

The ideal transforms of semigroups

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By a *semigroup* we mean a submonoid of a torsion-free abelian (additive) group in this paper. Let S be a semigroup with the quotient group $q(S)$, that is, $q(S) = \{ s - s' \mid s, s' \in S \}$. Any semigroup T between S and $q(S)$ is called an *oversemigroup* of S .

Moreover, let $\mathbb{Z}$ be the set of all integers and let $\mathbb{Z}_n = \{ a \in \mathbb{Z} \mid a \geq n \}$ and $X \cdot \mathbb{Z}_m = \{ aX \mid a \in \mathbb{Z}_m \}$. And $S[X] = S + \mathbb{Z}_0 X = \{ s + nX \mid s \in S, n \in \mathbb{Z}_0 \}$ is called a *polynomial semigroup* over S (cf. [KOM]).

Let I be a subset of S . I is called an *ideal* of S if $I + S = I$, that is, $a + s \in I$ for each $a \in I$ and each $s \in S$. For any $a \in S$, put $(a) = a + S = \{ a + s \mid s \in S \}$. Then (a) is an ideal of S and it is called a *principal ideal* of S . For $a_1, a_2, \dots, a_n \in S$, we set $I = (a_1, a_2, \dots, a_n) = \cup_{i=1}^n (a_i) = \cup_{i=1}^n (a_i + S)$. The $(a_1, a_2, \dots, a_n)$ is an ideal of S and it is called an *ideal generated* by $a_1, a_2, \dots, a_n$ and $\{ a_1, a_2, \dots, a_n \}$ is called a *basis* of I .

An element u of S is called a *unit* if $u + v = 0$ for some $v \in S$. Let $U(S)$ be the set of units in S . Note that $U(S)$ be a subgroup of $q(S)$.

If we put $M = S - U(S)$, then M is an ideal of S . Moreover, if I is an ideal of S such that $M \subset I$, then $M = I$ or $I = S$. M is called the *maximal ideal* of S . A proper ideal P of S is called a *prime ideal* of S if $a + b \in P$ with $a, b \in S$ implies either $a \in P$ or $b \in P$. We note that the maximal ideal of S is a prime ideal, ϕ is a prime ideal of S and S has the only one maximal ideal.

We give semigroup versions of some results in [F].

Let S be a semigroup. Also, let $\text{Spec}(S)$ be the set of all prime ideals of S . For an ideal I of S , we put $V(I) = \{ P \in \text{Spec}(S) \mid P \supset I \}$ and $D(I) = \{ P \in \text{Spec}(S) \mid P \not\supset I \} = \text{Spec}(S) - V(I)$. In particular, put $D((a)) = D(a)$ for $a \in S$.

Lemma 1. *Let $\{ I_\lambda \}_{\lambda \in \Lambda}$ is a family of ideals of S and let I and J are ideals of S . Then we have the following statements.*

- (1) $\cap \{ I_\lambda \mid \lambda \in \Lambda \}$ is an ideal of S .
- (2) $\cup \{ I_\lambda \mid \lambda \in \Lambda \}$ is an ideal of S .
- (3) $I+J = \{ a+b \mid a \in I, b \in J \}$ is an ideal of S such that $I+J \subset I \cap J$.
- (4) If $P = I \cap J$ is a prime ideal of S , then $I = P$ or $J = P$.
- (5) If P and Q are two prime ideals of S , then $P \cup Q$ is also a prime ideal of S .

Lemma 2. Let S be a semigroup. Then the following statements hold.

- (1) $V(\phi) = \text{Spec}(S)$, $V(S) = \phi$.
- (2) If $I \subset J$, then $V(I) \supset V(J)$.
- (3) $V(I_1) \cap V(I_2) = V(I_1 \cup I_2)$.
- (4) $V(\cup \{ I_\lambda \mid \lambda \in \Lambda \}) = \cap \{ V(I_\lambda) \mid \lambda \in \Lambda \}$.

We make $\text{Spec}(S)$ into a topological space; the topology is called the *Zariski topology*. The closed sets are defined by the $V(I) = \{ P \in \text{Spec}(S) \mid P \supset I \}$.

Then $D(I)$ is an open set of $\text{Spec}(S)$ and the $D(f) = \{ P \in \text{Spec}(S) \mid f \notin P \}$ is an open basis of $\text{Spec}(S)$. For this topology, we give the following statement.

Proposition 3. $\text{Spec}(S)$ is a Kolmogoroff space (T_0 -space) and a quasi-compact space.

Definition 1. We call the *ideal transform of S with respect to an ideal I of S* the following oversemigroup of S :

$$T_S(I) := \{ z \in q(S) \mid (S :_S z + S) \supset nI \text{ for some } n \geq 1 \}$$

Also, we call the *Kaplansky ideal transform of S with respect to an ideal I of S* the following oversemigroup of S :

$$\Omega_S(I) := \{ z \in q(S) \mid \text{rad}(S :_S z + S) \supset I \}.$$

where $\text{rad}(J) = \{ a \in S \mid na \in J \text{ for some positive integer } n \}$.

Note that $\Omega_S(I)$ is an oversemigroup of $T_S(I)$ and note that if I is finitely generated, then $\Omega_S(I) = T_S(I)$. For a principal ideal I , we have that $I + T_S(I) = T_S(I)$.

Proposition 4. *Let I be a principal ideal of S and P be a prime ideal of S . Then the following results are hold.*

- (1) $I + T_S(I) = T_S(I)$, $I + \Omega_S(I) = \Omega_S(I)$.
- (2) $P \in V(I)$ if and only if $P + T_S(I) = T_S(I)$ if and only if $P + \Omega_S(I) = \Omega_S(I)$.

Definition 2 ([K],[KB],[KM] and [MK]). A semigroup S is a *valuation semigroup* if $\alpha \in q(S)$ then $\alpha \in S$ or $-\alpha \in S$.

Also, we say that S is a *seminormal semigroup* if $2\alpha, 3\alpha \in S$ for $\alpha \in q(S)$, we have $\alpha \in S$.

It is clear that valuation semigroups are seminormal.

Definition 3. A non-empty subset N of a semigroup S is called an *additive system* of S if $a, b \in N$ implies $a + b \in N$ and $0 \in N$.

Put $S_N = \{s - t \mid s \in S, t \in N\}$. Then S_N is an oversemigroup of S and is called the *quotient semigroup* of S . If P is a prime ideal of S , then $T = S - P$ is an additive system of S and the quotient semigroup S_T is denoted by S_P .

Definition 4 ([OK]). Let T be an oversemigroup of S . Then T is said to be *flat* over S if for any prime ideal P of S , either $P + T = T$ or $T \subset S_P$. Put $\text{Flat}(T) = \{P \in \text{Spec}(S) \mid P + T = T \text{ or } T \subset S_P\}$.

Example 1. Let $S = (\mathbb{Z}_1 + \mathbb{Z}_1X) \cup \{0\}$. Then $U(S) = \{0\}$ and $M = \mathbb{Z}_1 + \mathbb{Z}_1X = S - U(S)$ is the maximal ideal of S . Also, $\text{Krull dim } S = 1$ and S is not valuation semigroup.

Also, let $T = (\mathbb{Z}_1 + \mathbb{Z}_1X) \cup \mathbb{Z}_0$. Then T is not a valuation semigroup and $U(T) = \{0\}$. Put $N = \mathbb{Z}_1 + \mathbb{Z}_1X$. Then $N \notin \text{Flat}(T)$.

Theorem 5 ([OK]). *Let T be an oversemigroup of S . Then the following statements are equivalent.*

- (1) T is flat over S .
- (2) $T = S_{N \cap S}$ for the maximal ideal N of T .
- (3) For any two ideals I, J of S , $(I \cap J) + T = (I + T) \cap (J + T)$.

Definition 5. Let S be a semigroup and let T be an oversemigroup of S . Then T is said to be *LCM-stable* over S if $((a + S) \cap (b + S)) + T = (a + T) \cap (b + T)$ for each $a, b \in S$.

A flat oversemigroup T over S is LCM-stable over S .

Theorem 6. *Assume that S be a Noetherian semigroup. Let T be an oversemigroup of S . Then T is flat over S if and only if T is LCM-stable over S .*

Corollary 7. *If S is a valuation semigroup and a proper principal ideal $I = (a)$ of S , then $P \in D(I)$ if and only if $T_S(I) = \Omega_S(I) \subset S_P$.*

Proposition 8. *The following statements are hold.*

- (1) $S_a = \Omega_S((a))$ for a non-unit $a \in S$.
- (2) If I and J are ideals of S such that $I \subset J$, then $\Omega_S(I) \supset \Omega_S(J)$.
- (3) $\Omega_S(I) = \cap \{S_P \mid P \in D(I)\} = \cap \{\Omega_S(I + S_P) \mid P \in \text{Spec}(S)\}$.
- (4) If I is a proper ideal of S , then $\Omega_S(I) = \cap \{\Omega_S(a + S) \mid a \in I\} = \cap \{S_a \mid a \in S_a\}$.

Definition 6. $x \in G$ is called an *almost integral element* of S if there exists an element $a \in S$ such that $a + nx \in S$ for each positive integer n . Also, S is a *completely integrally closed* if x is almost integral over S then $x \in S$.

Theorem 9 ([K]). *Let S be a valuation semigroup such that $S \neq q(S)$. Then $\text{Krull dim } S = 1$ if and only if S is a completely integrally closed semigroup.*

Theorem 10 ([KHF]). (1) $\text{Spec}(\mathbb{Z}_0[X]) = \{ (X), (1), (1, X), \phi \}$.

(2) The primary ideals of $\mathbb{Z}_0[X]$ are the following:

- (i) All the ideals that contains both elements of $\mathbb{Z}_0$ and $\mathbb{Z}_0X$.
- (ii) $\mathbb{Z}_k + \mathbb{Z}_0X = (k)$ with $k \in \mathbb{Z}_0$.
- (iii) $\mathbb{Z}_0 + \mathbb{Z}_kX = (kX)$ with $k \in \mathbb{Z}_0$.

Example 2. Let $S = \mathbb{Z}_0 \cup (\mathbb{Z} + \mathbb{Z}_1X)$. Put $P = \mathbb{Z} + \mathbb{Z}_1X$ and $M = (1) = 1 + S = P \cup \mathbb{Z}_1$. Then $\text{Spec}(S) = \{ \phi, P, M \}$. Since $\phi \subset P \subset M$, we have that $\text{Krull dim } S = 2$. It is clear that S is a valuation semigroup. Since P is not finitely generated, S is not Noetherian semigroup.

Theorem 11. *Let the notation be as in Example 2 and let I be an ideal of S . Then the following statements hold.*

(1) Let $I = (f)$ be a principal ideal of S . If $f = 0$, then $T_S(I) = \Omega_S(I) = S$. Also, if $f \in M - P$, then $T_S(I) = \Omega_S(I) = S_f = \mathbb{Z}[X] = \mathbb{Z} + \mathbb{Z}_0 X$. Next, if $f \in P$, then $T_S(I) = \Omega_S(I) = S_f = q(S)$.

(2) If I is not a finitely generated ideal of S , then $I = \mathbb{Z} + \mathbb{Z}_n X$ ($n \geq 1$) and $\Omega_S(I) = q(S)$.

(3) Let $I \neq S$. Then $\text{Spec}(\Omega_S(I)) \cong D(I)$ if and only if $I + \Omega_S(I) = \Omega_S(I)$.

(4) S is not a completely integrally closed and each oversemigroup of S is a flat semigroup over S , and so $\text{Flat}(T) = \text{Spec}(S)$ for each oversemigroup T of S .

Theorem 12. Let S be a semigroup and I an ideal of S . Then the following statements are equivalent.

(1) $D(I)$ is an affine open subspace of $\text{Spec}(S)$.

(2) $\Omega_S(I)$ is flat over S and, for each $P \in \text{Spec}(S)$ with $P \supset I$, $P + \Omega_S(I) = \Omega_S(I)$.

(3) $I + \Omega_S(I) = \Omega_S(I)$.

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