

Bilinear estimates and critical Sobolev inequality in BMO , with applications to the Navier-Stokes and the Euler equations

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Introduction.

In this paper we prove that the BMO norm of the velocity and the vorticity controls the blow-up phenomena of smooth solutions to the Navier-Stokes and the Euler equations. Our result is applied to the criterion on regularity of weak solutions to the Navier-Stokes equations.

We consider the Navier-Stokes and the Euler equations in $\mathbf{R}^n$, $n \geq 3$:

$$(N-S) \quad \begin{cases} \frac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = 0, & \operatorname{div} u = 0 \quad \text{in } x \in \mathbf{R}^n, t > 0, \\ u|_{t=0} = a, \end{cases}$$

$$(E) \quad \begin{cases} \frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = 0, & \operatorname{div} u = 0 \quad \text{in } x \in \mathbf{R}^n, t > 0, \\ u|_{t=0} = a \end{cases}$$

where $u = (u^1(x, t), u^2(x, t), \dots, u^n(x, t))$ and $p = p(x, t)$ denote the unknown velocity vector and the unknown pressure of the fluid at the point $(x, t) \in \mathbf{R}^n \times (0, \infty)$, respectively, while $a = (a^1(x), a^2(x), \dots, a^n(x))$ is the given initial velocity vector.

It is proved by Fujita-Kato [10] that for every $a \in H_\sigma^s \equiv \{v \in H^s; \operatorname{div} v = 0\}$ with $s > n/2 - 1$, there exist $T > 0$ and a unique solution $u(t)$ of (N-S) on $[0, T)$ in the class

$$(CN)_s \quad u \in C([0, T); H_\sigma^s) \cap C^1((0, T); H^s) \cap C((0, T); H^{s+2}).$$

Concerning the Euler equations, Kato-Lai [15] and Kato-Ponce [16] proved that for every $a \in W_\sigma^{s,p}$ for $s > n/p + 1$, $1 < p < \infty$, there are $T > 0$ and a unique solution u of (E) on the interval $[0, T)$ in the class

$$(CE)_{s,p} \quad u \in C([0, T); W_\sigma^{s,p}) \cap C^1([0, T); W_\sigma^{s-2,p}),$$

where subindex σ means the divergence free. It is an interesting question whether the solution $u(t)$ really blows up as $t \uparrow T$. Giga [11] showed that if the strong solution u in $(CN)_s$ satisfies

$$(Se) \quad \int_0^T \|u(t)\|_{L^r}^\kappa dt < \infty \quad \text{for } 2/\kappa + n/r = 1 \text{ with } n < r \leq \infty,$$

then u can be continued to the solution in the class $(CN)_s$ beyond $t = T$. Concerning the Euler equations, Beale-Kato-Majda [1] dealt with the vorticity $\omega = \text{rot } u$ and proved that under the condition

$$\int_0^T \|\omega(t)\|_{L^\infty} dt < \infty$$

$u(t)$ can never break down its regularity at $t = T$. To prove this assertion, in [1] they made use of the logarithmic inequality such as

$$(0.1) \quad \|\nabla u\|_{L^\infty} \leq C(1 + \|\text{rot } u\|_{L^\infty}(1 + \log^+ \|u\|_{W^{s+1,p}}) + \|\text{rot } u\|_{L^2}), \quad sp > n$$

for all vector functions u with $\text{div } u = 0$, where $\log^+ a = \log a$ if $a \geq 1$, $= 0$ if $0 < a < 1$.

The purpose of this paper is to extend these results to the marginal space BMO which is larger than L^∞ .

1 Results.

Before stating our results, we introduce some function spaces. Let $C_{0,\sigma}^\infty$ denote the set of all C^∞ vector functions $\phi = (\phi^1, \phi^2, \dots, \phi^n)$ with compact support in $\mathbf{R}^n$, such that $\text{div } \phi = 0$. L_σ^r is the closure of $C_{0,\sigma}^\infty$ with respect to the L^r -norm $\|\cdot\|_r$; $(\cdot, \cdot)$ denotes the duality pairing between L^r and $L^{r'}$, where $1/r + 1/r' = 1$. L^r stands for the usual (vector-valued) L^r -space over $\mathbf{R}^n$, $1 \leq r \leq \infty$. H_σ^s denotes the closure of $C_{0,\sigma}^\infty$ with respect to the H^s -norm $\|\phi\|_{H^s} = \|(1 - \Delta)^{\frac{s}{2}} \phi\|_2$, $s \geq 0$.

Our result on continuation of strong solutions of (N-S) now reads:

Theorem 1 *Let $s > n/2 - 1$ and let $a \in H_\sigma^s$. Suppose that u is the strong solution of (N-S) in the class $(CN)_s$ on $(0, T)$. If*

$$(1.1) \quad \int_{\varepsilon_0}^T \|u(t)\|_{BMO}^2 dt < \infty \quad \text{for some } 0 < \varepsilon_0 < T,$$

then u can be continued to the strong solution in the class $(CN)_s$ on $(0, T')$ for some $T' > T$.

Corollary 1 *Let u be the strong solution of (N-S) in the class $(CN)_s$ on $(0, T)$ for $s > n/2 - 1$. Suppose that T is maximal, i.e., u cannot be continued in the class $(CN)_s$ on $(0, T')$ for any $T' > T$. Then*

$$(1.2) \quad \int_\varepsilon^T \|u(t)\|_{BMO}^2 dt = \infty \quad \text{for all } 0 < \varepsilon < T.$$

For the space BMO , we refer to Stein [24]. Since $s > n/2 - 1$, there holds $H^{s+2} \subset BMO$, and hence for every u in the class $(CN)_s$ on $(0, T)$, we have $u \in C((0, T); BMO)$.

We next consider a criterion on uniqueness and regularity of weak solutions to (N-S). Our definition of a weak solution is as follows.

Definition 1. Let $a \in L^2_\sigma$. A measurable function u on $\mathbb{R}^n \times (0, T)$ is called a weak solution of (N-S) on $(0, T)$ if

- (i) $u \in L^\infty(0, T; L^2_\sigma) \cap L^2(0, T; H^1_\sigma)$;
- (ii) $u(t)$ is continuous on $[0, T]$ in the weak topology of L^2_σ ;
- (iii)

$$(1.3) \quad \int_s^t \{-(u, \partial_\tau \Phi) + (\nabla u, \nabla \Phi) + (u \cdot \nabla u, \Phi)\} d\tau = -(u(t), \Phi(t)) + (u(s), \Phi(s))$$
 for every $0 \leq s \leq t < T$ and every $\Phi \in H^1((s, t); H^1_\sigma \cap L^n)$.

Our result on weak solutions of (N-S) now reads:

Theorem 2 (1) (uniqueness) Let $a \in L^2_\sigma$ and let u, v be two weak solutions of (N-S) on $(0, T)$. Suppose that

$$(1.4) \quad u \in L^2(0, T; BMO)$$

and that v satisfies the energy inequality

$$(1.5) \quad \|v(t)\|_2^2 + 2 \int_0^t \|\nabla v\|_2^2 d\tau \leq \|a\|_2^2, \quad 0 < t < T.$$

Then we have $u \equiv v$ on $[0, T]$.

(2) (regularity) Let $a \in L^2_\sigma$ and let u be a weak solution with the additional property (1.4). Then for every $0 < \varepsilon < T$, u is actually a strong solution of (N-S) on (ε, T) in the class $(CN)_s$ for $s > n/2 - 1$.

Remark. Theorem 2 may be regarded as an extension of Serrin's criterion [22], [23] on uniqueness and regularity of weak solutions u in the class

$$(1.6) \quad u \in L^\kappa(0, T; L^r) \quad \text{for } 2/\kappa + n/r = 1 \text{ with } n < r \leq \infty.$$

Our class (1.4) is larger than the marginal case $L^2(0, T; L^\infty)$ in (1.6). Moreover, by virtue of the estimate $\|u\|_{BMO} \leq C\|\nabla u\|_{M^n}$ of John-Nirenberg [13], we see that the weak solution u with $\nabla u \in L^2(0, T; M^n)$ becomes regular, where M^n denotes the Morrey space which is larger than L^n . See Beirão da Veiga [2].

We shall next investigate continuation of the strong solution in terms of the vorticity $\omega = \text{rot } u \equiv (\partial_j u^k - \partial_k u^j)_{1 \leq j, k \leq n}$ and the deformation tensor $\text{Def } u \equiv (\partial_j u^k + \partial_k u^j)_{1 \leq j, k \leq n}$.

Theorem 3 Let $s > n/2 - 1$. Suppose that u is the strong solution of (N-S) in the class $(CN)_s$ on $(0, T)$. If either

$$(1.7) \quad \int_{\varepsilon_0}^T \|\omega(t)\|_{BMO} dt < \infty$$

or

$$(1.8) \quad \int_{\varepsilon_0}^T \|\text{Def } u(t)\|_{BMO} dt < \infty$$

holds for some $0 < \varepsilon_0 < T$, then u can be continued to the strong solution in the class $(CN)_s$ on $(0, T')$ for some $T' > T$.

Corollary 2 Suppose that u is the strong solution of (N-S) in the class $(CN)_s$ on $(0, T)$ for $s > n/2 - 1$. Assume that T is maximal in the same sense as in Corollary 1. Then both

$$(1.9) \quad \int_{\varepsilon}^T \|\omega(t)\|_{BMO} dt = \infty \text{ and } \int_{\varepsilon}^T \|\text{Def } u(t)\|_{BMO} dt = \infty$$

hold for all $0 < \varepsilon < T$.

Theorem 3 yields the following regularity criterion on weak solutions of (N-S) by mean of $\text{rot } u$ and $\text{Def } u$.

Theorem 4 Let $a \in L^2_{\sigma}$. Suppose that u is a weak solution of (N-S) on $(0, T)$. If either

$$(1.10) \quad \omega \in L^1(0, T; BMO) \quad \text{or} \quad \text{Def } u \in L^1(0, T; BMO)$$

holds, then for every $0 < \varepsilon < T$, u is actually a strong solution of (N-S) in the class $(CN)_s$ on (ε, T) for $s > n/2 - 1$.

Remark. Beirão da Veiga [2] proved the regularity criterion in the class $\nabla u \in L^{\kappa}(0, T; L^r)$ for $2/\kappa + n/r = 2$ with $1 < \kappa < \infty$, $n/2 < r < \infty$. Theorem 4 covers the borderline case $\kappa = 1$ and $r = \infty$.

Our result on (E) reads as follows.

Theorem 5 Let $1 < p < \infty$, $s > n/p + 1$. Suppose that u is the solution of (E) in the class $(CE)_{s,p}$ on $(0, T)$. If either

$$(1.11) \quad \int_0^T \|\omega(t)\|_{BMO} dt (\equiv M_0) < \infty$$

or

$$(1.12) \quad \int_0^T \|\text{Def } u(t)\|_{BMO} dt (\equiv M_1) < \infty$$

holds, then u can be continued to the solution in the class $(CE)_{s,p}$ on $(0, T')$ for some $T' > T$.

Corollary 3 Let u be the solution of (E) in the class $(CE)_{s,p}$ on $(0, T)$ for $1 < p < \infty$, $s > n/p + 1$. Assume that T is maximal, i.e., u cannot be continued to the solution in the class $(CE)_{s,p}$ on $(0, T')$ for any $T' > T$. Then both

$$\int_0^T \|\text{rot } u(t)\|_{BMO} dt = \infty \quad \text{and} \quad \int_0^T \|\text{Def } u(t)\|_{BMO} dt = \infty$$

hold.

2 Bilinear estimates and critical Sobolev inequality in BMO .

In this section we shall prepare some lemmas. In what follows we shall denote by C various constants. In particular, $C = C(*, \dots, *)$ denotes constants depending only on the quantities appearing in the parenthesis.

We first prove the following key estimate.

Lemma 2.1 (Bilinear estimates) *Let $1 < r < \infty$. Then we have*

$$(i) \quad (2.1) \quad \|f \cdot \nabla g\|_r \leq C(\|f\|_r \|(-\Delta)^{\frac{1}{2}} g\|_{BMO} + \|(-\Delta)^{\frac{1}{2}} f\|_{BMO} \|g\|_r)$$

for all $f, g \in W^{1,r}$ with $\nabla f, \nabla g \in BMO$ with $C = C(n, r)$.

(ii) Let $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$ be multi-indices with $|\alpha| = \alpha_1 + \dots + \alpha_n \geq 1$ and $|\beta| = \beta_1 + \dots + \beta_n \geq 1$. Then

$$(2.2) \quad \|\partial^\alpha f \cdot \partial^\beta g\|_2 \leq C(\|f\|_{BMO} \|(-\Delta)^{\frac{|\alpha|+|\beta|}{2}} g\|_2 + \|(-\Delta)^{\frac{|\alpha|+|\beta|}{2}} f\|_2 \|g\|_{BMO})$$

for all $f, g \in BMO \cap H^{|\alpha|+|\beta|}$ with $C = C(n, \alpha, \beta)$, where $\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$.

The proof of this lemma is based on the following proposition due to Coifman-Meyer [6, Chapter V. Proposition 2].

Proposition 2.1 (Coifman-Meyer) *Let $\sigma = \sigma(\xi, \eta) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n \setminus \{(0, 0)\})$ satisfy*

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi, \eta)| \leq C(|\xi| + |\eta|)^{-|\alpha|-|\beta|}, \quad (\xi, \eta) \in \mathbf{R}^n \times \mathbf{R}^n \setminus \{(0, 0)\}$$

for all multi-indices α, β with $C = C(\alpha, \beta)$. Suppose that

$$\sigma(\xi, 0) = 0.$$

Then the bilinear operator $\sigma(D)(\cdot, \cdot)$ defined by

$$(2.3) \quad \sigma(D)(f, g)(x) \equiv \iint_{\mathbf{R}^n \times \mathbf{R}^n} e^{ix \cdot (\xi + \eta)} \sigma(\xi, \eta) \widehat{f}(\xi) \widehat{g}(\eta) d\xi d\eta, \quad x \in \mathbf{R}^n$$

satisfies

$$(2.4) \quad \|\sigma(D)(f, g)\|_p \leq C\|f\|_p \|g\|_{BMO} \quad (1 < p < \infty)$$

with $C = C(n, p)$.

Proof of Lemma 2.1. Here we prove only (2.2). The proof of (2.1) is similar to that of (2.2). Let Φ_1 be a C^∞ -function on $[0, \infty)$ such that $\text{supp } \Phi_1 \subset [0, 1]$, $0 \leq \Phi_1 \leq 1$, $\Phi_1(t) \equiv 1$ for $0 \leq t \leq 1/2$, and let $\Phi_2 = 1 - \Phi_1$. Then we have

$$\begin{aligned} & \partial^\alpha f(x) \partial^\beta g(x) \\ &= C \iint_{\mathbf{R}^n \times \mathbf{R}^n} e^{ix \cdot (\xi + \eta)} \xi^\alpha \eta^\beta \widehat{f}(\xi) \widehat{g}(\eta) d\xi d\eta \end{aligned}$$

$$\begin{aligned}
&= C \iint_{\mathbf{R}^n \times \mathbf{R}^n} e^{ix \cdot (\xi + \eta)} \times \\
&\quad \left(\frac{\xi^\alpha \eta^\beta}{|\eta|^{|\alpha|+|\beta|}} \Phi_1(|\xi|/|\eta|) \widehat{f}(\xi) |\eta|^{|\alpha|+|\beta|} \widehat{g}(\eta) + \frac{\xi^\alpha \eta^\beta}{|\xi|^{|\alpha|+|\beta|}} \Phi_2(|\xi|/|\eta|) |\xi|^{|\alpha|+|\beta|} \widehat{f}(\xi) \widehat{g}(\eta) \right) d\xi d\eta \\
&= C \left(\sigma_1(D)(f, (-\Delta)^{\frac{|\alpha|+|\beta|}{2}} g)(x) + \sigma_2(D)((-\Delta)^{\frac{|\alpha|+|\beta|}{2}} f, g)(x) \right),
\end{aligned}$$

where

$$\sigma_1(\xi, \eta) = \frac{\xi^\alpha \eta^\beta}{|\eta|^{|\alpha|+|\beta|}} \Phi_1(|\xi|/|\eta|), \quad \sigma_2(\xi, \eta) = \frac{\xi^\alpha \eta^\beta}{|\xi|^{|\alpha|+|\beta|}} \Phi_2(|\xi|/|\eta|).$$

Since $|\alpha| \geq 1$ and $|\beta| \geq 1$, we see that

$$\sigma_1(0, \eta) = 0, \quad \sigma_2(\xi, 0) = 0$$

and that σ_1 and σ_2 satisfy the hypotheses of Proposition 2.1. Hence there holds

$$\begin{aligned}
\|\sigma_1(D)(f, (-\Delta)^{\frac{|\alpha|+|\beta|}{2}} g)\|_2 &\leq C \|f\|_{BMO} \|(-\Delta)^{\frac{|\alpha|+|\beta|}{2}} g\|_2, \\
\|\sigma_2(D)((-\Delta)^{\frac{|\alpha|+|\beta|}{2}} f, g)\|_2 &\leq C \|(-\Delta)^{\frac{|\alpha|+|\beta|}{2}} f\|_2 \|g\|_{BMO},
\end{aligned}$$

which yields (2.2). This proves Lemma 2.1.

The next lemma plays an important role to show the energy identity of weak solutions in the class (1.4) and (1.10).

Lemma 2.2 (i) Let $w \in L^\infty(0, T; L_\sigma^2) \cap L^2(0, T; H_\sigma^1)$ and $u \in L^2(0, T; H_\sigma^1 \cap BMO)$. Then we have

$$(2.5) \quad \int_0^T (w \cdot \nabla u, u) d\tau = 0.$$

(ii) Let $w, u \in L^\infty(0, T; L_\sigma^2) \cap L^2(0, T; H_\sigma^1)$. Suppose that either

$$\text{rot } w, \text{rot } u \in L^1(0, T; BMO)$$

or

$$\text{Def } w, \text{Def } u \in L^1(0, T; BMO)$$

holds. Then we have

$$(2.6) \quad \int_0^T (w \cdot \nabla u, u) d\tau = 0.$$

To prove (2.5), we use the estimate of Coifman-Lions-Meyer-Semmes [5]:

$$(2.7) \quad w \cdot \nabla u \in \mathcal{H}^1 \quad \text{with } \|w \cdot \nabla u\|_{\mathcal{H}^1} \leq C \|w\|_2 \|\nabla u\|_2,$$

where $\mathcal{H}^1$ denotes the Hardy space on $\mathbf{R}^n$. For detail, see [5].

To prove (2.6), we use (2.1) and the Biot-Savart law. Indeed, by the Biot-Savart law, we have the representation

$$(2.8) \quad \frac{\partial u}{\partial x_j} = R_j(R \times \omega), \quad j = 1, \dots, n, \quad \text{where } \omega = \text{rot } u;$$

$$(2.9) \quad \frac{\partial u^l}{\partial x_j} = R_j \left(\sum_{k=1}^n R_k \text{Def } u_{kl} \right), \quad j, l = 1, \dots, n, \quad \text{where } \text{Def } u_{kl} = \frac{\partial u^k}{\partial x_l} - \frac{\partial u^l}{\partial x_k}.$$

Here $R = (R_1, \dots, R_n)$, and $R_j = \frac{\partial}{\partial x_j}(-\Delta)^{-\frac{1}{2}}$ denote the Riesz transforms. Since R is a bounded operator in BMO , we have by (2.8), (2.9) and assumption that

$$(2.10) \quad \nabla u, \nabla w \in L^1(0, T; BMO).$$

It follows from Lemma 2.1 (2.1) and (2.10) that $\int_0^T (w \cdot \nabla u, u) d\tau$ is well-defined. For details of the proof of Lemma 2.2 we refer to [17].

Using the usual mollifier argument, by Lemma 2.2, we have the following energy identity for weak solutions with (1.4) or (1.10).

Lemma 2.3 *Let $n \geq 3$ and let $a \in L_\sigma^2$. Suppose that u is a weak solution of (N-S) on $(0, T)$ satisfying one of the additional conditions (1.4) and (1.10). Then u fulfills the energy identity*

$$(2.11) \quad \|u(t)\|_2^2 + 2 \int_s^t \|\nabla u\|_2^2 d\tau = \|u(s)\|_2^2 \quad \text{for all } 0 \leq s \leq t < T.$$

Now we prove the following lemma which is an extension of (0.1).

Lemma 2.4 (Critical Sobolev Inequality) *Let $1 < p < \infty$ and let $s > n/p$. There is a constant $C = C(n, p, s)$ such that the estimate*

$$(2.12) \quad \|f\|_\infty \leq C (1 + \|f\|_{BMO} (1 + \log^+ \|f\|_{W^{s,p}}))$$

holds for all $f \in W^{s,p}$.

Remark. Compared with (0.1), we do not need to add $\|f\|_{L^2}$ to the right hand side of (2.12). This makes it easier to derive an apriori estimate of solutions to the Euler equations than Beale-Kato-Majda [1].

Proof of Lemma 2.4.

We shall make use of the Littlewood-Paley decomposition; there exists a non-negative function $\varphi \in \mathcal{S}$ ($\mathcal{S}$; the Schwartz class) such that $\text{supp } \varphi \subset \{2^{-1} \leq |\xi| \leq 2\}$ and such that $\sum_{k=-\infty}^{\infty} \varphi(2^{-k}\xi) = 1$ for $\xi \neq 0$. See Bergh-Löfström [3, Lemma 6.1.7]. Let us define ϕ_0 and ϕ_1 as

$$\phi_0(\xi) = \sum_{k=1}^{\infty} \varphi(2^k \xi) \quad \text{and} \quad \phi_1(\xi) = \sum_{k=-\infty}^{-1} \varphi(2^k \xi),$$

respectively. Then we have that $\phi_0(\xi) = 1$ for $|\xi| \leq 1/2$, $\phi_0(\xi) = 0$ for $|\xi| \geq 1$ and that $\phi_1(\xi) = 0$ for $|\xi| \leq 1$, $\phi_1(\xi) = 1$ for $|\xi| \geq 2$. It is easy to see that for every positive integer N there holds the identity

$$(2.13) \quad \phi_0(2^N \xi) + \sum_{k=-N}^N \varphi(2^k \xi) + \phi_1(2^{-N} \xi) = 1, \quad \xi \neq 0.$$

Since C_0^∞ is dense in $W^{s,p}$ and since $W^{s,p}$ is continuously embedded in BMO , implied by $s > n/p$, it suffices to prove (2.12) for $f \in C_0^\infty$. For such f we have the representation

$$f(x) = \int_{y \in \mathbf{R}^n} K(x-y) \cdot \nabla f(y) dy \quad \text{with} \quad K(y) = \frac{1}{n\omega_n} \frac{y}{|y|^n},$$

for all $x \in \mathbf{R}^n$, where ω_n denotes the volume of the unit ball in $\mathbf{R}^n$. By (2.13) we decompose f into three parts:

$$\begin{aligned}
 f(x) &= \int_{y \in \mathbf{R}^n} K(x-y) \times \\
 &\quad \times \left(\phi_0(2^N(x-y)) + \sum_{k=-N}^N \varphi(2^{-k}(x-y)) + \phi_1(2^{-N}(x-y)) \right) \cdot \nabla f(y) dy \\
 (2.14) \quad &\equiv f_0(x) + g(x) + f_1(x)
 \end{aligned}$$

for all $x \in \mathbf{R}^n$.

We can show that

$$(2.15) \quad |f_0(x)| \leq C 2^{-\beta N} \|f\|_{W^{s,p}}$$

for all $x \in \mathbf{R}^n$, where $\beta = \beta(n, p, s)$ is a positive constant. For detail, see [18].

By integration by parts we have

$$g(x) = \sum_{k=-N}^N (\operatorname{div} \Psi)_{2^k} * f(x), \quad x \in \mathbf{R}^n,$$

where $\Psi(x) = K(x)\varphi(x)$ and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. Since $\Psi \in \mathcal{S}$ with the property that

$$\int_{\mathbf{R}^n} \operatorname{div} \Psi(x) dx = 0,$$

it follows from Stein [24, Chap. IV, 4.3.3] that

$$\begin{aligned}
 \|g\|_\infty &\leq \sum_{k=-N}^N \|(\operatorname{div} \Psi)_{2^k} * f\|_\infty \\
 &\leq \sum_{k=-N}^N \sup_{t>0} \|(\operatorname{div} \Psi)_t * f\|_\infty \\
 (2.16) \quad &\leq CN \|f\|_{BMO},
 \end{aligned}$$

where $C = C(n)$ is independent of N .

Integrating by parts, we have by a direct calculation

$$\begin{aligned}
 |f_1(x)| &= \left| \int_{y \in \mathbf{R}^n} \operatorname{div}_y \left(K(x-y) \phi_1(2^{-N}(x-y)) \right) f(y) dy \right| \\
 (2.17) \quad &\leq C 2^{-N \cdot \frac{n}{p}} \|f\|_p
 \end{aligned}$$

for all $x \in \mathbf{R}^n$, where $C = C(n, p)$ is independent of N .

Now it follows from (2.14) and (2.15)-(2.17) that

$$(2.18) \quad \|f\|_\infty \leq C(2^{-\gamma N} \|f\|_{W^{s,p}} + N \|f\|_{BMO})$$

with $\gamma = \min\{\beta, n/p\}$, where $C = C(n, s, p)$ is independent of N and f . If $\|f\|_{W^{s,p}} \leq 1$, then we may take $N = 1$; otherwise, we take N so large that the first term of the right hand

side of (2.18) is dominated by 1, i.e., $N \equiv \left\lceil \frac{\log \|f\|_{W^{s,p}}}{\gamma \log 2} \right\rceil + 1$ ($\lceil \cdot \rceil$; Gauss symbol) and (2.18) becomes

$$\|f\|_{\infty} \leq C \left\{ 1 + \|f\|_{BMO} \left(\frac{\log \|f\|_{W^{s,p}}}{\gamma \log 2} + 1 \right) \right\}.$$

In both cases, (2.12) holds. This proves Lemma 2.4.

3 Proof of Theorems 1-4

3.1 Proof of Theorem 1.

It is proved by Kato [14] and Giga [11] that, for the initial data $a \in H^s$ with $s > n/2 - 1$, the local existence time interval T of the strong solution u of (N-S) in the class $(CN)_s$ can be estimated from below as

$$(3.1) \quad T \geq \frac{C}{\|a\|_{H^s}^{\frac{2}{s-(n/2-1)}}},$$

where $C = C(n, s)$. Actually, for $a \in L^r$ with $r > n$, Giga [11, Theorem 1 (ii)] gave T in such a way that

$$(3.2) \quad T = \frac{C}{\|a\|_r^{2r/(r-n)}},$$

so from the continuous embedding $H^s \subset L^r$ for $1/r = 1/2 - s/n$, we obtain (3.1). Hence by the standard argument of continuation of local solutions, it suffices to prove the following apriori estimate

$$(3.3) \quad \sup_{\varepsilon_0 < t < T} \|u(t)\|_{H^{[s]+1}} \leq \|u(\varepsilon_0)\|_{H^{[s]+1}} \exp \left(C \int_{\varepsilon_0}^T \|u\|_{BMO}^2 dt \right),$$

where $C = C(n, s)$ is independent of T .

Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a multi-index with $|\alpha| \equiv \alpha_1 + \dots + \alpha_n \leq [s] + 1$, and let $v = \partial^{\alpha} u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$. Applying ∂^{α} to (N-S), we have for v the equation

$$(3.4) \quad \frac{\partial v}{\partial t} - \Delta v + u \cdot \nabla v + \nabla q = F, \quad \varepsilon_0 < t < T,$$

where $q = \partial^{\alpha} p$ and

$$(3.5) \quad F = - \sum_{|\beta| \leq |\alpha|-1} \alpha C_{\beta} \partial^{\alpha-\beta} u \cdot \nabla (\partial^{\beta} u).$$

Taking the inner product in L^2 between (3.4) and v , and then integrating the result identity on the time interval (ε_0, t) , we obtain

$$(3.6) \quad \|v(t)\|_2^2 + 2 \int_{\varepsilon_0}^t \|\nabla v\|_2^2 d\tau \leq \|v(\varepsilon_0)\|_2^2 + 2 \int_{\varepsilon_0}^t \|F\|_2 \|v\|_2 d\tau.$$

On the other hand, by (2.2), we have

$$\|F\|_2 \leq C \|u\|_{BMO} \|(-\Delta)^{\frac{|\alpha|+1}{2}} u\|_2,$$

from which and (3.6) it follows that

$$\begin{aligned} & \|\partial^\alpha u(t)\|_2^2 + 2 \int_{\varepsilon_0}^t \|\nabla(\partial^\alpha u)\|_2^2 d\tau \\ & \leq \|\partial^\alpha u(\varepsilon_0)\|_2^2 + \int_{\varepsilon_0}^t \|(-\Delta)^{\frac{|\alpha|+1}{2}} u\|_2^2 d\tau + C \int_{\varepsilon_0}^t \|u\|_{BMO}^2 \|\partial^\alpha u\|_2^2 d\tau \end{aligned}$$

with C independent of t . Summing over α with $0 \leq |\alpha| \leq [s] + 1$, we have

$$\|u(t)\|_{H^{[s]+1}}^2 \leq \|u(\varepsilon_0)\|_{H^{[s]+1}}^2 + C \int_{\varepsilon_0}^t \|u\|_{BMO}^2 \|u\|_{H^{[s]+1}}^2 d\tau$$

for all $\varepsilon_0 \leq t < T$. Now the Gronwall inequality yields (3.3). This proves Theorem 1.

3.2 Proof of Theorem 2.

(1) Let us first prove uniqueness. We follow the argument of Masuda [20, Theorems 2, 3]. We can show that

$$(3.7) \quad \int_0^t \{2(\nabla u, \nabla v) + (v \cdot \nabla v, u) - (u \cdot \nabla v, u)\} d\tau = -(u(t), v(t)) + \|a\|_2^2.$$

See Masuda [20, p.640 (4.4)]. By Lemma 2.3, u satisfies the energy identity

$$(3.8) \quad \|u(t)\|_2^2 + 2 \int_0^t \|\nabla u\|_2^2 d\tau = \|a\|_2^2.$$

Addition of (3.7) (multiplied by -2), (3.8) and (1.5) yields

$$(3.9) \quad \|w(t)\|_2^2 + 2 \int_0^t \|\nabla w\|_2^2 d\tau \leq 2 \int_0^t (w \cdot \nabla v, u) d\tau = 2 \int_0^t (w \cdot \nabla w, u) d\tau,$$

where $w = v - u$. In the last identity, we have used (2.5). By (2.7) we have

$$\begin{aligned} \text{RHS of (3.9)} & \leq C \int_0^t \|w \cdot \nabla w\|_{\mathcal{H}^1} \|u\|_{BMO} d\tau \\ & \leq C \int_0^t \|w\|_2 \|\nabla w\|_2 \|u\|_{BMO} d\tau \\ & \leq \int_0^t \|\nabla w\|_2^2 d\tau + C \int_0^t \|w\|_2^2 \|u\|_{BMO}^2 d\tau. \end{aligned}$$

Hence by (3.9)

$$\|w(t)\|_2^2 \leq C \int_0^t \|w\|_2^2 \|u\|_{BMO}^2 d\tau, \quad 0 \leq t < T.$$

Since $u \in L^2(0, T; BMO)$, the Gronwall inequality yields

$$\|w(t)\|_2^2 = 0, \quad 0 \leq t < T$$

from which we get the desired uniqueness.

(2) We next prove regularity. Since $u \in L^2(0, T; H_\sigma^1 \cap BMO)$, for every $0 < \varepsilon < T$, there is $0 < \delta < \varepsilon$ such that $u(\delta) \in H_\sigma^1 \cap BMO \subset L_\sigma^2 \cap L_\sigma^r$ for $n < r < \infty$. Hence it follows from the local existence theorem of Kato [14] and Giga [11] that there are $T_* > \delta$ and a unique solution $\tilde{u}$ on $[\delta, T_*)$ with $\tilde{u}|_{t=\delta} = u(\delta)$, such that

$$(3.10) \quad \tilde{u} \in C([\delta, T_*); H_\sigma^1 \cap L_\sigma^r) \cap C^1((\delta, T_*); H^{s+2}) \quad \text{for } s > n/2 - 1.$$

Since u satisfies the energy identity

$$(3.11) \quad \|u(t)\|_2^2 + 2 \int_\delta^t \|\nabla u\|_2^2 d\tau = \|u(\delta)\|_2^2, \quad \delta \leq t < T,$$

implied by Lemma 2.3, we have by the uniqueness criterion of Serrin-Masuda [23], [20]

$$(3.12) \quad u \equiv \tilde{u} \quad \text{on } [\delta, T_*).$$

By (3.10) and (3.12), we may regard u as a strong solution in the class $(CN)_s$ on (δ', T_*) for $\delta < \delta' < \varepsilon$.

In fact, there holds $T_* = T$. Suppose that $T_* < T$. Then there exists $T_0 < T$ such that u is a strong solution in the class $(CN)_s$ on (δ', T_0) , but cannot be continued in the class $(CN)_s$ on $(\delta', \tilde{T})$ for any $\tilde{T} > T_0$. By assumption, we have

$$(3.13) \quad \int_{\delta'}^{T_0} \|u\|_{BMO}^2 d\tau \leq \int_0^T \|u\|_{BMO}^2 d\tau < \infty.$$

This contradicts Corollary 1, so we get $T_* = T$. This proves Theorem 2.

3.3 Proof of Theorems 3-4.

Proof of Theorem 3:

On account of (3.2), it suffices to prove

$$(3.14) \quad \sup_{\varepsilon_0 < t < T} \|u(t)\|_r \leq \|u(\varepsilon_0)\|_r \exp \left(C \int_{\varepsilon_0}^T \|\nabla u\|_{BMO} d\tau \right), \quad r > n.$$

In the same way as in (2.10), we see that the hypothesis (1.7) or (1.8) yields

$$(3.15) \quad \int_{\varepsilon_0}^T \|\nabla u\|_{BMO} dt < \infty.$$

Since $u \in C([\varepsilon_0, T]; H^{s+2}) \subset C([\varepsilon_0, T]; W^{1,\infty})$, u is actually the solution in $C([\varepsilon_0, T]; W^{1,r}) \cap C^1((\varepsilon_0, T); W^{1,r}) \cap C((\varepsilon_0, T); W^{3,r})$ for all $2 \leq r < \infty$ and has the integral representation:

$$(3.16) \quad u(t) = e^{(t-\varepsilon_0)\Delta} u(\varepsilon_0) - \int_{\varepsilon_0}^t e^{(t-s)\Delta} P(u \cdot \nabla u)(s) ds, \quad \varepsilon_0 < t < T.$$

See Kato [14]. Here $e^{t\Delta}$ is the well-known heat operator and $P = \{P_{kl}\}_{k,l=1,\dots,n}$ is the Helmholtz projection defined by $P_{kl} = \delta_{kl} + R_k R_l$.

Since $\|e^{t\Delta}\|_{B(L^r, L^r)} \leq 1$ for all $t > 0$, it follows from Lemma 2.1(i) and (3.16) that

$$\begin{aligned} \|u(t)\|_r &\leq \|u(\varepsilon_0)\|_r + C \int_{\varepsilon_0}^t \|u \cdot \nabla u\|_r d\tau \\ &\leq \|u(\varepsilon_0)\|_r + C \int_{\varepsilon_0}^t \|\nabla u\|_{BMO} \|u\|_r d\tau, \quad \varepsilon_0 < t < T. \end{aligned}$$

From this and the Gronwall inequality, we obtain the desired apriori estimate (3.14), which proves Theorem 3.

Proof of Theorem 4:

The proof of Theorem 4 is parallel to that of Theorem 2.

4 Proof of Theorem 5.

We follow the argument of Beale-Kato-Majda [1]. It is proved by Kato-Lai [15] and Kato-Ponce [16] that for the given initial data $a \in W^{s,p}$ for $s > 1 + n/p$, the time interval T of the existence of the solution u to (E) in the class $(CE)_{s,p}$ depends only on $\|a\|_{W^{s,p}}$. Hence by the standard argument of continuation of local solutions, it suffices to establish an apriori estimate for u in $W^{s,p}$ in terms of a, T, M_0 or a, T, M_1 according to (1.11) or (1.12). Indeed, we shall show that the solution $u(t)$ in the class $(CE)_{s,p}$ on $(0, T)$ is subject to the following estimate:

$$(4.17) \quad \sup_{0 < t < T} \|u(t)\|_{W^{s,p}} \leq (\|a\|_{W^{s,p}} + e)^{\alpha_j} \exp(CT\alpha_j) \quad \text{with } \alpha_j = e^{CM_j}, \quad j = 0, 1,$$

where $C = C(n, p, s)$ is a constant independent of a and T .

We shall first prove (4.17) under (1.11). It follows from the commutator estimate in L^p given by Kato-Ponce [16, Proposition 4.2] that

$$(4.18) \quad \|u(t)\|_{W^{s,p}} \leq \|a\|_{W^{s,p}} \exp\left(C \int_0^t \|\nabla u(\tau)\|_\infty d\tau\right), \quad 0 < t < T,$$

where $C = C(n, p, s)$.

By the Biot-Savard law (2.8), we have

$$(4.19) \quad \|\nabla u\|_{BMO} \leq C\|\omega\|_{BMO}$$

with $C = C(n)$. Hence it follows from (4.19) and Lemma 2.4 that

$$(4.20) \quad \|\nabla u(t)\|_\infty \leq C(1 + \|\omega(t)\|_{BMO}(1 + \log^+ \|u(t)\|_{W^{s,p}}))$$

for all $0 < t < T$ with $C = C(n, p, s)$. Substituting (4.20) to (4.18), we have

$$\begin{aligned} &\|u(t)\|_{W^{s,p}} + e \\ &\leq (\|a\|_{W^{s,p}} + e) \exp\left(C \int_0^t \{1 + \|\omega(\tau)\|_{BMO} \log(\|u(\tau)\|_{W^{s,p}} + e)\} d\tau\right) \end{aligned}$$

for all $0 < t < T$. Defining $z(t) \equiv \log(\|u(t)\|_{W^{s,p}} + e)$, we obtain from the above estimate

$$z(t) \leq z(0) + CT + C \int_0^t \|\omega(\tau)\|_{BMO} z(\tau) d\tau, \quad 0 < t < T.$$

Now (1.11) and the Gronwall inequality yield

$$\begin{aligned} z(t) &\leq (z(0) + CT) \exp \left(C \int_0^t \|\omega(\tau)\|_{BMO} d\tau \right) \\ &\leq (z(0) + CT) \alpha_0 \end{aligned}$$

for all $0 < t < T$ with $C = C(n, p, s)$, which implies (4.17) for $j = 0$.

Similarly we prove (4.17) for $j = 1$ under (1.12). This proves Theorem 5.

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