

Triple product p -adic L -functions attached to p -adic families of modular forms

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1 Introduction

In this paper, we present the result [Fuk19, Theorem 5.2.1]. Let p be an odd prime. In [Hsi17], Hsieh constructed three-variable p -adic triple product L -functions attached to triples of Hida families. We generalize the result [Hsi17, (1) of Theorem 7.1] axiomatically and construct three-variable p -adic triple product L -functions in the unbalanced case attached to triples $(F, G^{(2)}, G^{(3)})$. Here, F is a Hida family and $G^{(i)}$ is a more general p -adic family for $i = 2, 3$. For example, we can take Hida families, Coleman families or CM-families as $G^{(i)}$.

To state our theorem precisely, we prepare some notation. We denote by $\mathbb{Q}$, $\mathbb{Q}_p$ and $\mathbb{C}$ the fields of rational numbers, p -adic rational numbers and complex numbers respectively. Let $\mathbb{Z}$ and $\mathbb{Z}_p$ be the rings of integers and p -adic integers respectively. Throughout this paper, we fix an isomorphism $i_p : \overline{\mathbb{Q}_p} \cong \mathbb{C}$ over $\overline{\mathbb{Q}}$. Here, $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}_p}$ are the algebraic closures of the fields $\mathbb{Q}$ and $\mathbb{Q}_p$ respectively. We denote by $\mathbb{A}$ the adèle over $\mathbb{Q}$. Let A be a ring. We denote by $a(n, f)$ the n -th coefficient of a formal power series $f \in A[[q]]$, where n is a non-negative integer. Let ω_p be the Teichmüller character mod p . Let (N_1, N_2, N_3) be a triple of positive integers which are prime to p and (ψ_1, ψ_2, ψ_3) a triple of Dirichlet characters of modulo (N_1p, N_2p, N_3p) which satisfies the following hypothesis.

Hypothesis (1). *There exists an integer $a \in \mathbb{Z}$ such that $\psi_1\psi_2\psi_3 = \omega_p^{2a}$.*

Let K be a finite extension of $\mathbb{Q}_p$ and $\mathcal{O}_K$ the ring of integers of K . We denote by $\Lambda_K := \mathcal{O}_K[[\Gamma]]$ the Iwasawa algebra over $\mathcal{O}_K$, where $\Gamma := 1 + p\mathbb{Z}_p$. Let $\mathbf{I}_i$ be a normal finite flat extension of Λ_K for $i = 1, 2, 3$. We fix a set of non-zero $\mathcal{O}_K$ -algebraic homomorphisms

$$\mathfrak{X}^{(i)} := \{Q_m^{(i)} : \mathbf{I}_i \rightarrow \overline{\mathbb{Q}_p}\}_{m \geq 1}$$

for $i = 1, 2, 3$. Let $G^{(i)} \in \mathbf{I}_i[[q]]$ be a formal series such that the specialization

$$G^{(i)}(m) := \sum Q_m^{(i)}(a(n, G^{(i)}))q^n \in \overline{\mathbb{Q}_p}[[q]]$$

is the Fourier expansion of a normalized cuspidal Hecke eigenform of weight $k^{(i)}(m)$, level $N_i p^{e^{(i)}(m)}$ and Nebentypus $\psi_i \omega_p^{-k^{(i)}(m)} \epsilon_m^{(i)}$ which is primitive outside of p for each positive integer m . Here, $k^{(i)}(m)$ and $e^{(i)}(m)$ are positive integers and $\epsilon_m^{(i)}$ is a finite character of Γ . Let $\mathfrak{X}_{\mathbf{I}_1}$ be the set of arithmetic points Q with weight $k_Q \geq 2$ and a finite part ϵ_Q defined in Definition 2.0.1. We take the pair $(\mathfrak{X}^{(1)}, G^{(1)})$ to be the pair $(\mathfrak{X}_{\mathbf{I}_1}, F)$, where F is a primitive Hida family F of tame level N_1 and Nebentypus ψ_1 defined in Definition 2.0.3. We denote by F_Q the specialization of F at Q for each $Q \in \mathfrak{X}_{\mathbf{I}_1}$. Let $R := \mathbf{I}_1 \widehat{\otimes}_{\mathcal{O}_K} \mathbf{I}_2 \widehat{\otimes}_{\mathcal{O}_K} \mathbf{I}_3$ be the complete tensor product of $\mathbf{I}_1, \mathbf{I}_2$ and $\mathbf{I}_3$ over $\mathcal{O}_K$. We define an unbalanced domain of interpolation points of R to be

$$\mathfrak{X}_R^F := \left\{ \underline{Q} = (Q_1, Q_{m_2}^{(2)}, Q_{m_3}^{(3)}) \in \mathfrak{X}_{\mathbf{I}_1} \times \mathfrak{X}^{(2)} \times \mathfrak{X}^{(3)} \left| \begin{array}{l} k_{Q_1} + k^{(2)}(m_2) + k^{(3)}(m_3) \equiv 0 \pmod{2}, \\ k_{Q_1} \geq k^{(2)}(m_2) + k^{(3)}(m_3) \end{array} \right. \right\}.$$

For each $\underline{Q} = (Q_1, Q_{m_2}^{(2)}, Q_{m_3}^{(3)}) \in \mathfrak{X}_R^F$, we denote by $(F, G^{(2)}, G^{(3)})(\underline{Q})$ the specialization of the triple $(F, G^{(2)}, G^{(3)})$ at $\underline{Q}$. We define a representation $\Pi'_Q = \pi_{Q_1} \boxtimes \pi_{Q_{m_2}^{(2)}} \boxtimes \pi_{Q_{m_3}^{(3)}}$ of $(\mathrm{GL}_2(\mathbb{A}))^3$, where $(\pi_{Q_1}, \pi_{Q_{m_2}^{(2)}}, \pi_{Q_{m_3}^{(3)}})$ is the triple of automorphic representation attached to the triple $(F, G^{(2)}, G^{(3)})(\underline{Q})$. Let $(\chi_Q)_\mathbb{A}$ be the adelization of the following Dirichlet character

$$\chi_Q := \omega_p^{\frac{1}{2}(2a - k_{Q_1} - k^{(2)}(m_2) - k^{(3)}(m_3))} (\epsilon_{Q_1} \epsilon_{m_2}^{(2)} \epsilon_{m_3}^{(3)})^{\frac{1}{2}}$$

for each $\underline{Q} = (Q_1, Q_{m_2}^{(2)}, Q_{m_3}^{(3)}) \in \mathfrak{X}_R^F$. We set $\Pi_Q = \Pi'_Q \otimes (\chi_Q)_\mathbb{A}$ for each $\underline{Q} \in \mathfrak{X}_R^F$. Let $\epsilon_l(s, \Pi_Q)$ be the local epsilon factor of Π_Q defined in [Ike92, page 227] for each finite prime l . We set $N = N_1 N_2 N_3$. Let $\mathfrak{m}_1$ be the unique maximal ideal of $\mathbf{I}_1$. We summarize some hypotheses to state Main Theorem.

Hypothesis (2). *The residual Galois representation $\bar{\rho}_F := \rho_F \bmod \mathfrak{m}_1 : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ attached to F is absolutely irreducible as $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -module and p -distinguished in the sense that the semi-simplification of $\bar{\rho}_F$ restricted to $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ -module is a sum of two different characters.*

Hypothesis (3). *The number $\gcd(N_1, N_2, N_3)$ is square free.*

Hypothesis (4). *For each $\underline{Q} \in \mathfrak{X}_R^F$ and for each prime $l|N$, we have $\epsilon_l(1/2, \Pi_Q) = 1$.*

Hypothesis (5). *Let $i = 2, 3$ and n a positive integer which is prime to p . There exists an element $\langle n \rangle^{(i)} \in \mathbf{I}_i$ which satisfies*

$$Q_m^{(i)}(\langle n \rangle^{(i)}) = \epsilon_m^{(i)}(n) (n \omega_p^{-1}(n))^{k^{(i)}(m)}$$

for each positive integer m .

Hypothesis (6). *Let $i = 2, 3$. We have $a(p, G^{(i)}(m)) \neq 0$ or $G^{(i)}(m)$ is primitive for each positive integer m .*

Hypothesis (7). *For each prime $l|N$, the l -th Fourier coefficients of $F, G^{(2)}$ and $G^{(3)}$ are non-zero.*

Let $L(s, \Pi_Q)$ be the triple product L -function attached to Π_Q defined in §3. Let $\Omega_{F_{Q_1}}$ be the canonical period defined in [Hsi17, (1.3)] and $\mathcal{E}_{F_{Q_1}, p}(\Pi_Q)$ the modified p -Euler factor defined in [Hsi17, (1.2)]. Our main theorem is as follows.

Main Theorem. *Let us assume Hypotheses (1)~(7). Then, there exists an element $\mathcal{L}_{G^{(2)}, G^{(3)}}^F \in R$ such that we have the interpolation property :*

$$(\mathcal{L}_{G^{(2)}, G^{(3)}}^F(\underline{Q}))^2 = \mathcal{E}_{F_{Q_1}, p}(\Pi_Q) \cdot \frac{L(\frac{1}{2}, \Pi_Q)}{(\sqrt{-1})^{2k_{Q_1}} \Omega_{F_{Q_1}}^2}$$

for every $\underline{Q} = (Q_1, Q_{m_2}^{(2)}, Q_{m_3}^{(3)}) \in \mathfrak{X}_R^F$.

Let $\langle \cdot \rangle_{\Lambda_K} : \mathbb{Z}_p^\times \rightarrow \Lambda_K^\times$ be a group homomorphism defined by $\langle z \rangle_{\Lambda_K} = [z \omega_p^{-1}(z)]$, where $[z \omega_p^{-1}(z)]$ is the group-like element of $z \omega_p(z)^{-1} \in \Gamma$ in $\Lambda_K^\times$. Let n be a positive integer which is prime to p . We have $Q(\langle n \rangle_{\Lambda_K}) = \epsilon_Q(n) (n \omega_p^{-1}(n))^{k_Q}$ for each arithmetic point $Q \in \mathfrak{X}_{\mathbf{I}_1}$. Then, if we take a Hida family as $G^{(i)}$, $\langle n \rangle_{\Lambda_K}$ satisfies the Hypothesis (5).

2 p -adic families of modular forms

Let K be a finite extension of $\mathbb{Q}_p$ and $\mathcal{O}_K$ the ring of integers of K . Let $\mathbf{I}$ be a normal finite flat extension of the Iwasawa algebra Λ_K over $\mathcal{O}_K$. In this section, we recall the definitions of ordinary $\mathbf{I}$ -adic cusp forms, primitive Hida families and congruence numbers attached to Hida families. Let N be a positive integer which is prime to p . Throughout this section, we assume that $\mathbb{Q}_p(\chi) \subset K$ for each Dirichlet character χ modulo Np . Let A be a subring of $\overline{\mathbb{Q}}$. We denote by $\mathcal{S}_k(M, \psi, A)$ the A -module of cusp forms of weight k , level M and Nebentypus ψ whose Fourier coefficients at ∞ are included in A , where k, M are positive integers and ψ is a Dirichlet character modulo M . We set $\mathcal{S}_k(M, \psi, B) := \mathcal{S}_k(M, \psi, A) \otimes_A B$ for each A -algebra B .

Definition 2.0.1. We call a continuous $\mathcal{O}_K$ -algebra homomorphism $Q : \mathbf{I} \rightarrow \overline{\mathbb{Q}_p}$ an arithmetic point of weight $k_Q \geq 2$ and a finite part $\epsilon_Q : \Gamma \rightarrow \overline{\mathbb{Q}_p}^\times$ if the restriction $Q|_\Gamma : \Gamma \rightarrow \overline{\mathbb{Q}_p}^\times$ is given by $Q(x) = x^{k_Q} \epsilon_Q(x)$ for each $x \in \Gamma$. Here, $\epsilon_Q : \Gamma \rightarrow \overline{\mathbb{Q}_p}^\times$ is a finite character.

Let $\mathfrak{X}_{\mathbf{I}}$ be the set of arithmetic points of $\mathbf{I}$. We denote by e the ordinary projection defined in [Hid85, (4.3)]. We recall the definition of ordinary $\mathbf{I}$ -adic cusp forms defined in [Wil88].

Definition 2.0.2. Let χ be a Dirichlet character modulo Np . We call a formal power series $\mathbf{f} \in \mathbf{I}[[q]]$ an ordinary $\mathbf{I}$ -adic cusp form of tame level N and Nebentypus χ if the specialization $\mathbf{f}_Q := \sum_{n \geq 0} Q(a(n, \mathbf{f}))q^n \in Q(\mathbf{I})[[q]]$ of $\mathbf{f}$ is the Fourier expansion of an element of $e\mathcal{S}_{k_Q}(Np^{e_Q}, \chi\omega_p^{-k_Q}\epsilon_Q, Q(\mathbf{I}))$ with $e_Q \geq 1$ for all but a finite number of $Q \in \mathfrak{X}_{\mathbf{I}}$.

Let $\mathbf{S}^{\text{ord}}(N, \chi, \mathbf{I})$ be the $\mathbf{I}$ -module consisting of ordinary $\mathbf{I}$ -adic cusp forms of tame level N and Nebentypus χ . Next, we recall the definition of the Hecke algebra of $\mathbf{S}^{\text{ord}}(N, \chi, \mathbf{I})$. For each prime $l \nmid Np$, we define the Hecke operator $T_l \in \text{End}_{\mathbf{I}}(\mathbf{S}^{\text{ord}}(N, \chi, \mathbf{I}))$ at l to be

$$T_l(f) = \sum_{n \geq 1} a(n, T_l(f))q^n$$

for each $f \in \mathbf{S}^{\text{ord}}(N, \chi, \mathbf{I})$, where

$$a(n, T_l(f)) = \sum_{b|(n, l)} \langle b \rangle_{\Lambda_K} \chi(b) b^{-1} a(ln/b^2, f).$$

For each prime $l|Np$, we define the Hecke operator $T_l \in \text{End}_{\mathbf{I}}(\mathbf{S}^{\text{ord}}(N, \chi, \mathbf{I}))$ at l to be

$$T_l(f) = \sum_{n \geq 1} a(ln, f)q^n$$

for each $f \in \mathbf{S}^{\text{ord}}(N, \chi, \mathbf{I})$. The Hecke algebra $\mathbf{T}^{\text{ord}}(N, \chi, \mathbf{I})$ is defined by the sub-algebra of $\text{End}_{\mathbf{I}}(\mathbf{S}^{\text{ord}}(N, \chi, \mathbf{I}))$ generated by T_l for all primes l . Next, we recall the definition of primitive Hida families.

Definition 2.0.3. We call an element $\mathbf{f} \in \mathbf{S}^{\text{ord}}(N, \chi, \mathbf{I})$ a primitive Hida family of tame level N and Nebentypus χ if the specialization $\mathbf{f}_Q$ is the Fourier expansion of an ordinary p -stabilized cuspidal newform for all but a finite number of $Q \in \mathfrak{X}_{\mathbf{I}}$.

Next, we recall the definition of the congruence number. Let $F \in \mathbf{S}^{\text{ord}}(N, \chi, \mathbf{I})$ be a primitive Hida family which satisfies Hypothesis (2). Let $\lambda_F : \mathbf{T}^{\text{ord}}(N, \chi, \mathbf{I}) \rightarrow \mathbf{I}$ be an $\mathbf{I}$ -algebra homomorphism defined by $\lambda_F(T) = a(1, T(F))$ for each $T \in \mathbf{T}^{\text{ord}}(N, \chi, \mathbf{I})$. Let $\mathfrak{m}_F$ be a unique maximal ideal of $\mathbf{T}^{\text{ord}}(N, \chi, \mathbf{I})$ which contains $\text{Ker} \lambda_F$. Let $\mathbf{T}^{\text{ord}}(N, \chi, \mathbf{I})_{\mathfrak{m}_F}$ be the localization of

$\mathbf{T}^{\text{ord}}(N, \chi, \mathbf{I})$ by $\mathbf{m}_F$. Let $\lambda_{\mathbf{m}_F} : \mathbf{T}^{\text{ord}}(N, \chi, \mathbf{I})_{\mathbf{m}_F} \rightarrow \mathbf{I}$ be the restriction of λ_F to $\mathbf{T}^{\text{ord}}(N, \chi, \mathbf{I})_{\mathbf{m}_F}$. By [Hid88a, Corollary 3.7], there exists a finite dimensional $\text{Frac}\mathbf{I}$ -algebra B and an isomorphism

$$\lambda : \mathbf{T}^{\text{ord}}(N, \chi, \mathbf{I})_{\mathbf{m}_F} \otimes_{\mathbf{I}} \text{Frac}\mathbf{I} \cong \text{Frac}\mathbf{I} \oplus B$$

such that $(\text{pr}_{\text{Frac}\mathbf{I}} \circ \lambda)|_{\mathbf{T}^{\text{ord}}(N, \chi, \mathbf{I})_{\mathbf{m}_F}} = \lambda_{\mathbf{m}_F}$, where $\text{pr}_{\text{Frac}\mathbf{I}} : \text{Frac}\mathbf{I} \oplus B \rightarrow \text{Frac}\mathbf{I}$ is the projection to the first part.

Definition 2.0.4. Let $\text{pr}_{\text{Frac}\mathbf{I}}$ (resp. pr_B) be the projection from $\text{Frac}\mathbf{I} \oplus B$ to $\text{Frac}\mathbf{I}$ (resp. B). We put $h(\text{Frac}\mathbf{I}) := \text{pr}_{\text{Frac}\mathbf{I}} \circ \lambda(\mathbf{T}^{\text{ord}}(N, \chi, \mathbf{I})_{\mathbf{m}_F})$ and $h(B) := \text{pr}_B \circ \lambda(\mathbf{T}^{\text{ord}}(N, \chi, \mathbf{I})_{\mathbf{m}_F})$. We define the module of congruence for F to be

$$C(F) := h(\text{Frac}\mathbf{I}) \oplus h(B) / \lambda(\mathbf{T}^{\text{ord}}(N, \chi, \mathbf{I})_{\mathbf{m}_F}).$$

Let

$$1_F \in \mathbf{T}^{\text{ord}}(N, \chi, \mathbf{I})_{\mathbf{m}_F} \otimes_{\mathbf{I}} \text{Frac}\mathbf{I}$$

be the idempotent element corresponded to $(1, 0) \in \text{Frac}\mathbf{I} \oplus B$ by λ . Let $\text{Ann}(C(F)) := \{a \in \mathbf{I} \mid aC(F) = \{0\}\}$ be the annihilator of $C(F)$. By [Wil95, Corollary 2, page 482], $\mathbf{T}^{\text{ord}}(N, \chi, \mathbf{I})_{\mathbf{m}_F}$ is a Gorenstein ring. Hence, by [Hid88b, Theorem 4.4], the annihilator $\text{Ann}(C(F))$ is generated by an element.

Definition 2.0.5. We call a generator η_F of $\text{Ann}(C(F))$ a congruence number of F .

Next, we introduce general p -adic families of modular forms. We fix a set of non-zero continuous $\mathcal{O}_K$ -algebraic homomorphisms

$$\mathfrak{X} := \{Q_m : \mathbf{I} \rightarrow \overline{\mathbb{Q}_p}\}_{m \geq 1}.$$

Then, we define the specialization of an element $G = \sum_{n \geq 0} a(n, G)q^n \in \mathbf{I}[[q]]$, at $Q_m \in \mathfrak{X}$ to be

$$G_{Q_m} := \sum_{n \geq 0} Q_m(a(n, G))q^n \in Q_m(\mathbf{I})[[q]]. \text{ Let } \chi \text{ be a Dirichlet character modulo } Np.$$

Definition 2.0.6. We call an element $G \in \mathbf{I}[[q]]$ a primitive p -adic families of tame level N and Nebentypus χ attached to $\mathfrak{X}$ if G_{Q_m} is the Fourier expansion of a cuspidal Hecke eigenform of weight k_{Q_m} , level $Np^{e_{Q_m}}$ and Nebentypus $\chi\omega_p^{-k_{Q_m}}\epsilon_{Q_m}$ which is primitive outside of p for each positive integer $m \geq 1$. Here, k_{Q_m} and e_{Q_m} are positive integers and ϵ_{Q_m} is a finite character of Γ .

3 Triple product L -functions

Let (g_1, g_2, g_3) be a triple of primitive forms of weight (k_1, k_2, k_3) , level (M_1, M_2, M_3) and Nebentypus (χ_1, χ_2, χ_3) . We assume that there exists a Dirichlet character χ such that $\chi_1\chi_2\chi_3 = \chi^2$. Let (π_1, π_2, π_3) be a triple of automorphic representations of $\text{GL}_2(\mathbb{A})$ attached to (g_1, g_2, g_3) . In this section, we recall the definition of the triple product L -function attached to the automorphic representation

$$\Pi := \pi_1 \otimes (\chi)_{\mathbb{A}} \boxtimes \pi_2 \boxtimes \pi_3,$$

where $(\chi)_{\mathbb{A}}$ is the adelization of χ . We define the triple product L -function $L(s, \Pi)$ to be

$$L(s, \Pi) = \prod_{v:\text{place}} L_v(s, \Pi), \quad \text{Re}(s) > 1,$$

where $L_v(s, \Pi)$ is the GCD local triple product L -function defined in [PSR87] and [Ike92]. Let l be a prime. The local L -function $L_l(s, \Pi)$ at l can be written by the form $1/P(p^{-s})$, where

$P(T) \in \mathbb{C}[T]$ such that $P(0) = 1$. By the result of [Ike98], the archimedean factor $L_\infty(s, \Pi)$ can be written by the form

$$L_\infty(s, \Pi) := \Gamma_{\mathbb{C}}\left(s + \frac{w}{2}\right) \prod_{i=1}^3 \Gamma_{\mathbb{C}}(s + 1 - k_i^*),$$

where $w = k_1 + k_2 + k_3 - 2$, $k_i^* = \frac{k_1 + k_2 + k_3}{2} - k_i$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s} \Gamma(s)$. By [Ike92, Proposition 2.5], the function $L(s, \Pi)$ is continued to the entire $\mathbb{C}$ -plane analytically and by [Ike92, Proposition 2.4], the function $L(s, \Pi)$ satisfies the functional equation

$$L(s, \Pi) = \epsilon(s, \Pi) L(1 - s, \Pi),$$

where $\epsilon(s, \Pi)$ is the global epsilon factor defined in [Ike92, page 230]. The epsilon factor $\epsilon(s, \Pi)$ can be decomposed by the product of the local epsilon factors

$$\epsilon(s, \Pi) = \prod_{v:\text{place}} \epsilon_v(s, \Pi)$$

and it is known that $\epsilon_v(\frac{1}{2}, \Pi) \in \{\pm 1\}$.

4 Construction of p -adic triple product L -functions

Let K be a finite extension of $\mathbb{Q}_p$ and $\mathbf{I}_i$ a normal finite flat extension of Λ_K for $i = 1, 2, 3$. We fix a triple of Dirichlet characters (ψ_1, ψ_2, ψ_3) of modulo $(N_1 p, N_2 p, N_3 p)$, where N_i is a positive integer which is prime to p for $i = 1, 2, 3$. Let $F \in \mathcal{S}^{\text{ord}}(N_1, \psi_1, \mathbf{I}_1)$ be a primitive Hida family defined in Definition 2.0.3. Let $G^{(i)} \in \mathbf{I}_i[[q]]$ be a p -adic family of tame level N_i and Nebentypus ψ_i attached to

$$\mathfrak{X}^{(i)} := \{Q_m^{(i)} : \mathbf{I}_i \rightarrow \overline{\mathbb{Q}_p}\}_{m \geq 1}$$

for $i = 2, 3$. In this section, we prove Main theorem and construct the p -adic triple product L -function attached to $(F, G^{(2)}, G^{(3)})$. For simplicity, we assume $N_1 = N_2 = N_3 = 1$. Further, we assume that the triple $(F, G^{(2)}, G^{(3)})$ satisfies Hypothesis (1)~(7). We set $R := \mathbf{I}_1 \hat{\otimes}_{\mathcal{O}_K} \mathbf{I}_2 \hat{\otimes}_{\mathcal{O}_K} \mathbf{I}_3$ and

$$\mathfrak{X}_R^F := \left\{ \underline{Q} = (Q_1, Q_{m_2}^{(2)}, Q_{m_3}^{(3)}) \in \mathfrak{X}_{\mathbf{I}_1} \times \mathfrak{X}^{(2)} \times \mathfrak{X}^{(3)} \left| \begin{array}{l} k_{Q_1} + k^{(2)}(m_2) + k^{(3)}(m_3) \equiv 0 \pmod{2}, \\ k_{Q_1} \geq k^{(2)}(m_2) + k^{(3)}(m_3) \end{array} \right. \right\}.$$

We define a formal operator $\mathbf{U}_{R,p} \in \text{End}_R(R[[q]])$ to be

$$\mathbf{U}_{R,p}(f) = \sum_{n \geq 0} a(pn, f) q^n$$

for each $f = \sum_{n \geq 0} a(n, f) q^n \in R[[q]]$. Let $\Theta : \mathbb{Z}_p^\times \rightarrow R^\times$ be a character defined by

$$\Theta(z) = \psi_1 \omega_p^{-a}(z) \langle z \rangle_{\mathbf{I}_1}^{\frac{1}{2}} (\langle z \rangle^{(2)} \langle z \rangle^{(3)})^{-\frac{1}{2}},$$

for each $z \in \mathbb{Z}_p^\times$, where $\langle z \rangle_{\mathbf{I}_1}$ is the image of $\langle z \rangle_{\Lambda_K}$ by the natural inclusion $\Lambda_K \hookrightarrow \mathbf{I}_1$. For each $f \in \sum_{n \geq 0} a(n, f) q^n \in R[[q]]$, we define a Θ -twisted form $f|[\Theta] \in R[[q]]$ to be

$$f|[\Theta] = \sum_{p \nmid n} \Theta(n) \cdot a(n, f) q^n.$$

We set $d := \frac{d}{d_Q}$. For each $\underline{Q} = (Q_1, Q_{m_2}^{(2)}, Q_{m_3}^{(3)}) \in \mathfrak{X}_R^F$, we have $f|[\Theta](\underline{Q}) = d^{r_Q}(f(\underline{Q})|[\Theta_{\underline{Q}}])$ with the Dirichlet character

$$\Theta_{\underline{Q}} = \psi_1 \omega_p^{-a-r_Q} \epsilon_{Q_1}^{\frac{1}{2}} \epsilon_{m_2}^{(2)-\frac{1}{2}} \epsilon_{m_3}^{(3)-\frac{1}{2}},$$

where $r_Q = \frac{1}{2}(k_{Q_1} - k^{(2)}(m_2) - k^{(3)}(m_3))$. Here, $f(\underline{Q})|[\Theta_{\underline{Q}}]$ is the twisted cusp form by the Dirichlet character $\Theta_{\underline{Q}}$. We regard $G^{(2)}$ and $G^{(3)}$ as elements of $R[[q]]$ by natural embeddings $\mathbf{I}_2 \hookrightarrow R$ and $\mathbf{I}_3 \hookrightarrow R$. We set $H := G^{(2)} \cdot (G^{(3)}|[\Theta]) \in R[[q]]$. We define the Maass-Shimura differential operator δ_k to be

$$\delta_k := \frac{1}{2\pi\sqrt{-1}} \left(\frac{\partial}{\partial z} + \frac{k}{2\sqrt{-1}\text{Im}(z)} \right)$$

for each non-negative integer k . Further, we set $\delta_k^m := \delta_{k+2m-2} \dots \delta_{k+2}\delta_k$, where m is a non-negative integer. We denote by $\mathcal{H}$ the holomorphic projection from the space of nearly holomorphic modular forms to modular forms defined in [Shi76]. Let $\mathfrak{m}_R$ be the maximal ideal of R .

Lemma 4.0.1. *Let $\underline{Q} = (Q_1, Q_{m_2}^{(2)}, Q_{m_3}^{(3)}) \in \mathfrak{X}_R^F$. We fix a finite extension L of K such that $\mathcal{O}_L$ contains $Q_1(\mathbf{I}_1)$, $Q_{m_2}^{(2)}(\mathbf{I}_2)$ and $Q_{m_3}^{(3)}(\mathbf{I}_3)$. Then, the sequence $\{U_{\mathbf{R},p}^n H(\underline{Q})\}_{n \geq 1}$ converges in $\mathcal{O}_L[[q]]$ by the $\mathfrak{m}_R$ -adic topology and the limit of the sequence equals to the Fourier expansion of $e\mathcal{H}(G^{(2)}(m_2)\delta_{k^{(3)}(m_3)}^{r_Q} G^{(3)}(m_3)|\Theta_{\underline{Q}}) \in eS_{k_{Q_1}}(p^{e_{Q_1}}, \psi_1 \omega_p^{k_{Q_1}} \epsilon_{Q_1}, L)$, with $e_{Q_1} := \max\{1, m_{\epsilon_{Q_1}}\}$. Here, $m_{\epsilon_{Q_1}}$ is the p -power of the conductor of ϵ_{Q_1} .*

Proof. It is known that $H(\underline{Q})$ is a Fourier expansion of a p -adic modular form and by [Hid85, Lemma 5.2], we have

$$H(\underline{Q}) = \mathcal{H}(G^{(2)}(m_2)\delta_{k^{(3)}(m_3)}^{r_Q} G^{(3)}(m_3)|\Theta_{\underline{Q}}) + d(g'_{\underline{Q}}) \in L[[q]],$$

where $g'_{\underline{Q}} \in L[[q]]$ is a p -adic modular form. By [Hid85, (6.12)], $ed = 0$ and we have $eH(\underline{Q}) = e\mathcal{H}(G^{(2)}(m_2)\delta_{k^{(3)}(m_3)}^{r_Q} G^{(3)}(m_3)|\Theta_{\underline{Q}})$. Further, by [Hid85, (4.3)], the sequence $\{U_{\mathbf{R},p}^n H(\underline{Q})\}_{n \geq 1}$ converges in $\mathcal{O}_L[[q]]$ by the $\mathfrak{m}_R$ -adic topology and the limit of the sequence equals to $eH(\underline{Q})$. We have completed the proof. $\square$

To construct a triple product p -adic L -function $L_{G^{(2)}, G^{(3)}}^F \in R$, we prove the following lemma and proposition.

Lemma 4.0.2. *There exists a unique element $H^{\text{ord}} \in R[[q]]$ such that the specialization of H^{ord} at each $\underline{Q} = (Q_1, Q_{m_2}^{(2)}, Q_{m_3}^{(3)}) \in \mathfrak{X}_R^F$ equals to the Fourier expansion of the modular form $e\mathcal{H}(G^{(2)}(m_2)\delta_{k^{(3)}(m_3)}^{r_Q} G^{(3)}(m_3)|\Theta_{\underline{Q}})$.*

Proof. Let $I_{\underline{Q}}$ be the ideal of R generalized by $\text{Ker} Q_1, \text{Ker} Q_{m_2}^{(2)}$ and $\text{Ker} Q_{m_3}^{(3)}$ for each $\underline{Q} = (Q_1, Q_{m_2}^{(2)}, Q_{m_3}^{(3)}) \in \mathfrak{X}_R^F$. We denote by $\mathfrak{B}$ the set of finite intersections of $I_{\underline{Q}}$ for $\underline{Q} \in \mathfrak{X}_R^F$. Then, we can easily check that $\bigcap_{J \in \mathfrak{B}} J = \{0\}$. Further, we have the natural isomorphism $R \cong \varprojlim_{J \in \mathfrak{B}} (R/J)$. In particular, we have

$$R[[q]] \cong \varprojlim_{J \in \mathfrak{B}} R[[q]] \otimes_R (R/J).$$

For each $J = \bigcap_{i=1}^m I_{\underline{Q}_i} \in \mathfrak{B}$, it suffices to prove that there exists a unique element $H_J^{\text{ord}} \in R[[q]] \otimes_R (R/J)$ such that the image of H_J^{ord} by the natural embedding $i_J : R[[q]] \otimes_R (R/J) \hookrightarrow$

$\prod_{i=1}^m (R[[q]] \otimes_R R/I_{\underline{Q}_i})$ equals to $\left[e(H(\underline{Q}_i)) \right]_{i=1}^m$. The uniqueness of H_J^{ord} is trivial. We prove the existence of H_J^{ord} .

Let $p_J : R[[q]] \rightarrow R \otimes_R (R/J)$ be the natural projection. If $J = I_{\underline{Q}}$ for $\underline{Q} \in \mathfrak{X}_R^F$, we have $\lim_{n \rightarrow \infty} p_J(U_{R,p}^{n!} H) = e\mathcal{H}(\underline{Q})$ by Lemma 4.0.1. We assume that there exist elements $H_J^{\text{ord}} = \lim_{n \rightarrow \infty} p_J(U_{R,p}^{n!} H) \in R[[q]] \otimes (R/J)$ and $H_{J'}^{\text{ord}} = \lim_{n \rightarrow \infty} p_{J'}(U_{R,p}^{n!} H) \in R[[q]] \otimes (R/J')$ for a pair $(J, J') \in \mathcal{B} \times \mathcal{B}$. We define the R -linear map:

$$\begin{array}{ccc} (R[[q]] \otimes_R (R/J)) \times (R[[q]] \otimes_R (R/J')) & \xrightarrow{i_{J,J'}} & (R[[q]] \otimes_R (R/J + J')) \\ \downarrow \Psi & & \downarrow \Psi \\ (a, b) & \longmapsto & a - b \end{array}.$$

Then, we have $i_{J,J'}(H_J^{\text{ord}}, H_{J'}^{\text{ord}}) = \lim_{n \rightarrow \infty} i_{J,J'}(p_J(U_{R,p}^{n!} H), p_{J'}(U_{R,p}^{n!} H)) = 0$. Further, since $\text{Ker } i_{J,J'} \cong R[[q]] \otimes_R (R/J \cap J')$, there exists a unique element $H_{J \cap J'}^{\text{ord}} \in R[[q]] \otimes_R (R/J \cap J')$ such that the image of $H_{J \cap J'}^{\text{ord}}$ in $(R[[q]] \otimes_R (R/J)) \times (R[[q]] \otimes_R (R/J'))$ equals to $(H_J^{\text{ord}}, H_{J'}^{\text{ord}})$. In particular, we have $H_{J \cap J'}^{\text{ord}} = \lim_{n \rightarrow \infty} p_{J \cap J'}(U_{R,p}^{n!} H)$. Then, for each $J = \cap_{i=1}^m I_{\underline{Q}_i} \in \mathcal{B}$, there exists a unique element $H_J^{\text{ord}} \in R[[q]] \otimes_R (R/J)$ such that the image of H_J^{ord} by the natural embedding $i_J : R[[q]] \otimes_R (R/J) \hookrightarrow \prod_{i=1}^m (R[[q]] \otimes_R R/I_{\underline{Q}_i})$ equals to $\left[e(H(\underline{Q}_i)) \right]_{i=1}^m$. We have completed the proof. $\square$

Proposition 4.0.3. *The power series H^{ord} is an element of $\mathbf{S}^{\text{ord}}(N, \psi_1, \mathbf{I}_1) \hat{\otimes}_{\mathbf{I}_1} R$.*

Proof. We identify the Iwasawa algebra Λ_K with $\mathcal{O}_K[[X]]$ by the isomorphism $[1+p] \mapsto 1+X$ and we regard $\mathbf{I}_i$ as the normal finite flat extension of $\mathcal{O}_K[[X_i]]$ for $i = 1, 2, 3$. Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be a base of R over $R_0 = \mathcal{O}_K[[X_1, X_2, X_3]]$. We put

$$H^{\text{ord}} = \sum_{i=1}^n H^{(i)} \alpha_i,$$

where $H^{(i)} \in R_0[[q]]$ for each $i = 1, \dots, n$. We put $L = \text{Frac} R$ and $L_0 = \text{Frac} R_0$. Let $\text{Tr}_{L/L_0} : L \rightarrow L_0$ be the trace map and $\alpha_1^*, \alpha_2^*, \dots, \alpha_n^*$ be the dual base of $\alpha_1, \alpha_2, \dots, \alpha_n$ with respect to Tr_{L/L_0} . Then, we have

$$H^{(i)}(\underline{Q}) = \text{Tr}(H(\underline{Q}) \alpha_i^*(\underline{Q}))$$

for all but a finite number of $\underline{Q} = (Q_1, Q_{m_2}^{(2)}, Q_{m_3}^{(3)}) \in \mathfrak{X}_R^F$. Further, $\text{Tr}(H(\underline{Q}) \alpha_i^*(\underline{Q}))$ is the Fourier expansion of an element of $eS_{k_{Q_1}}(Np^{e_{Q_1}}, \epsilon_{Q_1} \psi_1 \omega_p^{-k_{Q_1}}, \underline{Q}(R))$. It suffices to prove

$$H^{(i)} \in \mathbf{S}^{\text{ord}}(1, \psi_1, \mathcal{O}_K[[X_1]]) \hat{\otimes}_{\mathcal{O}_K[[X_1]]} R_0$$

for each $i = 1, \dots, n$.

For each positive integers m_2, m_3 , let $H_{m_2, m_3}^{(i)} \in \mathcal{O}_K[b_{m_2}^{(2)}, b_{m_3}^{(3)}][[X_1]][[q]]$ be the specialization of $H^{(i)}$ at $(Q_{m_2}^{(2)}, Q_{m_3}^{(3)})$, where $b_{m_2}^{(2)} := Q_{m_2}^{(2)}(X_2)$ and $b_{m_3}^{(3)} := Q_{m_3}^{(3)}(X_3)$. First, we prove $H_{m_2, m_3}^{(i)} \in \mathbf{S}^{\text{ord}}(1, \psi_1, \mathcal{O}_K[b_{m_2}^{(2)}, b_{m_3}^{(3)}][[X_1]])$. We define a subset $\mathfrak{X}_{m_2, m_3}^F$ of arithmetic points of $\mathbf{I}_1$ to be

$$\mathfrak{X}_{m_2, m_3}^F := \left\{ Q \in \mathfrak{X}_{\mathbf{I}_1} \mid (Q, Q_{m_2}^{(2)}, Q_{m_3}^{(3)}) \in \mathfrak{X}_R^F \right\}.$$

For each positive integer k , there exists an arithmetic point $Q \in \mathfrak{X}_{\mathbf{I}_1}$ with $k_Q = k$. Then, we have $\#\mathfrak{X}_{m_2, m_3}^F = \infty$. Let $\mathbf{S}_{m_2, m_3}^{\text{ord}} \subset \mathcal{O}_K[b_{m_2}^{(2)}, b_{m_3}^{(3)}][[X_1]][[q]]$ be an $\mathcal{O}_K[b_{m_2}^{(2)}, b_{m_3}^{(3)}][[X_1]]$ -module consisting

of elements $f \in \mathcal{O}_K[b_{m_2}^{(2)}, b_{m_3}^{(3)}][[X_1]][[q]]$ such that, for all but a finite number of $Q \in \mathfrak{X}_{m_2, m_3}^F$, $f(Q)$ equals to the specialization of an element of $\mathbf{S}^{\text{ord}}(1, \psi_1, \mathcal{O}_K[b_{m_2}^{(2)}, b_{m_3}^{(3)}][[X_1]])$ at Q . Then, we have $\mathbf{S}^{\text{ord}}(1, \psi_1, \mathcal{O}_K[b_{m_2}^{(2)}, b_{m_3}^{(3)}][[X_1]]) \subset \mathbf{S}_{m_2, m_3}^{\text{ord}}$ and $H_{m_2, m_3}^{(i)} \in \mathbf{S}_{m_2, m_3}^{\text{ord}}$. It suffices to prove that we have $\mathbf{S}^{\text{ord}}(1, \psi_1, \mathcal{O}_K[b_{m_2}^{(2)}, b_{m_3}^{(3)}][[X_1]]) = \mathbf{S}_{m_2, m_3}^{\text{ord}}$. Let $g_1, \dots, g_d$ be elements of $\mathbf{S}_{m_2, m_3}^{\text{ord}}$ which are $\mathcal{O}_K[b_{m_2}^{(2)}, b_{m_3}^{(3)}][[X_1]]$ -linear independent. Then, there are positive integers $m_1, \dots, m_d$ such that

$$d = \det(a(m_i, g_j))_{1 \leq i, j \leq d} \neq 0 \in \mathcal{O}_K[b_{m_2}^{(2)}, b_{m_3}^{(3)}][[X_1]].$$

Since $\#\mathfrak{X}_{m_2, m_3}^F = \infty$, there exists an element $Q \in \mathfrak{X}_{m_2, m_3}^F$ such that $d(Q) \neq 0$. Then, we have

$$\text{rank}_{\mathcal{O}_K[b_{m_2}^{(2)}, b_{m_3}^{(3)}][[X_1]]} \mathbf{S}_{m_2, m_3}^{\text{ord}} = \text{rank}_{\mathcal{O}_K[b_{m_2}^{(2)}, b_{m_3}^{(3)}][[X_1]]} \mathbf{S}^{\text{ord}}(1, \psi_1, \mathcal{O}_K[b_{m_2}^{(2)}, b_{m_3}^{(3)}][[X_1]]).$$

Then, if we take an element $f \in \mathbf{S}_{m_2, m_3}^{\text{ord}}$, there exists an element $a \in \mathcal{O}_K[b_{m_2}^{(2)}, b_{m_3}^{(3)}][[X_1]] \setminus \{0\}$ such that $af \in \mathbf{S}^{\text{ord}}(1, \psi_1, \mathcal{O}_K[b_{m_2}^{(2)}, b_{m_3}^{(3)}][[X_1]])$. Since a has only finite roots, we have $f \in \mathbf{S}^{\text{ord}}(1, \psi_1, \mathcal{O}_K[b_{m_2}^{(2)}, b_{m_3}^{(3)}][[X_1]])$. Then, we have $\mathbf{S}^{\text{ord}}(1, \psi_1, \mathcal{O}_K[b_{m_2}^{(2)}, b_{m_3}^{(3)}][[X_1]]) = \mathbf{S}_{m_2, m_3}^{\text{ord}}$.

For each positive integer m_3 , let $H^{(i), m_3} \in \mathcal{O}_K[b_{m_3}^{(3)}][[X_1, X_2]]$ be the specialization of $H^{(i)}$ at $Q_{m_3}^{(3)}$. Next, we prove $H^{(i), m_3} \in \mathbf{S}^{\text{ord}}(1, \psi_1, \mathcal{O}_K[b_{m_3}^{(3)}][[X_1]]) \hat{\otimes}_{\mathcal{O}_K[b_{m_3}^{(3)}]} \mathcal{O}_K[b_{m_3}^{(3)}][[X_2]]$. We define an $\mathcal{O}_K[b_{m_3}^{(3)}][[X_1, X_2]]$ -module $\mathbf{S}_{m_3}^{\text{ord}} \subset \mathcal{O}_K[b_{m_3}^{(3)}][[X_1, X_2]]$ consisting of elements $f(X_1, X_2)$ such that $f(X_1, b_m^{(2)}) \in \mathbf{S}^{\text{ord}}(1, \psi_1, \mathcal{O}_K[[X_1]]) \otimes_{\mathcal{O}_K} \mathcal{O}_{\overline{\mathbb{Q}_p}}$ for each positive integer m . We have already proved that $H^{(i), m_3} \in \mathbf{S}_{m_3}^{\text{ord}}$. It is clear that $\mathbf{S}^{\text{ord}}(1, \psi_1, \mathcal{O}_K[b_{m_3}^{(3)}][[X_1]]) \hat{\otimes}_{\mathcal{O}_K[b_{m_3}^{(3)}]} \mathcal{O}_K[b_{m_3}^{(3)}][[X_2]] \subset \mathbf{S}_{m_3}^{\text{ord}}$. Further, if $g_1, \dots, g_d \in \mathbf{S}_{m_3}^{\text{ord}}$ are linear independent, there exist positive integers $m_1, \dots, m_d$ such that

$$d = \det(a(m_i, g_j))_{1 \leq i, j \leq d} \neq 0 \in \mathcal{O}_K[b_{m_3}^{(3)}][[X_1, X_2]].$$

We can take a positive integer m_2 such that $d(X_1, b_{m_2}^{(2)}) \neq 0$. Then, $\text{rank}_{\mathcal{O}_K[b_{m_3}^{(3)}][[X_1, X_2]]} \mathbf{S}_{m_3}^{\text{ord}} = \text{rank}_{\mathcal{O}_K[b_{m_3}^{(3)}][[X_1]]} \mathbf{S}^{\text{ord}}(1, \psi_1, \mathcal{O}_K[b_{m_3}^{(3)}][[X_1]])$. We take an element $a \in \mathcal{O}_K[b_{m_3}^{(3)}][[X_1, X_2]] \setminus \{0\}$ such that $aH^{(i), m_3} \in \mathbf{S}^{\text{ord}}(1, \psi_1, \mathcal{O}_K[b_{m_3}^{(3)}][[X_1]]) \hat{\otimes}_{\mathcal{O}_K[b_{m_3}^{(3)}]} \mathcal{O}_K[b_{m_3}^{(3)}][[X_2]]$. Since we have $a(X_1, p^m) \neq 0$ for almost all positive integers m , there exists a positive integer k_{m_3} such that $H^{(i), m_3}(X_1, p^{m'}) \in \mathbf{S}^{\text{ord}}(1, \psi_1, \mathcal{O}_K[b_{m_3}^{(3)}][[X_1]])$ for each positive integer $m' \geq k_{m_3}$.

We put $H_0^{(i), m_3} := H^{(i), m_3}$ and $c_m = p^{k_{m_3} + m}$ for each non-negative integer m . We define a power series $H_m^{(i), m_3} \in \mathcal{O}_K[b_{m_3}^{(3)}][[X_1, X_2]][[q]]$ inductively for each positive integer m to be

$$H_m^{(i), m_3}(X_1, X_2) := (H_{m-1}^{(i), m_3}(X_1, X_2) - H_{m-1}^{(i), m_3}(X_1, c_m))(X_2 - c_m)^{-1} \in \mathcal{O}_K[b_{m_3}^{(3)}][[X_1, X_2]][[q]].$$

By the induction of m , we have $H_m^{(i), m_3}(X_1, c_l) \in \mathbf{S}^{\text{ord}}(1, \psi_1, \mathcal{O}_K[b_{m_3}^{(3)}][[X_1]])$ for each non-negative integer m and $l \geq m + 1$. In particular, if we put $H_{m, m+1}^{(i), m_3} := H_m^{(i), m_3}(X_1, c_{m+1})$, we have

$$H^{(i), m_3} = \sum_{m=1}^{\infty} H_{m, m+1}^{(i), m_3} \prod_{j=1}^m (X_2 - c_j) \in \mathbf{S}^{\text{ord}}(1, \psi_1, \mathcal{O}_K[b_{m_3}^{(3)}][[X_1]]) \hat{\otimes}_{\mathcal{O}_K[b_{m_3}^{(3)}]} \mathcal{O}_K[b_{m_3}^{(3)}][[X_2]].$$

Next, we prove $H^{(i)} \in \mathbf{S}^{\text{ord}}(1, \psi_1, \mathcal{O}_K[[X_1]]) \hat{\otimes}_{\mathcal{O}_K} \mathcal{O}_K[[X_2, X_3]]$. By the same way as above, we can take a non-zero element $a \in \mathcal{O}_K[[X_1, X_2, X_3]] \setminus \{0\}$ such that $aH^{(i)}$ is an element of $\mathbf{S}^{\text{ord}}(1, \psi_1, \mathcal{O}_K[[X_1]]) \hat{\otimes}_{\mathcal{O}_K} \mathcal{O}_K[[X_2, X_3]]$. Further, there exists a positive integer k which satisfies $H^{(i)}(X_1, X_2, p^m) \in \mathbf{S}^{\text{ord}}(1, \psi_1, \mathcal{O}_K[[X_1]]) \hat{\otimes}_{\mathcal{O}_K} \mathcal{O}_K[[X_2]]$ for each $m \geq k$. We put $H_0^{(i)} := H^{(i)}$ and $c'_m = p^{k+m}$ for each non-negative integer m . We define a power series $H_m^{(i)} \in \mathcal{O}_K[[X_1, X_2, X_3]][[q]]$ inductively for each positive integer m to be

$$H_m^{(i)} := (H_{m-1}^{(i)}(X_1, X_2, X_3) - H_{m-1}^{(i)}(X_1, X_2, c'_m))(X_3 - c'_m)^{-1} \in \mathcal{O}_K[[X_1, X_2, X_3]][[q]].$$

Then, we have

$$H^{(i)} = \sum_{m=0}^{\infty} H_m^{(i)}(X_1, X_2, c'_{m+1}) \prod_{j=1}^m (X_3 - c'_j) \in \mathbf{S}^{\text{ord}}(1, \psi_1, \mathcal{O}_K[[X_1]]) \hat{\otimes}_{\mathcal{O}_K} \mathcal{O}_K[[X_2, X_3]].$$

We have completed the proof. $\square$

Definition 4.0.4. We define an element $L_{G^{(2)}, G^{(3)}}^F \in R$ to be

$$L_{G^{(2)}, G^{(3)}}^F := a(1, \eta_F 1_F(H^{\text{ord}})).$$

Here, 1_F is the idempotent element defined in §2 and η_F is the congruence number defined in Definition 2.0.5.

By [Hid85, Proposition 4.5] and [Ich08, Theorem 1.1], we have the interpolation formula of $L_{G^{(2)}, G^{(3)}}$. However, we omit the detail of the proof of the interpolation formula. Let $\Omega_{F_{Q_1}}$ be the canonical period defined in [Hsi17, (1.3)] and $\mathcal{E}_{F_{Q_1}, p}(\Pi_Q)$ the modified p -Euler factor defined in [Hsi17, (1.2)].

Proposition 4.0.5. We assume Hypotheses (1)~(7). Then, there exists an element $\mathcal{L}_{G^{(2)}, G^{(3)}}^F \in R$ such that we have the interpolation property :

$$(\mathcal{L}_{G^{(2)}, G^{(3)}}^F(Q))^2 = \mathcal{E}_{F_{Q_1}, p}(\Pi_Q) \cdot \frac{L(\frac{1}{2}, \Pi_Q)}{(\sqrt{-1})^{2k_{Q_1}} \Omega_{F_{Q_1}}^2}$$

for every $Q = (Q_1, Q_{m_2}^{(2)}, Q_{m_3}^{(3)}) \in \mathfrak{X}_R^F$.

5 Examples

In this subsection, we give examples of the triple $(\mathbf{I}_i, \mathfrak{X}^{(i)}, G^{(i)})$ which satisfy Hypothesis (5), (6) and (7). As a first example, we can take families of CM forms of weight 1. Let L be a quadratic imaginary extension of $\mathbb{Q}$ with a discriminant D . We assume that D is square-free and prime to p . Let $\mathfrak{f}$ be an integral ideal of $\mathcal{O}_L$ such that $\mathfrak{f}$ is prime to Dp . We assume that $N(\mathfrak{f})$ is square-free, where N is the absolute norm. Let $\mathfrak{C}(\mathfrak{f}(p)^j)$ be the class ray group modulo $\mathfrak{f}(p)^j$ over L for each $j \geq 0$. By the class field theory, $\mathfrak{C}(\mathfrak{f}(p)^\infty) = \varprojlim_{j \geq 0} \mathfrak{C}(\mathfrak{f}(p)^j)$ is a $\mathbb{Z}_p$ -module of rank

2. Let $\Delta_{\mathfrak{f}}$ be the torsion part of $\mathfrak{C}(\mathfrak{f}(p)^\infty)$ and $\chi : \Delta_{\mathfrak{f}} \rightarrow \mathbb{C}^\times$ be a primitive character. Here, a primitive character means that it is not induced by any character from $\Delta_{\mathfrak{f}'}$ for $\mathfrak{f} \subsetneq \mathfrak{f}'$. Let L_∞^-/L be the anticyclotomic extension of L . By the class field theory, the Galois group $\text{Gal}(L_\infty^-/L)$ is a direct summand of the $\mathbb{Z}_p$ -torsion free part of $\mathfrak{C}(\mathfrak{f}(p)^\infty)$. Let $\text{pr}_{\mathfrak{f}} : \mathfrak{C}(\mathfrak{f}(p)^\infty) \rightarrow \Delta_{\mathfrak{f}}$ and $\text{pr}_- : \mathfrak{C}(\mathfrak{f}(p)^\infty) \rightarrow \text{Gal}(L_\infty^-/L)$ be the natural projections to $\Delta_{\mathfrak{f}}$ and $\text{Gal}(L_\infty^-/L)$ respectively. Let E be a finite Galois extension of $\mathbb{Q}_p$ such that the image of $\Delta_{\mathfrak{f}}$ by χ is contained in E . We define a group homomorphism

$$\Psi : \mathfrak{C}(\mathfrak{f}(p)^\infty) \rightarrow \mathcal{O}_E[\text{Gal}(L_\infty^-/L)]^\times$$

to be $\Psi(a) = \chi(\text{pr}_{\mathfrak{f}}(a))[\text{pr}_-(a)]$ for $a \in \mathfrak{C}(\mathfrak{f}(p)^\infty)$. Let $J_{\mathfrak{f}(p)}$ be the group which consists of fractional ideals $\mathfrak{a}$ of L which is prime to $\mathfrak{f}(p)$. For each finite prime ideal $\mathfrak{l}$, we denote by $L_{\mathfrak{l}}$ the completion of L by $\mathfrak{l}$. Let $\mathcal{O}_{L_{\mathfrak{l}}}$ be the integers of $L_{\mathfrak{l}}$ and $\pi_{\mathfrak{l}}$ a generator of the maximal ideal of $\mathcal{O}_{L_{\mathfrak{l}}}$. We define a group homomorphism

$$\Psi^* : J_{\mathfrak{f}(p)} \rightarrow \mathcal{O}_E[\text{Gal}(L_\infty^-/L)]^\times$$

to be $\Psi^*(\mathfrak{a}) = \prod_{l \nmid f(p)} \Psi_l(\pi_l^{m_l})$, where $\Psi = \prod_l \Psi_l$ and $\mathfrak{a} = \prod_{l \nmid f(p)} l^{m_l}$. We put

$$F_\Psi = \sum_{\mathfrak{a} \nmid f(p)} \Psi^*(\mathfrak{a}) q^{N(\mathfrak{a})},$$

where $\mathfrak{a}$ runs through integral ideals of L which are prime to $f(p)$. Let $\epsilon : \text{Gal}(L_\infty^-/L) \rightarrow \overline{\mathbb{Q}}^\times$ be a finite character. We denote by $P_\epsilon : \mathcal{O}_E[\text{Gal}(L_\infty^-/L)] \rightarrow \overline{\mathbb{Q}}_p$ the $\mathcal{O}_E$ -algebra homomorphism defined by $P_\epsilon([w]) = \epsilon(w)$ for $w \in \text{Gal}(L_\infty^-/L)$. It is known that for each finite character $\epsilon : \text{Gal}(L_\infty^-/L) \rightarrow \overline{\mathbb{Q}}^\times$, the series $f_\epsilon := P_\epsilon(F_\Psi) \in P_\epsilon(\mathcal{O}_E[\text{Gal}(L_\infty^-/L)])[q]$ is the Fourier expansion of a classical modular form of weight 1 and level $(-D)N(\mathfrak{f})p^{e_\epsilon}$, where e_ϵ is a positive integer (cf. [Miy06, Theorem 4.8.2]). By the definition, f_ϵ is the CM-form. We remark that the p -th coefficient $a(p, F_\Psi) \in \mathcal{O}_E[\text{Gal}(L_\infty^-/L)]$ of F_Ψ is zero by the definition. However, if $\epsilon : \text{Gal}(L_\infty^-/L) \rightarrow \overline{\mathbb{Q}}^\times$ is primitive and the conductor is sufficiently large, it is known that f_ϵ is a primitive form (cf. [Miy06, Theorem 4.8.2]). Then, if we put $\mathfrak{X} := \{\text{Ker } P_\epsilon \mid f_\epsilon \text{ is primitive}\}$, the cardinality of $\mathfrak{X}$ is not finite, and the triple $(\mathcal{O}_E[\text{Gal}(L_\infty^-/L)], \mathfrak{X}, F_\Psi)$ satisfies the condition (6). Further, it is not difficult to prove that the triple $(\mathcal{O}_E[\text{Gal}(L_\infty^-/L)], \mathfrak{X}, F_\Psi)$ satisfies the condition (5). Let $\text{pr}_{\mathbb{A}^\times} : \mathbb{A}^\times \rightarrow \mathfrak{C}(f(p)^\infty)$ be the natural projection defined by the class field theory. We denote by $j_p : \mathbb{Q}_p^\times \hookrightarrow \mathbb{A}^\times$ the natural injection. If we put $\langle n \rangle = n\omega_p(n)^{-1}\Psi([\text{pr}_{\mathbb{A}^\times} \circ j_p(n\omega_p(n)^{-1})])^{-1} \in \mathcal{O}_E[\text{Gal}(L_\infty^-/L)]^\times$ for each positive integer n which is prime to p , $\langle n \rangle$ satisfies the condition of (5). Since $DN(\mathfrak{f})$ is square-free, by [Miy06, Theorem 4.6.17], F_Ψ satisfies Hypothesis (7).

As a second example of $(\mathbf{I}_i, \mathfrak{X}^{(i)}, G^{(i)})$, we give Coleman families. For an element $x \in K$ and $\epsilon \in p^\mathbb{Q}$, we denote by $\mathcal{B}[x, \epsilon]_K$ the closed ball of radius ϵ and center x , seen as a K -affinoid space. We denote by $\mathcal{A}_{\mathcal{B}[x, \epsilon]_K}$ the ring of analytic functions on $\mathcal{B}[x, \epsilon]_K$ and by $\mathcal{A}_{\mathcal{B}[x, \epsilon]_K}^0$ the subring of power bounded elements of $\mathcal{A}_{\mathcal{B}[x, \epsilon]_K}$. We remark that if $\epsilon \in K$, the ring $\mathcal{A}_{\mathcal{B}[x, \epsilon]_K}^0$ is isomorphic to the ring

$$\mathcal{O}_K \langle \epsilon^{-1}(T - x) \rangle = \left\{ \sum_{n \geq 0} a_n (\epsilon^{-1}(T - x))^n \in \mathcal{O}_K[[\epsilon^{-1}(T - x)]] \mid \lim_{n \rightarrow \infty} |a_n|_p = 0 \right\}.$$

Let M be a positive integer which is prime to p and square-free. Let ϵ_M be a Dirichlet character mod M . Let f be a p -stabilized newform of weight k_0 , level Mp , slope $\alpha < k_0 - 1$ and Nebentypus $\epsilon_M \omega_p^{i-k_0}$ where $0 \leq i \leq p-1$. Further, we assume that $a(p, f)^2 \neq \epsilon_M(p)p^{k_0-1}$ if $i = 0$. Then, by Coleman in [Col97], there exists an element $\epsilon \in p^\mathbb{Q} \cap K$ and a series

$$G \in \mathcal{A}_{\mathcal{B}[k_0, \epsilon]_K}^0[[q]]$$

such that the specialization $G(k)$ of G at k is the Fourier expansion of a normalized Hecke eigenform of weight k , level Mp , slope α and Nebentypus $\epsilon_M \omega_p^{i-k}$ for each positive integer $k \in \mathcal{B}[k_0, \epsilon]_K(K)$ which is greater than $\alpha + 1$. Further, we prove in [Fuk19, A2.7] that we can take a sufficiently small ϵ such that $G(k)$ is a p -stabilized newform for each positive integer $k \in \mathcal{B}[k_0, \epsilon]_K(K)$ which is greater than $\alpha + 1$. If we put $X = \epsilon^{-1}(T - k_0)$, we can regard the Coleman series G as a series $G(X)$ in $\mathcal{O}_K[[X]]$. Let $k \in \mathcal{B}[k_0, \epsilon]_K(K)$ be a positive integer which is greater than $\alpha + 1$. If we put $b_k = \epsilon^{-1}(k - k_0)$, $G(b_k)$ is the Fourier expansion of a p -stabilized newform of weight k , level Mp , slope α and Nebentypus $\epsilon_M \omega_p^{i-k}$. We denote by $P_k : \mathcal{O}_K[[X]] \rightarrow K$ the continuous $\mathcal{O}_K$ -algebra homomorphism defined by $P_k(X) = b_k$. We define $\mathfrak{X}$ to be the set consisting of P_k for each positive integer $k \in \mathcal{B}[k_0, \epsilon]_K(K)$ which is greater than $\alpha + 1$. Then, the triple $(\mathcal{O}_K[[X]], \mathfrak{X}, G(X))$ satisfies Hypothesis (6). We check that the triple $(\mathcal{O}_K[[X]], \mathfrak{X}, G(X))$ satisfies Hypothesis (5). Let $\exp(x)$ and $\log(x)$ be the formal exponential

series and log series in $K[[x]]$ defined by

$$\exp(x) = \sum_{n \geq 0} \frac{1}{n!} x^n,$$

$$\log(x) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} x^n.$$

We fix an isomorphism $\Lambda_K \cong \mathcal{O}_K[[X]]$ defined by $[1+p] \mapsto X+1$ and we define a formal series

$$\langle n \rangle' := \langle n \rangle_{\Lambda_K} ((1+p)^{k_0} \exp(\epsilon X \log(1+p)) - 1)$$

for each positive integer n which is prime to p . We remark that since we have $|p^m|_p \leq |m!|_p$ for each positive integer m , the series $\langle n \rangle'$ is contained in $\mathcal{O}_K[[X]]$. Further, for each positive integer n which is prime to p , the series $\langle n \rangle'$ satisfies the condition of Hypothesis (5). Since M is square-free, by [Miy06, Theorem 4.6.17], $G(X)$ satisfies Hypothesis (7).

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