ω , 0 $\bar{x}$ $\bar{x}$ $\bar{x}$ $\bar{\omega}$ $\bar{\omega}$ 0, 1 $\bar{x}$ $\bar{x}$ $\bar{x}$ $\bar{\omega}$ $\bar{x}$ 1, 1 $\bar{\omega}$ 00 $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 0, 11 $\bar{x}$ $\bar{\omega}$ 0101
 D_{36} : 00000111, 00001011, 0 $\bar{x}$ $\bar{x}$ $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 0 ω , ω 0 $\bar{x}$ $\bar{x}$ $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 0, $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 0 $\bar{x}$ $\bar{\omega}$ $\bar{x}$ 0, $\bar{x}$ $\bar{x}$ 0 $\bar{x}$ $\bar{\omega}$ $\bar{x}$ 0 $\bar{\omega}$, $\bar{\omega}$ 1 ω 10011, 1 $\bar{\omega}$ 1 ω 0011
 D_{37} : 00000111, 000 $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 1, 0 $\bar{x}$ $\bar{\omega}$ 01 $\bar{\omega}$ $\bar{x}$ 0, ω 0 $\bar{\omega}$ 00 $\bar{\omega}$ 1 ω , $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 1 $\bar{x}$ $\bar{\omega}$ 1 $\bar{x}$, $\bar{x}$ $\bar{x}$ 0 $\bar{x}$ $\bar{\omega}$ $\bar{x}$ 0, $\bar{\omega}$ 1 $\bar{x}$ $\bar{\omega}$ $\bar{x}$ $\bar{x}$ 1, 1 $\bar{\omega}$ 1 ω 0101
 D_{38} : 00000111, 000 $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 1, 0 $\bar{x}$ $\bar{\omega}$ 00 $\bar{x}$ $\bar{\omega}$ 1, ω 0 $\bar{x}$ $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 0, ω 1 $\bar{\omega}$ $\bar{x}$ $\bar{\omega}$ 1 ω , $\bar{x}$ $\bar{x}$ $\bar{x}$ 11 $\bar{x}$ $\bar{x}$, $\bar{\omega}$ $\bar{\omega}$ $\bar{x}$ 0 $\bar{\omega}$ $\bar{x}$ 0, 1 $\bar{x}$ $\bar{\omega}$ $\bar{x}$ $\bar{\omega}$ $\bar{x}$ 1
 D_{39} : 00000111, 00 $\bar{x}$ $\bar{x}$ 10 $\bar{\omega}$ $\bar{x}$, 0 ω 0 ω 1 $\bar{\omega}$ 0 $\bar{x}$, ω 0 $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 0 $\bar{x}$, ω 1 $\bar{x}$ $\bar{\omega}$ $\bar{x}$ $\bar{\omega}$ 1, $\bar{x}$ $\bar{\omega}$ 001 ω 0 ω , $\bar{\omega}$ $\bar{x}$ $\bar{x}$ 0 $\bar{\omega}$ 0 $\bar{\omega}$, 1 ω 1 ω 10 $\bar{x}$ $\bar{x}$
 D_{40} : 00000111, 000 $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 1, 0 $\bar{x}$ $\bar{\omega}$ 01 $\bar{\omega}$ $\bar{x}$ 0, ω 0 $\bar{\omega}$ 11110, $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 1 $\bar{\omega}$ 1 ω , $\bar{x}$ $\bar{x}$ 000 $\bar{x}$ $\bar{\omega}$ 1, $\bar{\omega}$ 1 $\bar{x}$ $\bar{\omega}$ $\bar{x}$ $\bar{\omega}$ 1, 1 $\bar{\omega}$ 1 $\bar{\omega}$ 1 $\bar{\omega}$ 0 $\bar{\omega}$
 D_{41} : 00000111, 00 $\bar{x}$ $\bar{x}$ 10 $\bar{\omega}$ $\bar{x}$, 0 ω 0 ω 1 $\bar{\omega}$ 0 $\bar{x}$, ω 00 ω 1 $\bar{\omega}$ $\bar{x}$ 0, $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 0 $\bar{x}$ $\bar{\omega}$ $\bar{x}$ 0, $\bar{x}$ $\bar{x}$ 1 $\bar{x}$ $\bar{\omega}$ 1 ω , $\bar{\omega}$ 1 $\bar{x}$ $\bar{x}$ 1 $\bar{x}$ $\bar{x}$, 1 $\bar{\omega}$ $\bar{x}$ $\bar{\omega}$ $\bar{x}$ 110
 D_{42} : 000 $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 1, 00 ω 0 $\bar{\omega}$ 1 $\bar{x}$ $\bar{x}$, 0 ω 00 $\bar{\omega}$ 1 $\bar{x}$ $\bar{x}$, ω 0001 $\bar{\omega}$ $\bar{x}$ $\bar{x}$, $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 1000 $\bar{x}$, $\bar{x}$ $\bar{x}$ 1 $\bar{\omega}$ 00 $\bar{x}$ 0, $\bar{\omega}$ 1 $\bar{x}$ $\bar{\omega}$ 0 ω 00, 1 $\bar{\omega}$ $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 000
 D_{43} : 000 $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 1, 00 ω 0 $\bar{\omega}$ 1 $\bar{x}$ $\bar{x}$, 0 ω 0 ω 1 $\bar{\omega}$ 0 $\bar{x}$, ω 0 ω 0 $\bar{\omega}$ 1 $\bar{\omega}$ 0, $\bar{x}$ $\bar{\omega}$ 1 $\bar{\omega}$ 00 $\bar{x}$ 0, $\bar{x}$ $\bar{x}$ $\bar{\omega}$ 1000 $\bar{x}$, $\bar{\omega}$ 10 $\bar{x}$ $\bar{\omega}$ 00 $\bar{x}$, 1 $\bar{\omega}$ $\bar{x}$ 00 $\bar{x}$ $\bar{\omega}$ 0

D_{44} : $000\omega\bar{\omega}\bar{\omega}1, 00\omega0\bar{\omega}1\bar{\omega}, 0\omega00\bar{\omega}1\omega\bar{\omega}, \omega0001\bar{\omega}\bar{\omega}\omega, \omega\bar{\omega}1\bar{\omega}\omega\bar{\omega}1, \bar{\omega}\omega\bar{\omega}1\omega\bar{\omega}1\bar{\omega}, \bar{\omega}1\bar{\omega}\omega1\bar{\omega}\omega\omega, 1\bar{\omega}\omega\bar{\omega}\bar{\omega}1\omega\omega$
 D_{45} : $000\omega\bar{\omega}\bar{\omega}1, 00\omega0\bar{\omega}\omega1\bar{\omega}, 0\omega0\omega1\bar{\omega}0\bar{\omega}, \omega0\omega0\bar{\omega}1\bar{\omega}0, \omega\bar{\omega}1\bar{\omega}\omega\bar{\omega}1, \bar{\omega}\omega\bar{\omega}1\omega\bar{\omega}1\bar{\omega}, \bar{\omega}10\bar{\omega}\bar{\omega}1\bar{\omega}1, 1\bar{\omega}\bar{\omega}01\bar{\omega}1\bar{\omega}$
 D_{46} : $000\omega\bar{\omega}\bar{\omega}1, 00\omega0\bar{\omega}\omega1\bar{\omega}, 0\omega00\bar{\omega}1\omega\bar{\omega}, \omega0\bar{\omega}10\omega0\bar{\omega}, \omega1\bar{\omega}\bar{\omega}\omega\bar{\omega}1, \bar{\omega}\bar{\omega}0\omega1\omega00, \bar{\omega}\bar{\omega}\omega\bar{\omega}0\bar{\omega}0\bar{\omega}, 1\omega1001\bar{\omega}1$
 D_{47} : $000\omega\bar{\omega}\bar{\omega}1, 00\omega0\bar{\omega}\omega1\bar{\omega}, 0\omega0\omega1\bar{\omega}0\bar{\omega}, \omega0\omega0\bar{\omega}1\bar{\omega}0, \omega1\bar{\omega}\omega\bar{\omega}\bar{\omega}1, \bar{\omega}\bar{\omega}\omega\omega\bar{\omega}1\bar{\omega}1, \bar{\omega}\bar{\omega}\omega\omega1\omega1\bar{\omega}, 1\omega\omega\bar{\omega}\bar{\omega}1\bar{\omega}$
 D_{48} : $000\omega\bar{\omega}\bar{\omega}1, 00\omega01\bar{\omega}\bar{\omega}\omega, 0\omega0\bar{\omega}\bar{\omega}0\omega1, \omega0\bar{\omega}01\omega0\bar{\omega}, \omega1\bar{\omega}10101, \bar{\omega}\bar{\omega}0\omega10\omega0, \bar{\omega}\bar{\omega}\omega00\omega01, 1\omega1\bar{\omega}1010$

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