

Toeplitz Ψ^* -algebras via unitary group representations

W. BAUER *

Science University of Tokyo,

Department of Mathematics,

Noda, Chiba (278-8510)

Japan

Email: bauerwolfram@web.de

Abstract

As it was pointed out in [12] there are construction methods for spectral invariant Fréchet operator algebras such as Ψ^* - and Ψ_0 -algebras in the bounded operators on a Hilbert space having prescribed properties. For the Segal-Bargmann space H and using systems of unbounded closable Toeplitz operators T_f where f is in a certain class $\text{SP}_{\text{Lip}}(\mathbb{C}^n)$ of symbols we show that these algebras contain all Toeplitz operators T_h with $h \in L^\infty(\mathbb{C}^n)$. Let ρ be the Segal-Bargmann representation of the Heisenberg group $\mathbb{H}_n$ in the bounded operators on H . As an application of our results above we characterize a class of smooth Toeplitz operators in the Ψ^* -algebra of smooth elements with respect to ρ .

1 Introduction

Subsequent to the results in [12] it frequently has been remarked that the abstract concept of (locally) spectral invariant Fréchet algebras such as Ψ_0 - and Ψ^* -algebras successfully can be applied to the structural analysis of certain algebras of pseudo-differential operators. Applications arise in complex analysis, analytic perturbation theory of Fredholm operators and non-abelian cohomology for analyzing isomorphisms of abelian groups in K -theory. By generalizing a characterization of the Hörmander classes $\Psi_{\rho,\delta}^0$ ¹ by commutator conditions (see Theorem 2.1) a construction method for algebras of the above mentioned type with prescribed properties have been given in [12].

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¹ $0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$

Let $H := H^2(\mathbb{C}^n, \mu)$ be the Segal-Bargmann space of Gaussian square integrable entire functions on $\mathbb{C}^n$. We denote by P the orthogonal projection from $L^2(\mathbb{C}^n, \mu)$ onto H and we write M_f for the multiplication with a measurable symbol f . In the initial stage of this paper we consider iterated commutators of closable Toeplitz operators $T_f := PM_f$ on H having symbols in a certain class $\text{SP}_{\text{Lip}}(\mathbb{C}^n)$ of measurable and in general unbounded functions on $\mathbb{C}^n$. For a system $\mathcal{S}_m := [T_{f_1}, \dots, T_{f_m}]$ of operators with $f_j \in \text{SP}_{\text{Lip}}(\mathbb{C}^n)$ and in the sense of [12] the Ψ_0 -algebra $\Psi_\infty^{\mathcal{S}_m}$ in the bounded operators $\mathcal{L}(H)$ on H can be defined by commutator methods with respect to $\mathcal{S}_m$. We show that $\Psi_\infty^{\mathcal{S}_m}$ contains all Toeplitz operators with bounded measurable symbols. More precisely:

Theorem A *The symbols map $L^\infty(\mathbb{C}^n) \ni h \mapsto T_h \in \Psi_\infty^{\mathcal{S}_m}$ is well-defined and continuous.*

Let $\mathbb{H}_n$ be the Heisenberg group and α be the Segal-Bargmann representation of $\mathbb{H}_n$ in $\mathcal{L}(H)$, c.f. [10]. The map α is well-known to be unitary, irreducible and strongly continuous. In particular, the Ψ^* -algebra $\Psi^\infty(\mathbb{H}_n) \subset \mathcal{L}(H)$ of smooth elements with respect to α arise in a natural way and it can be characterized by commutator methods. We describe a symmetric subspace $\mathcal{S}_s \subset L^\infty(\mathbb{C}^n)$ with the induced topology such that:

Theorem B *The symbols map $\mathcal{S}_s \ni h \mapsto T_h \in \Psi^\infty(\mathbb{H}_n)$ is well-defined and continuous.*

This result can be stated in terms of the algebra construction. Let $\mathcal{A}$ be the algebra of multiplication operators on $V := L^2(\mathbb{C}^n, \mu)$ with bounded measurable symbols. In a natural way α extends to a representation of $\mathbb{H}_n$ into $\mathcal{L}(V)$ and the corresponding operator algebras $\Psi^k(\mathcal{A}, \mathbb{H}_n)$ of C^k -elements in $\mathcal{A}$ form a decreasing scale. Note that $M_f \in \Psi^k(\mathcal{A}, \mathbb{H}_n)$ is related to the smoothness of the symbols $f \in L^\infty(\mathbb{C}^n)$. Clearly, $\mathcal{A}$ projects under P onto the space $\mathcal{A}_P := PAP$ of Toeplitz operators with bounded symbols. Theorem B states:

$$P \Psi^k(\mathcal{A}, \mathbb{H}_n) P = P \Psi^{k+1}(\mathcal{A}, \mathbb{H}_n) P \subset \mathcal{L}(H) \quad \text{for all } k \in \mathbb{N}.$$

Heuristically, the smoothness of f cannot be recovered by commutator methods from the Toeplitz operator T_f . We want to remark here that these results are related to an observation in [14], [3]. Let $\beta : L^2(\mathbb{R}^n) \rightarrow H$ be the Bargmann isometry and f a bounded measurable function on $\mathbb{C}^n$. The assignment $\beta^{-1} T_f \beta$ can be shown to be a pseudo-differential operator $W_{\sigma(f)}$ on $L^2(\mathbb{R}^n)$ in its Weyl quantization. By Identifying $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ the Weyl symbol $\sigma(f)$ and f are related via the heat equation on $\mathbb{R}^{2n}$. There is $t_0 > 0$ such that:

$$\sigma(f) = e^{-t_0 \Delta} f := \text{solution of the heat equation with initial data } f \text{ at the time } t_0.$$

Moreover, σ maps the space of continuous functions with compact support into the symbol class $\mathcal{S}_{\rho, \delta}^{-\infty}$, $0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$. Corresponding to Theorem A and B it can be checked that $f \mapsto \sigma(f)$ is continuous with respect to the $L^\infty(\mathbb{C}^n)$ topology and the usual Fréchet topology on $\mathcal{S}_{\rho, \delta}^{-\infty}$.

In our first section we remind of some basic definitions and results related to the construction of Ψ_0 - and Ψ^* -algebras. For Toeplitz operators having symbols of polynomial growth at infinity an invariant subspace $H_{\text{exp}}(\mathbb{C}^n)$ of H is defined in section 3. Moreover,

the existence of bounded extensions for a class of iterated commutators of Toeplitz operators on $H_{\text{exp}}(\mathbb{C}^n)$ and Theorem A are proved. Section 4 contains the proof of Theorem B and finally we have added some examples and applications in section 5.

2 Fréchet operator algebras with prescribed properties

The following definition due to B. Gramsch have been given in [11]:

Definition 2.1 Let $\mathcal{B}$ be a Banach-algebra with unit e and let $\mathcal{F}$ be a continuously embedded Fréchet algebra in $\mathcal{B}$ with $e \in \mathcal{F}$. Then $\mathcal{F}$ is called Ψ_0 -algebra if it is *locally spectral invariant* in $\mathcal{B}$, i.e. there is $\varepsilon > 0$ with

$$\{a \in \mathcal{F} : \|e - a\|_{\mathcal{B}} < \varepsilon\} \subset \mathcal{F}^{-1}.$$

Moreover, one defines:

- If $\mathcal{B}$ is a C^* -algebra and $\mathcal{F}$ is a symmetric Ψ_0 -algebra in $\mathcal{B}$, then $\mathcal{F}$ is called Ψ^* -algebra. ($\mathcal{F}$ automatically is *spectral invariant*, i.e. $\mathcal{F} \cap \mathcal{B}^{-1} = \mathcal{F}^{-1}$).
- If the topology of $\mathcal{F}$ is generated by a system $[q_j : j \in \mathbb{N}]$ of sub-multiplicative semi-norms with $q_j(e) = 1$ for $j \in \mathbb{N}$, then $\mathcal{F}$ is called *sub-multiplicative* or *locally m -convex* (E. Michael, 1952) Ψ_0 - resp. Ψ^* -algebra.

The concept of Ψ^* - and Ψ_0 -algebras allows to treat phenomenas of local structure. As it was observed for algebras of Pseudo-differential operators, C^∞ -properties such as pseudo- or micro-locality are preserved by taking closures in the Fréchet topology. Important examples of Ψ^* -algebras are given by the Hörmander classes $\Psi_{\rho,\delta}^0$ ² of zero order where $\mathcal{B} := \mathcal{L}(L^2(\mathbb{R}^n))$. It is known that $\Psi_{\rho,\delta}^0$ can be described in terms of commutator conditions.

Theorem 2.1 (R. Beals, '77, [6])

An operator $B : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ is of class $\Psi_{\rho,\delta}^0$ iff for $\alpha, \beta \in \mathbb{N}_0^n$ all iterated commutators:

$$ad[-ix]^\alpha ad[i\partial_x]^\beta(B) : H^{s-\rho|\alpha|+\delta|\beta|} \rightarrow H^s \quad (2.1)$$

admit bounded extensions between suitable Sobolev spaces to $L^2(\mathbb{R}^n)$.

On the one hand the spectral invariance of $\Psi_{\rho,\delta}^0$ follows from the commutator characterizations in Theorem 2.1, see [19], [20]. On the other hand, by replacing ix and $i\partial_x$ above with a system of closable and densely defined operators, conditions of the type (2.1) have been used to define (submultiplicative) Ψ_0 -algebras in a fairly general situation, see [12]. Below we give the definitions and remind of some basic results.

² $0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$

2.1 Commutator Methods

Given a topological vector space X we write $L(X)$ (resp. $\mathcal{L}(X)$) for the linear (resp. bounded linear) operators on X .

Definition 2.2 (Iterated commutators)

For a system $\mathcal{S}_m := [A_1, \dots, A_m]$ where $A_j, B \in L(X)$ we call m the *length* of $\mathcal{S}_m$. We inductively define the *iterated commutators* $\text{ad}[\emptyset](B) := B$ and:

- $\text{ad}[\mathcal{S}_1](B) := [A_1, B] = A_1B - BA_1,$
- $\text{ad}[\mathcal{S}_{j+1}](B) := \text{ad}[A_{j+1}](\text{ad}[\mathcal{S}_j](B))$ for $j = 1, \dots, m-1$.

In the case of $A = A_j$ where $j = 1, \dots, m$ we also write:

- $\text{ad}^0[A](B) := B$ and $\text{ad}^m[A](B) := \text{ad}[\mathcal{S}_m](B).$

With these notations it follows for finite systems $\mathcal{S}_j$ and $\mathcal{S}_k$ in $L(X)$:

$$\text{ad}[\mathcal{S}_j](\text{ad}[\mathcal{S}_k](B)) = \text{ad}[\mathcal{S}_k, \mathcal{S}_j](B).$$

Let H be a Hilbert space and $\mathcal{F} \subset \mathcal{L}(H)$ be a sub-multiplicative Ψ^* -algebra. Assume that the topology of $\mathcal{F}$ is generated by a sequence $(q_j)_{j \in \mathbb{N}}$ of semi-norms and without lost of generality let $q_0 := \|\cdot\|_{\mathcal{L}(H)}$. Given a finite system $\mathcal{V}$ of closed and densely defined operators $A : H \supset \mathcal{D}(A) \rightarrow H$ and following [12] we define:

- $\mathcal{I}(A) := \{a \in \mathcal{F} : a(\mathcal{D}(A)) \subset \mathcal{D}(A)\},$
- $\mathcal{B}(A) := \{a \in \mathcal{I}(A) : [A, a] \text{ extends to an element } \delta_A(a) \in \mathcal{F}\}.$

Inductively, one obtains:

- $\Psi_0^\mathcal{V} := \mathcal{F}$, with semi-norms $q_{0,j} := q_j$ for $j \in \mathbb{N}$,
- $\Psi_1^\mathcal{V} := \bigcap_{A \in \mathcal{V}} \mathcal{B}(A),$
- $\Psi_k^\mathcal{V} := \{a \in \Psi_{k-1}^\mathcal{V} : \delta_A a \in \Psi_{k-1}^\mathcal{V} \text{ for all } A \in \mathcal{V}\}$ where $k \geq 2$,
- $\Psi_\infty^\mathcal{V} := \bigcap_{k \in \mathbb{N}} \Psi_k^\mathcal{V}.$

This process leads to a decreasing scale of algebras in $\mathcal{F}$:

$$\mathcal{F} = \Psi_0^\mathcal{V} \supset \dots \Psi_n^\mathcal{V} \supset \Psi_{n+1}^\mathcal{V} \supset \dots \supset \Psi_\infty^\mathcal{V} := \bigcap_{k \in \mathbb{N}} \Psi_k^\mathcal{V}. \quad (2.2)$$

For $n \geq 1$, we inductively define a system $(q_{n,j})_{j \in \mathbb{N}}$ (resp. $(q_{n,j})_{j,n \in \mathbb{N}}$) of norms on $\Psi_n^\mathcal{V}$ (resp. on $\Psi_\infty^\mathcal{V}$) by:

$$q_{n,j}(a) := q_{n-1,j}(a) + \sum_{A \in \mathcal{V}} q_{n-1,j}(\delta_A a). \quad (2.3)$$

According to [12], $\Psi_\infty^\mathcal{V}$ is a sub-multiplicative Ψ_0 -algebra in $\mathcal{F}$. In the case where each $A \in \mathcal{V}$ is symmetric we replace $\mathcal{B}(A)$ by:

$$\mathcal{B}^*(A) := \{ a \in \mathcal{B}(A) : a^* \in \mathcal{B}(A) \}.$$

Then the algebras $\Psi_n^\mathcal{V}$ are symmetric and $\Psi_\infty^\mathcal{V}$ is a Ψ^* -algebra in $\mathcal{L}(H)$. Let $D \subset H$ be a *core* for $\mathcal{V}$, i.e. the inclusion $D \hookrightarrow \mathcal{D}(A)$ is dense with respect to the graph norm for all $A \in \mathcal{V}$. Then it was shown in [2], [3]:

Proposition 2.1 *Assume that $a \in \mathcal{F}$ and property (E_k) holds for $k \in \mathbb{N} \cup \{\infty\}$:*

(E_k) : D is invariant under all $A \in \mathcal{V}$ and $a \in \mathcal{F}$. Moreover, assume that for any system

$$\mathcal{A} \subset \mathcal{S}_k(\mathcal{V}) := \{ [A_1, \dots, A_j] : \text{where } A_l \in \mathcal{V} \text{ and } 1 \leq l \leq j \leq k \}.$$

$\text{ad}[\mathcal{A}](a) : H \supset D \rightarrow H$ has a continuous extensions to $C(\mathcal{A}, a) \in \mathcal{F}$.

Then $a \in \Psi_k^\mathcal{V}$ and $C(\mathcal{A}, a)$ is a bounded extension of $\text{ad}[\mathcal{A}](a) : H \subset \mathcal{D}(A) \rightarrow H$ to H for any operator $A \in \mathcal{V}$.

The (locally) spectral invariance of $\mathcal{A} \subset \mathcal{B}$ is preserved under projections $p = p^2 \in \mathcal{A}$. It is readily verified that $\mathcal{A}_p := p \mathcal{A} p$ is (locally) spectral invariant in $\mathcal{B}_p := p \mathcal{B} p$. If in addition $\mathcal{B}$ is a C^* -algebra, $\mathcal{A}$ is symmetric in $\mathcal{B}$ and $p = p^*$, then $\mathcal{A}_p$ is symmetric and spectral invariant in $\mathcal{B}_p$.

With (2.2) and an orthogonal projection $p \in \Psi_n^\mathcal{V}$, $n \in \mathbb{N} \cup \{\infty\}$ from H onto a closed subspace $H_0 \subset H$ there is a scale of projected algebras in $\mathcal{L}(H_0)$:

$$\mathcal{L}(H_0) \supset \mathcal{F}_p = \Psi_{0p}^\mathcal{V} \supset \dots \Psi_{n-1p}^\mathcal{V} \supset \Psi_{np}^\mathcal{V}. \quad (2.4)$$

It can be shown that (2.4) arises by commutator methods with a system $\mathcal{V}_p$ of closed operators on H_0 where $\mathcal{D}(A_p) := p[\mathcal{D}(A)]$ and

$$\mathcal{V}_p := \{ A_p := p A p : H_0 \supset \mathcal{D}(A_p) \rightarrow H_0 : A \in \mathcal{V} \}.$$

Defining (2.4) by commutator conditions with respect to $\mathcal{V}_p$ only requires that $p \in \Psi_1^\mathcal{V}$. Thus this method gives a natural extension of (2.4) to an infinite scale for $n \in \mathbb{N}$.

There is a corresponding scale of $\mathcal{V}$ -Sobolev spaces in H :

- $\mathcal{H}_\mathcal{V}^0 := H$ with the norm $p_0 := \|\cdot\|_H$.
- $\mathcal{H}_\mathcal{V}^1 := \bigcap_{A \in \mathcal{V}} \mathcal{D}(A)$.
- $\mathcal{H}_\mathcal{V}^k := \left\{ x \in \mathcal{H}_\mathcal{V}^{k-1} : Ax \in \mathcal{H}_\mathcal{V}^{k-1} \text{ for all } A \in \mathcal{V} \right\}, \quad k \geq 2.$
- $\mathcal{H}_\mathcal{V}^\infty := \bigcap_{k \in \mathbb{N}} \mathcal{H}_\mathcal{V}^k$.

We endow $\mathcal{H}_V^k$ with the norm

$$p_k(x) := p_{k-1}(x) + \sum_{A \in V} p_{k-1}(Ax), \quad x \in \mathcal{H}_V^k.$$

Let the topology of $\mathcal{H}_V^\infty$ be defined by the system of norms $(p_k)_{k \in \mathbb{N}_0}$. It can be shown that $(\mathcal{H}_V^k, p_k)$ is a Banach spaces and $(\mathcal{H}_V^\infty, (p_k)_{k \in \mathbb{N}})$ turns into a Fréchet space. Moreover, each $A \in V$ induces a bounded operator $A_k : \mathcal{H}_V^k \rightarrow \mathcal{H}_V^{k-1}$. For $n \in \mathbb{N} \cup \{\infty\}$ it was shown in [12] that all maps

$$\Psi_k^V \times \mathcal{H}_V^k \longrightarrow \mathcal{H}_V^k : (a, x) \mapsto a(x)$$

are bilinear and continuous. The following result on *regularity* was proved in [13]:

Theorem 2.2 *Let $A \in \Psi_\infty^V$ be a Fredholm operator and $u \in H$ with $Au = f \in \mathcal{H}_V^k$ for some $k \in \mathbb{N} \cup \{\infty\}$. Then it follows that $u \in \mathcal{H}_V^k$.*

3 On the Segal-Bargmann Projection

Throughout this paper we write $\langle x, y \rangle := x_1 \bar{y}_1 + \cdots x_n \bar{y}_n$ for the Hermitian inner product on $\mathbb{C}^n$ and $|x| := \sqrt{\langle x, x \rangle}$. For $c > 0$ and the Lebesgue measure v let us denote by μ_c the Gaussian measure on $\mathbb{C}^n$ given by:

$$d\mu_c = c^n \pi^{-n} \exp(-c|x|^2) dv.$$

With $\mu := \mu_1$ let $H^2(\mathbb{C}^n, \mu)$ be the *Segal-Bargmann space* of μ -square integrable entire functions on $\mathbb{C}^n$. We denote by P the orthogonal projection from $L^2(\mathbb{C}^n, \mu)$ onto $H^2(\mathbb{C}^n, \mu)$. The reproducing kernel K (resp. the normalized kernel k) corresponding to $H^2(\mathbb{C}^n, \mu)$ are known to be:

$$(a) \quad K(y, x) := \exp(\langle y, x \rangle),$$

$$(b) \quad k_x(y) := K(y, x) \|K(\cdot, x)\|^{-1} = \exp(\langle y, x \rangle - \frac{1}{2}|x|^2)$$

where $\|\cdot\|$ denotes the $L^2(\mathbb{C}^n, \mu)$ -norm. For $z, w \in \mathbb{C}^n$ we write $\tau_w(z) := z + w$ for the shift by w . Consider the space of measurable symbols on $\mathbb{C}^n$ given by:

$$\mathcal{T}(\mathbb{C}^n) := \{g : g \circ \tau_x \in L^2(\mathbb{C}^n, \mu) \text{ for all } x \in \mathbb{C}^n\}.$$

For $g \in \mathcal{T}(\mathbb{C}^n)$ and with the natural domain of definition

$$\mathcal{D}(T_g) := \{f \in H^2(\mathbb{C}^n, \mu) : gf \in L^2(\mathbb{C}^n, \mu)\} \quad (3.1)$$

the *Toeplitz operator* T_g on $H^2(\mathbb{C}^n, \mu)$ is densely defined by:

$$T_g : \mathcal{D}(T_g) \ni f \mapsto P(fg).$$

If g has *polynomial growth at infinity* we can determine an invariant subspace for T_g :

We inductively define a sequence $(a_n)_{n \in \mathbb{N}}$ with $a_1 := \frac{1}{4}$ and $a_{n+1} := [4 \cdot (1 - a_n)]^{-1}$ for all $n \geq 2$. It can be checked that:

- (a) $a_n < \frac{1}{2}, \quad \forall n \in \mathbb{N},$
- (b) $(a_n)_{n \in \mathbb{N}}$ is strictly increasing,
- (c) $\lim_{n \rightarrow \infty} a_n = \frac{1}{2}.$

Let $\mathbb{P}[\mathbb{C}^n]$ be the space of all *polynomials* on $\mathbb{C}^n$ in the variables $z := (z_1, \dots, z_n)$ and $\bar{z} := (\bar{z}_1, \dots, \bar{z}_n)$. We write $\mathbb{P}_a[\mathbb{C}^n]$ for all *analytic polynomials* and set:

$$L_{\exp}(\mathbb{C}^n) := \left\{ f \in L^2(\mathbb{C}^n, \mu) : \exists c < \frac{1}{2}, 0 < D \text{ s.t. } |f(z)| \leq D \exp(c|z|^2) \text{ a.e.} \right\}.$$

Because of $\mathbb{P}[\mathbb{C}^n] \subset L_{\exp}(\mathbb{C}^n)$ it follows that $L_{\exp}(\mathbb{C}^n)$ is dense in $L^2(\mathbb{C}^n, \mu)$. With the space $\mathcal{H}(\mathbb{C}^n)$ of entire functions on $\mathbb{C}^n$ we define a subspace of $H^2(\mathbb{C}^n, \mu)$ by:

$$H_{\exp}(\mathbb{C}^n) := \mathcal{H}(\mathbb{C}^n) \cap L_{\exp}(\mathbb{C}^n),$$

Consider the symbols having polynomial growth at ∞ :

$$\text{Pol}(\mathbb{C}^n) := \left\{ f : \exists j \in \mathbb{N} \text{ s.t. } |f(z)| (1 + |z|^2)^{-\frac{j}{2}} \in L^\infty(\mathbb{C}^n) \right\}.$$

Proposition 3.1 *It holds $P[L_{\exp}(\mathbb{C}^n)] \subset H_{\exp}(\mathbb{C}^n)$ and for f in $\text{Pol}(\mathbb{C}^n)$:*

$$T_f[H_{\exp}(\mathbb{C}^n)] \subset H_{\exp}(\mathbb{C}^n) \subset \mathcal{D}(T_f) \quad (3.2)$$

Proof: It is obvious that $H_{\exp}(\mathbb{C}^n) \subset \mathcal{D}(T_f)$. Because the multiplication by f clearly maps $H_{\exp}(\mathbb{C}^n)$ into $L_{\exp}(\mathbb{C}^n)$ it is sufficient to prove the first assertion of Proposition 3.1. For $g \in L_{\exp}(\mathbb{C}^n)$ there are $c < \frac{1}{2}$ and $D > 0$ such that a.e.:

$$|g(z)| \leq D \exp(c|z|^2).$$

By (a), (b) and (c) and with $(a_n)_{n \in \mathbb{N}}$ above we can choose $n_0 \in \mathbb{N}$ with $c < a_{n_0} < \frac{1}{2}$. Using the transformation formula and the reproducing property of K we obtain:

$$\begin{aligned} |[Pg](z)| &\leq \int_{\mathbb{C}^n} |g \exp\{\langle z, \cdot \rangle\}| d\mu \\ &\leq D \pi^{-n} \int_{\mathbb{C}^n} \exp\left\{ \text{Re}\langle z, \cdot \rangle - [1 - a_{n_0}]|\cdot|^2 \right\} dv \\ &= D (1 - a_{n_0})^{-n} \int_{\mathbb{C}^n} \exp\left\{ 2\text{Re}\langle 2^{-1}(1 - a_{n_0})^{-\frac{1}{2}}z, \cdot \rangle \right\} d\mu \\ &= D (1 - a_{n_0})^{-n} \exp\left\{ \underbrace{[4(1 - a_{n_0})]^{-1}}_{=a_{n_0+1}} |z|^2 \right\}. \end{aligned}$$

From (a) above we conclude that $Pg \in H_{\exp}(\mathbb{C}^n)$. □

Hence all finite products of Toeplitz operators with symbols in $\text{Pol}(\mathbb{C}^n)$ are well-defined on the dense subspace $H_{\exp}(\mathbb{C}^n)$ of $H^2(\mathbb{C}^n, \mu)$. In particular, all iterated commutators of P and multiplication operators M_f with $f \in \text{Pol}(\mathbb{C}^n)$ can be considered as elements in $L(L_{\exp}(\mathbb{C}^n))$. In fact, they can be written as integral operators and a standard application of the *Schur test* leads to a criterion for the boundedness.

Lemma 3.1 *Let $L : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}$ be a measurable function such that:*

$$|L(x, y)| \leq |F(x - y)| \exp \left\{ \operatorname{Re} \langle x, y \rangle \right\}$$

where $F \in L^1(\mathbb{C}^n, \mu_{\frac{1}{2}})$. Then the integral operator A on $L^2(\mathbb{C}^n, \mu)$ defined by

$$[Af](z) := \int_{\mathbb{C}^n} L(z, \cdot) f d\mu$$

is bounded on $L^2(\mathbb{C}^n, \mu)$ with $\|A\| \leq 2^n \|F\|_{L^1(\mathbb{C}^n, \mu_{\frac{1}{2}})}$.

Proof: With $p := q = \exp(\frac{1}{2}|\cdot|^2)$ on $\mathbb{C}^n$ it follows that:

$$\begin{aligned} \int_{\mathbb{C}^n} |L(\cdot, y)| p d\mu &\leq \frac{1}{\pi^n} \int_{\mathbb{C}^n} |F(\cdot - y)| \exp \left\{ \operatorname{Re} \langle \cdot, y \rangle - \frac{1}{2} |\cdot|^2 \right\} dv \\ &= \frac{1}{\pi^n} \int_{\mathbb{C}^n} |F| \exp \left\{ \operatorname{Re} \langle \cdot + y, y \rangle - \frac{1}{2} |\cdot + y|^2 \right\} dv \\ &= 2^n p(y) \|F\|_{L^1(\mathbb{C}^n, \mu_{\frac{1}{2}})}. \end{aligned}$$

Similarly, we get $\int |L(x, \cdot)| p d\mu \leq 2^n p(x) \|F\|_{L^1(\mathbb{C}^n, \mu_{\frac{1}{2}})}$. Applying the *Schur test* we obtain the desired result. $\square$

Consider the subspace $\operatorname{SP}_{\operatorname{Lip}}(\mathbb{C}^n)$ of $\operatorname{Pol}(\mathbb{C}^n)$ defined by:

$$\operatorname{SP}_{\operatorname{Lip}}(\mathbb{C}^n) := \{ f \in \operatorname{Pol}(\mathbb{C}^n) : \exists c, D > 0 \text{ s.t. } |f(z) - f(w)| \leq D \exp(c|z - w|) \}.$$

As an application of Lemma (3.1) we can prove:

Proposition 3.2 *Let $m \in \mathbb{N}$ and $\mathcal{S}_m := \{M_{f_1}, \dots, M_{f_m}\}$ with $f_j \in \operatorname{SP}_{\operatorname{Lip}}(\mathbb{C}^n)$. Then the commutator $\operatorname{ad}[\mathcal{S}_m](P) \in L(L_{\exp}(\mathbb{C}^n))$ has a continuous extension to $L^2(\mathbb{C}^n, \mu)$.*

Proof: It is easy to check that the commutator $\operatorname{ad}[\mathcal{S}_m](P)$ can be written as an integral operator on $L^2(\mathbb{C}^n, \mu)$ with kernel:

$$K_m(z, u) = \exp(\langle z, u \rangle) \prod_{j=1}^m \{f_j(z) - f_j(u)\}. \quad (3.3)$$

By (3.3) and our assumptions on $f_j \in \mathcal{S}_m$ we can choose $c, D > 0$ such that

$$|K_m(z, u)| \leq D \exp(c|z - u| + \operatorname{Re} \langle z, u \rangle).$$

Because of $F := D \exp(c|\cdot|) \in L^1(\mathbb{C}^n, \mu_{\frac{1}{2}})$ Lemma 3.1 implies the assertion. $\square$

We remark that by (3.3) the maps $\operatorname{ad}[\mathcal{S}_m](P)$ are invariant under permutations of the system $\mathcal{S}_m$. Now, we can prove the boundedness of a class of iterated commutators.

Corollary 3.1 *Let $g \in L^\infty(\mathbb{C}^n)$ and $\mathcal{S}_m := \{M_{f_1}, \dots, M_{f_m}\}$ with $f_j \in SP_{Lip}(\mathbb{C}^n)$. Then the commutator*

$$ad[\mathcal{S}_m]([P, M_g]) \in L(L_{\exp}(\mathbb{C}^n))$$

has a bounded extensions $A(\mathcal{S}_m, g)$ to $L^2(\mathbb{C}^n, \mu)$ and (3.4) below is continuous between Banach spaces:

$$L^\infty(\mathbb{C}^n) \ni g \mapsto A(\mathcal{S}_m, g) \in \mathcal{L}(L^2(\mathbb{C}^n, \mu)). \quad (3.4)$$

Proof: It can be checked by induction or our remark following Proposition 3.2 that:

$$ad[\mathcal{S}_m]([P, M_g]) = [ad[\mathcal{S}_m](P), M_g] \in L(L_{\exp}(\mathbb{C}^n)).$$

Because M_g is bounded and $ad[\mathcal{S}_m](P)$ has a bounded extension to $L^2(\mathbb{C}^n, \mu)$ by Proposition 3.2 we conclude the desired result. $\square$

Given a finite set $\mathbf{X} := \{X_1, \dots, X_n\} \subset \mathcal{L}(L^2(\mathbb{C}^n, \mu))$ we denote by $\mathcal{A}(\mathbf{X})$ the algebra generated by $\mathbf{X}$. Moreover, we write:

$$\mathcal{A}_P(\mathbf{X}) := P \mathcal{A}(\mathbf{X}) P := \{PAP : A \in \mathcal{A}(\mathbf{X})\}.$$

for the corresponding projected algebra in $\mathcal{L}(H^2(\mathbb{C}^n, \mu))$. By Proposition 3.1 and for all $m \geq 1$ it follows that the commutator:

$$ad[\mathcal{S}_{m-1}]([P, M_{f_m}]) = -ad[\mathcal{S}_m](P)$$

can be regarded as bounded operators on $L^2(\mathbb{C}^n, \mu)$.

Proposition 3.3 *Let $g \in L^\infty(\mathbb{C}^n)$ and $\mathcal{T}_m := \{T_{f_1}, \dots, T_{f_m}\}$ with $f_j \in SP_{Lip}(\mathbb{C}^n)$. Then*

$$ad[\mathcal{T}_m](T_g) \in L(H_{\exp}(\mathbb{C}^n))$$

is well-defined. More precisely, with $\mathcal{S}_m := \{M_{f_1}, \dots, M_{f_m}\}$ it holds:

$$ad[\mathcal{T}_m](T_g) \in \mathcal{A}_P \left\{ ad[\mathcal{N}](P), M_g : \text{with } \mathcal{N} \subset \mathcal{S}_m \right\} \quad (3.5)$$

and $ad[\mathcal{T}_m](T_g)$ has a bounded extension $C(\mathcal{T}_m, g)$ to $H^2(\mathbb{C}^n, \mu)$. Moreover, the symbols map

$$L^\infty(\mathbb{C}^n) \ni g \mapsto C(\mathcal{T}_m, g) \in \mathcal{L}(H^2(\mathbb{C}^n, \mu)) \quad (3.6)$$

is continuous between Banach spaces.

Proof: By Proposition 3.1 the iterated commutators $ad[\mathcal{T}_m](T_g)$ are well-defined. It is a straightforward computation that:

$$ad[\mathcal{T}_1](T_g) = P \left[[P, M_{f_1}], [P, M_g] \right] P$$

which proves (3.5) in the case $m = 1$. By induction assume $\text{ad}[\mathcal{T}_j](T_g)$ has the form:

$$\text{ad}[\mathcal{T}_j](T_g) = \sum_{l \in \mathcal{I}} P A_l M_g B_l P \quad (3.7)$$

where $\mathcal{I}$ is a finite index set, I the identity operator and

$$A_l, B_l \in \mathcal{A}(\mathcal{S}_j) := \mathcal{A}\left\{ \text{ad}[\mathcal{N}](P), I : \text{ with } \mathcal{N} \subset \mathcal{S}_j \right\}. \quad (3.8)$$

Then it follows that:

$$\text{ad}[\mathcal{T}_{j+1}](T_g) = \sum_{l \in \mathcal{I}} [T_{f_{j+1}}, P A_l M_g B_l P].$$

To prove (3.7) in the case $j + 1$ it is sufficient to show for all $l \in \mathcal{I}$ the existence of a finite set $\tilde{\mathcal{I}} \subset \mathbb{N}$ and operators $C_k, D_k \in \mathcal{A}(\mathcal{S}_{j+1})$ such that

$$[T_{f_{j+1}}, P A_l M_g B_l P] = \sum_{k \in \tilde{\mathcal{I}}} P C_k M_g D_k P. \quad (3.9)$$

Note that (3.9) follows from $T_{f_{j+1}} P A_l M_g B_l P = P M_{f_{j+1}} P A_l M_g B_l P$ and

$$[M_{f_{j+1}}, Q] \in \mathcal{A}(\mathcal{S}_{j+1})$$

for $Q \in \{P, A_l, B_l\}$. The continuity of (3.6) is a direct consequence of (3.7). $\square$

As an immediate consequence of Proposition 3.2 we remark:

Lemma 3.2 *Let $f \in SP_{Lip}(\mathbb{C}^n)$ and $\mathcal{D}(T_f)$ as in (3.1). Then the Toeplitz operator T_f is densely defined and closed on $\mathcal{D}(T_f)$.*

Proof: Because of $f \in \mathcal{T}(\mathbb{C}^n)$ it follows that T_f is densely defined. Moreover,

$$M_f = T_f + [M_f, P] : \mathcal{D}(T_f) \subset H^2(\mathbb{C}^n, \mu) \longrightarrow L^2(\mathbb{C}^n, \mu). \quad (3.10)$$

Proposition 3.2 with $j = 1$ shows that the commutator $[M_f, P]$ has a continuous extension to $H^2(\mathbb{C}^n, \mu)$. Choose a sequence $(h_n)_{n \in \mathbb{N}} \subset \mathcal{D}(T_f)$ such that:

- (i) $\lim_{n \rightarrow \infty} h_n = h \in H^2(\mathbb{C}^n, \mu)$,
- (ii) $\lim_{n \rightarrow \infty} T_f h_n = g \in H^2(\mathbb{C}^n, \mu)$.

Then we conclude from the continuity of $[M_f, P]$ and (3.10) that

$$fh = \lim_{n \rightarrow \infty} fh_n \in L^2(\mathbb{C}^n, \mu)$$

Hence $h \in \mathcal{D}(T_f)$ and $g = \lim_{n \rightarrow \infty} P(fh_n) = T_f h$. $\square$

Let $\mathcal{T}_m := \{T_{f_1}, \dots, T_{f_m}\}$ be a system of Toeplitz operators where $f_j \in \text{SP}_{\text{Lip}}(\mathbb{C}^n)$ for $j = 1, \dots, m$. From Lemma 3.2 it follows that the domains $\mathcal{D}(T_{f_j})$ are closed with respect to the graph norm $\|\cdot\|_{\text{gr}} := \|\cdot\| + \|T_{f_j} \cdot\|$. Consider $D_j \subset H^2(\mathbb{C}^n, \mu)$ defined by:

$$D_j := \|\cdot\|_{\text{gr}} - \text{closure of } H_{\text{exp}}(\mathbb{C}^n) \text{ in } \mathcal{D}(T_{f_j}).$$

If we consider T_{f_j} as a closed operator on D_j we can define a scale of algebras (2.2) by commutator methods with the system $\mathcal{S}_m$. By Lemma 2.1 with $D := H_{\text{exp}}(\mathbb{C}^n)$ our result in Proposition 3.3 can be formulated as follows:

Theorem 3.1 *The symbol map $L^\infty(\mathbb{C}^n) \ni h \mapsto T_h \in \Psi_\infty^{\mathcal{S}_m}$ is well-defined and continuous.*

Note that an application of Theorem 2.2 in the case of $\mathcal{V} := \mathcal{S}_m$ gives a regularity result for Fredholm Toeplitz operators with bounded symbols.

4 Toeplitz Ψ^* -algebras via the Segal-Bargmann representation

There is a unitary representation of the Heisenberg group $\mathbb{H}_n$ in $\mathcal{L}(L^2(\mathbb{C}^n, \mu))$. By identifying $\mathbb{H}_n$ with $\mathbb{C}^n \times \mathbb{R}$ the group law is given by, [10]:

$$(z, t) * (w, s) := (z + w, t + s + 2^{-1} \text{Im}\langle w, z \rangle).$$

For $z \in \mathbb{C}^n$ and $f \in L^2(\mathbb{C}^n, \mu)$ we define the operator $W_z f := k_z \cdot f \circ \tau_{-z}$. It follows by an easy calculation:

Lemma 4.1 *$H^2(\mathbb{C}^n, \mu)$ is an invariant subspace for all W_z where $z \in \mathbb{C}^n$. Moreover,*

- (1) W_z is unitary with $W_z^* = W_{-z} = W_z^{-1}$,
- (2) The commutator $\text{ad}[P] W_z$ vanishes,
- (3) For $z, w \in \mathbb{C}^n$: $W_z W_w = \exp(i \text{Im}\langle w, z \rangle) W_{z+w}$.

By Lemma 4.1 a unitary representation $\tilde{\rho} : \mathbb{H}_n \rightarrow \mathcal{L}(L^2(\mathbb{C}^n, \mu))$ of $\mathbb{H}_n$ is given by:

$$\tilde{\rho}(z, t) := e^{it} W_{\frac{z}{\sqrt{2}}}.$$

Moreover, the restriction of $\tilde{\rho}(z, t)$ to $H^2(\mathbb{C}^n, \mu)$ gives rise to a unitary representation ρ of $\mathbb{H}_n$ in $\mathcal{L}(H^2(\mathbb{C}^n, \mu))$. It is well-known that ρ is irreducible and strongly continuous and it is referred to as *Segal-Bargmann representation*, c.f. [10].

For any $A \in \mathcal{B} := \mathcal{L}(H^2(\mathbb{C}^n, \mu))$ we define the map:

$$\begin{aligned} \Phi_A : \mathbb{H}_n &\longrightarrow \mathcal{B} \\ (z, t) &\longmapsto \rho(z, t) A \rho(z, t)^{-1} = W_{\frac{z}{\sqrt{2}}} A W_{\frac{-z}{\sqrt{2}}}. \end{aligned} \tag{4.1}$$

In particular, note that for $f \in L^\infty(\mathbb{C}^n)$

$$\Phi_{T_f}(z, t) = T_{f \circ \tau_{-\frac{z}{\sqrt{2}}}}.$$

For $k \in \mathbb{N} \cup \{\infty\}$ we consider the C^k -elements

$$\Psi^k := \{ A \in \mathcal{B} : \Phi_A \in C^k(\mathbb{H}_n, \mathcal{B}) \}$$

defined via ρ . To any $z \in \mathbb{C}^n$ we associate $\varphi_A^z : \mathbb{R} \rightarrow \mathcal{B}$ by $\varphi_A^z(s) := W_{sz} A W_{-sz}$. According to (4.1) it follows that:

$$\Psi^k = \bigcap_{z \in \mathbb{C}^n} \Psi^{k,z} \text{ where } \Psi^{k,z} := \{ A \in \mathcal{B} : \varphi_A^z \in C^k(\mathbb{R}, \mathcal{B}) \}. \quad (4.2)$$

Here we characterize the C^k -Toeplitz operators (i.e. the Toeplitz operators $T_f \in \Psi^k$) in terms of their symbols. We use a characterization of Ψ^∞ by commutator conditions and apply our results of the previous section.

For all $z \in \mathbb{C}^n$ the map $(W_{sz})_{s \in \mathbb{R}} \subset \mathcal{B}$ defines a strongly continuous unitary group of operators. By V^z we denote its infinitesimal generator with domain of definition:

$$\mathcal{D}(V^z) := \{ h \in H^2(\mathbb{C}^n, \mu) : V^z h := \lim_{s \rightarrow 0} s^{-1}(W_{sz} - I)h \text{ exists} \}.$$

By *Stone's Theorem* iV^z is selfadjoint and associated to $\mathcal{V}^z := [iV^z]$ there is a scale:

$$\mathcal{B} := \Psi_0^{\mathcal{V}^z} \supset \dots \supset \Psi_n^{\mathcal{V}^z} \supset \Psi_{n+1}^{\mathcal{V}^z} \supset \dots \supset \Psi_\infty^{\mathcal{V}^z} := \bigcap_{k \in \mathbb{N}} \Psi_k^{\mathcal{V}^z} \quad (4.3)$$

of algebras in $\mathcal{B}$ defined by commutator methods with $\mathcal{V}^z$ as it was described in (2.2) of section 2.1. In particular, $\Psi_\infty^{\mathcal{V}^z}$ is a Ψ^* -algebra and it is well-known that (4.2) and (4.3) are related as follows, see [16]:

Proposition 4.1 *For $z \in \mathbb{C}^n$ let $\mathcal{V}^z := [iV^z]$ then:*

- (i) $\Psi^{k,z} \subset \Psi_k^{\mathcal{V}^z}$ for $k \in \mathbb{N}$,
- (ii) $\Psi_{k+1}^{\mathcal{V}^z} \subset \Psi^{k,z}$ for $k \in \mathbb{N}_0$ and $\Psi_\infty^{\mathcal{V}^z} = \Psi_\infty^{\mathcal{V}^z}$.

Using the fact that convergence in $H^2(\mathbb{C}^n, \mu)$ implies uniformly compact convergence on $\mathbb{C}^n$ we can calculate V^z explicitly. Let $h \in \mathcal{D}(V^z)$ and $w \in \mathbb{C}^n$:

$$[V^z h](w) = \frac{d}{ds} [k_{sz}(w) h(w - sz)]|_{s=0} = \left\{ \langle w, z \rangle - \sum_{j=1}^n z_j \frac{\partial}{\partial w_j} \right\} h(w). \quad (4.4)$$

It easily can be seen that all the monomials $m_\alpha(z) := z^\alpha$ for $\alpha \in \mathbb{N}_0^n$ are contained in the domain $\mathcal{D}(V^z)$. Moreover, from the standard identities $M_{w_j} := T_{w_j}$ and $\frac{\partial}{\partial w_j} := T_{\overline{w_j}}$ it follows that the restriction of V^z to $\mathbb{P}_a[\mathbb{C}^n]$ coincides with an unbounded Toeplitz operator:

$$V^z p := T_{\langle \cdot, z \rangle - \langle z, \cdot \rangle} p = 2i T_{\text{Im} \langle \cdot, z \rangle} p, \quad p \in \mathbb{P}_a[\mathbb{C}^n].$$

In the following we write:

$$g_z := 2i \operatorname{Im} \langle \cdot, z \rangle$$

for the symbol of the Toeplitz operator appearing above. Consider the space $\mathcal{D}(T_{g_z})$ with the graph norm $\| \cdot \|_{\text{gr}} := \| \cdot \| + \| T_{g_z} \cdot \|$. By Lemma 3.2 it follows that $(\mathcal{D}(T_{g_z}), \| \cdot \|_{\text{gr}})$ is a Banach space containing $\mathbb{P}_a[\mathbb{C}^n]$ and $H_{\text{exp}}(\mathbb{C}^n)$.

Lemma 4.2 *For all $z \in \mathbb{C}^n$ the embedding $\mathbb{P}_a[\mathbb{C}^n] \hookrightarrow H_{\text{exp}}(\mathbb{C}^n)$ is dense with respect to the graph norm topology. Moreover,*

$$H_{\text{exp}}(\mathbb{C}^n) \subset \mathcal{D}(V^z) \cap \mathcal{D}(T_{g_z}) \quad (4.5)$$

and the restrictions of V^z and T_{g_z} to $H_{\text{exp}}(\mathbb{C}^n)$ coincide.

Proof: For $f \in H_{\text{exp}}(\mathbb{C}^n)$ we can choose $c_1 \in (0, \frac{1}{2})$ and $D_1 > 0$ such that:

$$|f(w)| \leq D_1 \exp(c_1 |w|^2)$$

for all $z \in \mathbb{C}^n$. Hence, $f \in L^2(\mathbb{C}^n, \mu_r)$ for all $r \in (2c_1, 1)$. Fix c_2, c_3 with $2c_1 < c_2 < c_3 < 1$ and choose $D_2 > 0$ with

$$|w|^2 \leq D_2 \exp([c_3 - c_2] |w|^2)$$

for all $w \in \mathbb{C}^n$. Then we obtain for all $p \in \mathbb{P}_a[\mathbb{C}^n]$:

$$\begin{aligned} \|T_{g_z}(f - p)\|^2 &\leq \|g_z(f - p)\|^2 \\ &\leq 2|z|^2 \int_{\mathbb{C}^n} |\cdot|^2 |f - p|^2 d\mu \\ &\leq 2D_2 |z|^2 r^{-n} \|f - p\|_{L^2(\mathbb{C}^n, \mu_r)}^2 < \infty \end{aligned}$$

where $r = 1 - c_3 + c_2 \in (2c_1, 1)$. Because $\mathbb{P}_a[\mathbb{C}^n]$ is dense in $L^2(\mathbb{C}^n, \mu_r) \cap \mathcal{H}(\mathbb{C}^n)$ for all $r > 0$ the first assertion follows.

Now, (4.5) immediately can be derived from $T_{g_z}p = V^z p$ for $p \in \mathbb{P}_a[\mathbb{C}^n]$ and the density result above which implies that:

$$H_{\text{exp}}(\mathbb{C}^n) \subset \text{closure}(\mathbb{P}_a[\mathbb{C}^n], \| \cdot \|_{\text{gr}}) \subset \mathcal{D}(V^z) \cap \mathcal{D}(T_{g_z}).$$

Finally, we apply the continuity of $V^z, T_{g_z} : (\mathbb{P}_a[\mathbb{C}^n], \| \cdot \|_{\text{gr}}) \rightarrow H^2(\mathbb{C}^n, \mu)$. □

For $z \in \mathbb{C}^n$ we denote by $\tilde{V}^z$ the infinitesimal generator of $(W_{sz})_{s \in \mathbb{R}}$ considered as strongly continuous group of unitary operators on $L^2(\mathbb{C}^n, \mu)$. Let $\mathcal{D}(\tilde{V}^z)$ be its domain of definition, then V^z can be obtained by restricting $\tilde{V}^z$ to $\mathcal{D}(V^z)$. For $f \in \text{SP}_{\text{Lip}}(\mathbb{C}^n)$ and $r \in \mathbb{N}$ we write

$$\mathcal{A}_r(f) := \mathcal{A}(\underbrace{[M_f, \dots, M_f]}_{r\text{-times}}) \subset \mathcal{L}(L^2(\mathbb{C}^n, \mu))$$

where the algebra on the right hand side was defined in (3.8) of Proposition 3.3.

Lemma 4.3 *The domain $\mathcal{D}(\tilde{V}^z)$ is invariant under $A \in \mathcal{A}_r(f)$ where f is a linear function on $\mathbb{C}^n$. Moreover, the commutator $[A, \tilde{V}^z]$ vanishes as an operator on $\mathcal{D}(\tilde{V}^z)$.*

Proof: It is sufficient to show that for all $j \in \mathbb{N}$ the space $\mathcal{D}(\tilde{V}^z)$ is invariant under the operators

$$a_j(f) := \text{ad}^j [M_f] (P).$$

Note that $L_{\text{exp}}(\mathbb{C}^n)$ is an invariant under W_z and it holds $W_{-z} M_f W_z = M_{f \circ \tau_z}$. Because W_z commutes with P it follows that:

$$W_{-z} a_j(f) W_z = \text{ad}^j [M_{f \circ \tau_z}] (P) = a_j(f).$$

We have used the linearity of f for the second equality. Hence, the commutator $[A, W_z]$ vanishes for all $A \in \mathcal{A}_r(f)$. Fix $h \in \mathcal{D}(\tilde{V}^z)$ and $A \in \mathcal{A}_r(f)$, then:

$$\frac{1}{s} \{ W_{sz} - I \} A h = A \frac{1}{s} \{ W_{sz} - I \} h \rightarrow A \tilde{V}^z h$$

as s tends to 0. It follows that $Ah \in \mathcal{D}(\tilde{V}^z)$ with $\tilde{V}^z Ah = A \tilde{V}^z h$. $\square$

Remark 4.1 Let W be any subspace of $H := H^2(\mathbb{C}^n, \mu)$ such that $H_{\text{exp}}(\mathbb{C}^n) \subset W$. Consider the operators:

$$\mathcal{O}_W := \{ A \in L(W, H) : H_{\text{exp}}(\mathbb{C}^n) \text{ is an invariant space for } A \}.$$

Let $A \in \mathcal{O}_W$ and assume there is $A^* \in \mathcal{O}_W$ with $\langle Af, g \rangle = \langle f, A^*g \rangle$ for all $f, g \in W$. Because of $K(\cdot, \lambda) \in H_{\text{exp}}(\mathbb{C}^n)$ for all $\lambda \in \mathbb{C}^n$ it follows that A can be written as an integral operator with kernel:

$$K_A(z, w) = \overline{A^* K(\cdot, z)}(w). \quad (4.6)$$

In particular, A completely is determined by the restriction of A^* to $H_{\text{exp}}(\mathbb{C}^n)$. Assume that A has a continuous extensions $\tilde{A}$ from $H_{\text{exp}}(\mathbb{C}^n)$ to $H^2(\mathbb{C}^n, \mu)$. Fix $g \in H^2(\mathbb{C}^n, \mu)$ and a sequence $(g_n)_n \subset H_{\text{exp}}(\mathbb{C}^n)$ with $g = \lim_{n \rightarrow \infty} g_n$. Then it follows for $z \in \mathbb{C}^n$:

$$\begin{aligned} [\tilde{A}g](z) &= \lim_{n \rightarrow \infty} \langle Ag_n, K(\cdot, z) \rangle \\ &= \lim_{n \rightarrow \infty} \langle g_n, A^* K(\cdot, z) \rangle = \langle g, A^* K(\cdot, z) \rangle \end{aligned}$$

and $\tilde{A}$ is given by the same integral formula. In particular, A has a (unique) extension from W to $H^2(\mathbb{C}^n, \mu)$.

Let $h \in L^\infty(\mathbb{C}^n)$ and $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be a linear function. We write $C_j(f, h)$ for the continuous extensions of the commutators

$$\text{ad}^j [T_f] (T_h) \in L(H_{\text{exp}}(\mathbb{C}^n))$$

to $H^2(\mathbb{C}^n, \mu)$, (note that $f \in \text{SP}_{\text{Lip}}(\mathbb{C}^n)$ and Proposition 3.3).

Corollary 4.1 *Let $h \in L^\infty(\mathbb{C}^n)$. Assume that $\mathcal{D}(\tilde{V}^z)$ is invariant under the multiplication operator M_h . Then $\mathcal{D}(V^z)$ is invariant under $C_j(f, h)$ for all $j \in \mathbb{N}$.*

Proof: According to (3.7) there is a finite index set $\mathcal{I}$ and $A_l, B_l \in \mathcal{A}_j(f)$ such that

$$\text{ad}^j[T_f](T_h) = \sum_{l \in \mathcal{I}} P A_l M_h B_l P.$$

Due to our assumption on h and by Lemma 4.3 the assertion follows. $\square$

Now, we can proof our main result on the smoothness of Toeplitz operators with respect to the Segal-Bargmann representation ρ of the Heisenberg group:

Theorem 4.1 *Let $h \in \mathcal{S}_s := \mathcal{S} \cap \overline{\mathcal{S}}$ where $\overline{\mathcal{S}} = \{\bar{h} : h \in \mathcal{S}\}$ and*

$$\mathcal{S} := \{h \in L^\infty(\mathbb{C}^n) : \text{ s. t. } \mathcal{D}(\tilde{V}^z) \text{ is invariant under } M_h \text{ for all } z \in \mathbb{C}^n\}.$$

Then the symbol map into the Ψ^ -algebra Ψ^∞ given by:*

$$\mathcal{S}_s \ni h \mapsto T_h \in \Psi^\infty$$

is well-defined and continuous if $\mathcal{S}_s$ carries the $L^\infty(\mathbb{C}^n)$ -topology.

Proof: Using our notation in (4.2) and (4.3) we must show that $T_h \in \Psi^{\infty, z} = \Psi_\infty^{\mathcal{V}^z}$ for all complex directions $z \in \mathbb{C}^n$ and $\mathcal{V}^z := [iV^z]$:

$\mathcal{D}(V^z)$ is invariant under T_q for $q \in \{h, \bar{h}\} \subset \mathcal{S}_s$ and by Lemma 4.2 it follows that the commutators $A_1 := [iV^z, T_q]$ and $[T_{ig_z}, T_q]$ coincide on $H_{\text{exp}}(\mathbb{C}^n)$. Because iV^z is self-adjoint we can define $A_1^* := [T_q, iV^z]$ and $W := \mathcal{D}(V^z)$ in Remark 4.1. The operator $[T_{ig_z}, T_q]$ has a bounded extension $C_1(ig_z, q)$ from $H_{\text{exp}}(\mathbb{C}^n)$ to $H^2(\mathbb{C}^n, \mu)$. We conclude from Remark 4.1 that $C_1(ig_z, q)$ is an extension of A_1 from W to $H^2(\mathbb{C}^n, \mu)$ and $T_q \in \Psi_1^{\mathcal{V}^z}$. By induction we must prove for $j \in \mathbb{N}$:

- (1) The domain of definition $\mathcal{D}(V^z)$ is invariant under $C_j(ig_z, q)$,
- (2) The commutators $A_{j+1} := [iV^z, C_j(ig_z, q)]$ have the bounded extension $C_{j+1}(ig_z, q)$ from $\mathcal{D}(V^z)$ to $H^2(\mathbb{C}^n, \mu)$.

Assertion (1) is a direct consequence of Corollary 4.1 and (2) can be derived from Remark 4.1 with $A_{j+1}^* := [C_j(ig_z, q)^*, iV^z]$ on $W := \mathcal{D}(V^z)$ ³ and the fact that A_{j+1} has the continuous extension $C_{j+1}(ig_z, q)$ from $H_{\text{exp}}(\mathbb{C}^n)$ to $H^2(\mathbb{C}^n, \mu)$. The continuity of the symbols map follows from (2.3) together with the continuity of (3.6) in Proposition 3.3. $\square$

³Note that by Corollary 4.1 and the identity $C_j(ig_z, q)^* = (-1)^j C_j(ig_z, \bar{q})$ the commutator A_{j+1}^* is well-defined on $\mathcal{D}(V^z)$.

5 Examples and Applications

Let $\mathcal{A}$ denote the subalgebra of $\mathcal{L}(L^2(\mathbb{C}^n, \mu))$ of all multiplication operators with bounded symbols $h \in L^\infty(\mathbb{C}^n)$. For $z \in \mathbb{C}^n$ and with $\tilde{V}^z := [i\tilde{V}^z]$ there is a scale of algebras arising by commutator methods:

$$\mathcal{A} \supset \Psi_1^{\tilde{V}^z} \supset \dots \Psi_n^{\tilde{V}^z} \supset \Psi_{n+1}^{\tilde{V}^z} \supset \dots \Psi_\infty^{\tilde{V}^z} = \bigcap_{n \in \mathbb{N}} \Psi_n^{\tilde{V}^z}. \quad (5.1)$$

In general, the inclusions above will be proper. As an immediate consequence of Theorem 4.1 it follows for the projected scale of vector spaces:

$$\mathcal{A}_P \supset \Psi_1^{\tilde{V}^z}_P = \dots = \Psi_n^{\tilde{V}^z}_P = \Psi_{n+1}^{\tilde{V}^z}_P = \dots = \Psi_\infty^{\tilde{V}^z}_P. \quad (5.2)$$

Here $\mathcal{A}_P \subset \mathcal{L}(H^2(\mathbb{C}^n, \mu))$ is the space of Toeplitz operators with bounded measurable symbols. By passing from (5.1) to the scale (5.2) the underlying C^k -structure is lost.

We give an example of a class of bounded functions g such that $\mathcal{D}(\tilde{V}^z)$ is an invariant subspace for M_g and $M_{\bar{g}}$ for all $z \in \mathbb{C}^n$.

Example 5.1 Denote by $C_c^\infty(\mathbb{C}^n)$ the space of compactly supported smooth functions. For $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ we write $z_j := x_j + iy_j$ and with $\alpha, \beta \in \mathbb{N}_0^n$:

$$z^{\alpha, \beta} := x^\alpha y^\beta, \quad \partial^{\alpha, \beta} := \frac{\partial^{|\alpha|}}{\partial x^\alpha} \frac{\partial^{|\beta|}}{\partial y^\beta}.$$

Fix $h \in \mathcal{D}(\tilde{V}^z)$ and $z \in \mathbb{C}^n$. For $g \in C_c^\infty(\mathbb{C}^n)$ (real valued) and $s \neq 0$ we write:

$$\frac{1}{s} [W_{sz} - I] M_g h = \frac{1}{s} [M_{g \circ \tau_{-sz}} - M_g] W_{sz} h + M_g \frac{1}{s} [W_{sz} - I] h. \quad (5.3)$$

The second term converges in $L^2(\mathbb{C}^n, \mu)$ as $s \rightarrow 0$. Consider the smooth and compactly supported function $dg(z, \cdot) := -\langle \text{grad } g(\cdot), z \rangle_{\mathbb{R}^{2n}}$. Then:

$$\begin{aligned} C_{s,z} &:= \left\| \frac{1}{s} [M_{g \circ \tau_{-sz}} - M_g] - M_{dg(z, \cdot)} \right\| \\ &= \left\| \frac{1}{s} [g \circ \tau_{-sz} - g] - dg(z, \cdot) \right\|_\infty \leq \sum_{|\alpha|+|\beta|=2} \frac{|s|}{(\alpha+\beta)!} \|\partial^{\alpha, \beta} g\|_\infty |z^{\alpha, \beta}|. \end{aligned}$$

Hence $\lim_{s \rightarrow 0} C_{s,z} = 0$ and the right hand side of

$$\left\| \frac{1}{s} [M_{g \circ \tau_{-sz}} - M_g] W_{sz} h - M_{dg(z, \cdot)} h \right\| \leq C_{s,z} \|h\| + \|dg(z, \cdot)\|_\infty \|(W_{sz} - I)h\|$$

tends to 0 as $s \rightarrow 0$. It follows $gh \in \mathcal{D}(V^z)$. With our notation of Theorem 4.1 we conclude that $C_c^\infty(\mathbb{C}^n) \subset \mathcal{S}_s$. By the continuity of

$$L^\infty(\mathbb{C}^n) \subset \mathcal{S}_s \ni h \mapsto T_h \in \Psi^\infty$$

and the fact that $C_c^\infty(\mathbb{C}^n)$ is uniformly dense in the space $C_0(\mathbb{C}^n)$ of all continuous functions vanishing at infinity it follows that $\{T_h : h \in C_0(\mathbb{C}^n)\} \subset \Psi^\infty$.

In our second example we construct a compact operator $A \in \mathcal{B} := \mathcal{L}(H^2(\mathbb{C}, \mu))$ which is not contained in $\Psi^{1,z}$ for any $z \in \mathbb{C}$ (with our notation in (4.2)). As a consequence and using Example 5.1 A is not limit point of finite sums of finite products of Toeplitz operators with symbols in $C_0(\mathbb{C})$ and with respect to the Fréchet topology of $\Psi^{\infty,z}$. However, since A is compact it can be approximated by Toeplitz operators with smooth and compactly supported symbols in the topology of $\mathcal{B}$, c.f. [8].

Example 5.2 For $j \in \mathbb{N}_0$ let $P_j \in \mathcal{B}$ be the rank one projection onto $\text{span}\{m_j := z^j\}$. With a sequence $a := (a_n)_{n \in \mathbb{N}}$ tending to zero consider the compact diagonal operator:

$$A := \sum_{j \in \mathbb{N}} a_j P_j \in \mathcal{B}.$$

With $z \in \mathbb{C}$, $|z| = 1$ and $g_z := 2i \text{Im}\langle \cdot, z \rangle$ we compute $[T_{g_z}, A]m_j = [V^z, A]m_j$ explicitly for all $j \in \mathbb{N}$. By (4.4) one obtains that:

$$\begin{aligned} [T_{g_z}, A]m_j &= a_j T_{g_z} m_j - A[\bar{z} m_{j+1} - j z m_{j-1}] \\ &= a_j (\bar{z} m_{j+1} - j z m_{j-1}) - (a_{j+1} \bar{z} m_{j+1} - j a_{j-1} z m_{j-1}) \\ &= (a_j - a_{j+1}) \bar{z} m_{j+1} - j z (a_j - a_{j-1}) m_{j-1}. \end{aligned}$$

With $e_j := (j!)^{-\frac{1}{2}} z^j$ we have $\langle e_j, e_l \rangle_2 = \delta_{l,j}$ for all $j, l \in \mathbb{N}$. Hence it follows that

$$\|[T_{g_z}, A]e_j\|_2^2 = (j+1) |a_j - a_{j+1}|^2 + j |a_j - a_{j-1}|^2. \quad (5.4)$$

We choose a such that the right hand side of (5.4) tends to infinity for $j \rightarrow \infty$. This can be done by the choice of an oscillating sequence $a_j := (-1)^j j^{-\frac{1}{4}}$. Then it follows

$$(j+1) |a_j - a_{j+1}|^2 = (j+1) |j^{-\frac{1}{4}} + (j+1)^{-\frac{1}{4}}|^2 \geq \sqrt{j+1}$$

and so the right hand side of (5.4) is unbounded for $j \rightarrow \infty$. Hence $[T_{g_z}, A]$ has no bounded extension to $H^2(\mathbb{C}^n, \mu)$ and $A \notin \Psi^{1,z}$ by Proposition 4.1.

Let $\beta : L^2(\mathbb{R}^n) \rightarrow H^2(\mathbb{C}^n, \mu)$ denote the *Bargmann isometrie*, c.f. [10]. Our results on Toeplitz operators on $H^2(\mathbb{C}^n, \mu)$ can be used in the analysis of a class of *Gabor-Daubechies windowed localization operators* $L_h := \beta^{-1} T_h \beta$ ⁴ on $L^2(\mathbb{R}^n)$ where $h \in L^\infty(\mathbb{C}^n)$, c.f. [9]. It was remarked in [14] the operator L_h can be considered as a pseudodifferential operator $W_{\sigma(h)}$ in Weyl quantization with *Weyl symbol* $\sigma(h)$ on $\mathbb{R}^{2n}$. Via the identification of $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ the correspondence between h and $\sigma(h)$ can be expressed in terms of the heat equation on $\mathbb{R}^{2n}$. More precisely, $\sigma(h)$ is a solution with initial data h at a fixed time $t_0 > 0$. In the next example we describe how the operators introduced in the previous sections transform under β , c.f. [10].

⁴Here the window is a Hermite function on $\mathbb{R}^n$

Example 5.3 For $u \in L^2(\mathbb{R}^n)$ it is well-known that βu can be expressed by the integral:

$$[\beta u](z) = (2\pi)^{-\frac{n}{4}} \int_{\mathbb{R}^n} u(x) \exp \left\{ \langle x, z \rangle - \frac{1}{4}|x|^2 - \frac{1}{2}\langle z, \bar{z} \rangle \right\} dx.$$

Fix $a = p + iq \in \mathbb{C}^n$, then it can be checked that $W_a \in \mathcal{L}(H^2(\mathbb{C}^n, \mu))$ transform as:

$$B_a u := [\beta^{-1} W_a \beta](u) = u(\cdot - 2p) \exp \{ iq(p - \cdot) \}.$$

In particular, in the case $q = 0$ the unitary operator B_a is a usual shift in direction $2p$. For $j = 1, \dots, n$ it is readily verified that T_{z_j} and $T_{\bar{z}_j}$ transform in the following way:

$$(i) \quad \beta^{-1} T_{z_j} \beta = \frac{1}{2} x_j - \partial_{x_j},$$

$$(ii) \quad \beta^{-1} T_{\bar{z}_j} \beta = \frac{1}{2} x_j + \partial_{x_j}$$

From (i), (ii) and for $\alpha \in \mathbb{N}_0^n$ one obtains the identity:

$$\beta \partial_x^\alpha = (-1)^{|\alpha|} T_{i\text{Im}z_1}^{\alpha_1} \cdots T_{i\text{Im}z_n}^{\alpha_n} \beta =: (-1)^{|\alpha|} T_{i\text{Im}z}^\alpha \beta.$$

Let $g \in \mathcal{D}(\mathbb{R}^n)$ be a test function and fix $f \in H_{\text{exp}}(\mathbb{C}^n)$. It follows that:

$$\langle \beta^{-1} f, \partial_x^\alpha g \rangle_{L^2(\mathbb{R}^n)} = \langle f, \beta \partial_x^\alpha g \rangle = \langle \beta^{-1} T_{i\text{Im}z_1}^{\alpha_1} \cdots T_{i\text{Im}z_n}^{\alpha_n} f, g \rangle_{L^2(\mathbb{R}^n)}.$$

Here we have used the fact that $H_{\text{exp}}(\mathbb{C}^n)$ is invariant under all unbounded Toeplitz operators $T_{i\text{Im}z_j}$ which was proved in Proposition 3.1. It follows that:

$$\mathbf{D} := \beta^{-1} [H_{\text{exp}}(\mathbb{C}^n)] \subset H^\infty(\mathbb{R}^n) = \bigcap_{k \in \mathbb{N}} H^k(\mathbb{R}^n)$$

where $H^k(\mathbb{R}^n)$ denotes the k -th Sobolev space. Hence, for $\alpha, \beta \in \mathbb{N}_0^n$ the restriction of (2.1) in Theorem 2.1 to $\mathbf{D}$:

$$\text{ad}[-ix]^\alpha \text{ad}[i\partial_x]^\beta(B) : \mathbf{D} \rightarrow \mathbf{D} \quad (5.5)$$

is well-defined for any $B \in L(\mathbf{D})$. With the choice $h \in L^\infty(\mathbb{C}^n)$ and $L_h := \beta^{-1} T_h \beta \in L(\mathbf{D})$ we obtain by conjugating (5.5) with β and using (i), (ii) above:

$$\text{ad}[iT_{2\text{Re}z}]^\alpha \text{ad}[T_{\text{Im}z}]^\beta(T_h) : H_{\text{exp}}(\mathbb{C}^n) \rightarrow H_{\text{exp}}(\mathbb{C}^n). \quad (5.6)$$

It follows by Proposition 3.3 that the operators in (5.6) have bounded extensions to $H^2(\mathbb{C}^n, \mu)$ and so (5.5) can be extended continuously to $L^2(\mathbb{R}^n)$. Hence we have proved a weaker version of the defining property (2.1) for $\Psi_{\rho, \delta}^0$ in Theorem 2.1.

Since the Gaussian measure μ is invariant under unitary transformations of $\mathbb{C}^n$, there is a natural group representation of U_n in $\mathcal{L}(H^2(\mathbb{C}^n, \mu))$ generating Ψ^* -algebras of smooth elements. As a final example we want to remark:

Example 5.4 Let $A \in \mathbb{R}^{n \times n}$ be self-adjoint and consider the unitary group:

$$\mathbb{R} \ni t \mapsto e^{itA} \in U_n.$$

The group of unitary composition operators $C_t f := f \circ e^{itA}$ on $H^2(\mathbb{C}^n, \mu)$ can be shown to be strongly continuous, cf. [3]. The restriction of the infinitesimal generator L_A of $(C_t)_{t \in \mathbb{R}}$ to $\mathbb{P}_a[\mathbb{C}^n]$ coincides with an (unbounded) Toeplitz operator. More precisely, it was shown in [3] that:

$$L_A p = [T_{\langle Az, z \rangle} - n \cdot \text{trace}(A)]p, \quad p \in \mathbb{P}_a[\mathbb{C}^n].$$

Hence, in general the symbol of L_A regarded as a Toeplitz operator is a polynomial of degree 2, which is not globally lipschitz continuous on $\mathbb{C}^n$. Proposition 3.3 cannot be applied in this situation and the smoothness of a Toeplitz operator T_f with bounded symbols f with respect to $(C_t)_t$ requires further assumption on the symbol f . For a more detailed calculation we refer to [3].

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