

Equivariant homotopy groups, Power operations
and the equivariant Kahn-Priddy Theorem

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1. Introduction. Let G be a finite group and V a finite dimensional real G -module with an invariant metric. S^V and B^V will denote the unit sphere and ball in V , and $\Sigma^V = B^V/S^V$. Let X be a finite G -complex with base point $(\in X^G)$. The stable G -cohomotopy group $\omega_G^\alpha(X)$ is usually defined for $\alpha \in RO(G)$ by the formula

$$\omega_G^\alpha(X) = \operatorname{colim}_U [\Sigma^{U \oplus W} X, \Sigma^{U \oplus V}]^G,$$

where $\alpha = V - W$ and U runs through finite dimensional G -modules. But we are interested here in multiplicative structure with respect to smash products, so we restrict ourselves to $(R(G) + \mathbb{Z})$ -graded theory ~~.....~~, i.e., $\omega_G^\alpha(X)$ is defined only for $\alpha \in R(G) + \mathbb{Z}$ by the same formula as above restricting all G -modules U , V and W to complex ones up to oriented real trivial summands. Then the multiplication is always commutative in graded sense.

In equivariant homotopy theory there are two types of natural homomorphisms for $H < G$:

1.

$$\psi_H = \psi_H^G : \omega_G^\alpha(X) \rightarrow \omega_H^{\psi_H^\alpha}(X)$$

called the forgetful (or restriction) morphism, and

$$\phi_H = \phi_H^G : \omega_G^\alpha(X) \rightarrow \omega_{W(H)}^{\phi_H^\alpha}(X^H)$$

called the fixed-point morphism, where, for $\alpha = V-W \in R(G)+Z$,

$$\psi_H^\alpha = \text{res}_H^G V - \text{res}_H^G W \in R(H)+Z,$$

$$\phi_H^\alpha \cong V^H - W^H \in R(W(H))+Z, \quad W(H) = N(H)/H.$$

These are multiplicative as is easily seen.

In case $G = \mathbb{Z}/2$, there are exact sequences involving the forgetful and fixed-point morphisms (after Landweber), which played important roles in our previous work to compute $\pi_{p,q}^S$, $p+q \leq 13$ (jointly with K. Iriye). We observed also that the combination of these two exact sequences and squaring operation gives the Kahn-Priddy theorem for $\mathbb{Z}/2$.

Here we observe how these machines for $G = \mathbb{Z}/2$ can be generalized for more general groups, which implies the equivariant Kahn-Priddy theorem at least for $p = 2$.

2. The forgetful and fixed-point exact sequences.

Let $G > H$, and X be a finite H -complex. There holds the canonical isomorphism

$$c : \omega_G^\alpha(G_+ \wedge_H X) \approx \omega_H^{\psi_H^\alpha}(X).$$

When X is a G -complex, $G_+ \wedge_H X \approx_G (G/H)_+ \wedge X$ and the map $G/H \rightarrow \text{pt}$ induces the forgetful morphism

$$\psi_H : \omega_G^\alpha(X) \rightarrow \omega_H^{\psi_H^\alpha}(X).$$

Thus the G -cofibration $(G/H)_+ \rightarrow C(G/H)_+ \rightarrow \Sigma(G/H)$ (where C and Σ denote unreduced cone and suspension) induces an exact sequence involving the forgetful morphism, which may be called the forgetful exact sequence (even though $\omega_G^\alpha(\Sigma(G/H) \wedge X)$ might be generally not simple to discuss).

As to the fixed-point morphism

$$\phi_H : \omega_G^\alpha(X) \rightarrow \omega_{W(H)}^{\phi_H^\alpha}(X^H),$$

first we remark that it can be decomposed as

$$\phi_H = \phi_H^{N(H)} \circ \psi_{N(H)}^G,$$

so we would like to be satisfied if we get exact sequences involving $\psi_{N(H)}^G$ and $\phi_H^{N(H)}$ separately. Thus we consider only the case of a normal subgroup.

Let V and W be finite dimensional complex G -modules, and X be a pointed finite G -complex. The G -cofibration $S_+^V \rightarrow B_+^V \rightarrow \Sigma^V$ induces the exact sequence

$$\dots \rightarrow \omega_G^{\alpha+V-1}(S_+^V \wedge X) \xrightarrow{\delta_V} \omega_G^\alpha(X) \xrightarrow{\chi_V} \omega_G^{\alpha+V}(X) \xrightarrow{\beta_V} \omega_G^{\alpha+V}(S_+^V \wedge X) \rightarrow \dots,$$

where $\chi_V : \omega_G^\alpha(X) \rightarrow \omega_G^{\alpha+V}(X)$ is induced by the inclusion $i_V : \Sigma^0 \subset \Sigma^V$. $\chi_V = \chi_V^{*1} \in \omega_G^V$ is called the Euler class of V (after tom Dieck). When $V^G \neq \{0\}$, then $\chi_V = 0$.

So we observe only the case that V contains no trivial G -modules. The commutative diagram of G -cofibrations

$$\begin{array}{ccccc} S_+^{V \oplus W} & \rightarrow & B_+^{V \oplus W} & \rightarrow & \Sigma^{V \oplus W} \\ \downarrow & & \downarrow & & \parallel \\ S^{V \oplus W}/S^V \times B^W & \rightarrow & B^{V \oplus W}/S^V \times B^W & \rightarrow & \Sigma^{V \oplus W} \\ \parallel & & \parallel & & \nearrow \\ \Sigma^V(S_+^W) & \rightarrow & \Sigma^V(B_+^W) & \rightarrow & \Sigma^V(i_W) \end{array}$$

induces the following commutative diagram of exact sequences:

$$(2.1) \quad \begin{array}{ccccccc} \dots & \rightarrow & \omega_G^{\alpha+W-1}(S_+^W \wedge X) & \xrightarrow{\delta_W} & \omega_G^\alpha(X) & \xrightarrow{\chi_W} & \omega_G^{\alpha+W}(X) \rightarrow \dots \\ & & \downarrow \xi_{W, W \oplus V} & & \parallel & & \downarrow \chi_V \\ \dots & \rightarrow & \omega_G^{\alpha+V+W-1}(S_+^{V \oplus W} \wedge X) & \xrightarrow{\delta_{V+W}} & \omega_G^\alpha(X) & \xrightarrow{\chi_{V+W}} & \omega_G^{\alpha+V+W}(X) \rightarrow \dots \end{array}$$

Let $G \triangleright N$, a normal subgroup. Put $\text{Irr}(G)$ = the set of all isomorphism classes of complex irreducible G -modules.

Decompose

$$\text{Irr}(G) = A_N \amalg B_N$$

by " $V \in A_N \Leftrightarrow \text{res}_N^G V$ is non-trivial" ($\Leftrightarrow V^N = \{0\}$ as N is normal). Then $V \in B_N \Leftrightarrow V^N = V$. Put

$$\bar{A}_N = \{ \text{finite sums of elements of } A_N \},$$

$$\bar{B}_N = \{ \text{finite sums of elements of } B_N \cup \{R_{\text{triv}}\} \}.$$

Define

$$\lambda_{G,N}^\alpha(X) = \text{colim}_{V \in \bar{A}_N} \{ \omega_G^{\alpha+V-1}(S_+^V \wedge X), \chi_V \}, \quad V, V+W, V, W \in \bar{A}_N.$$

Taking the colimit of the diagram (2.1) with respect to $V \in \bar{A}_N$ we get the exact sequence

$$(2.2) \quad \dots \rightarrow \lambda_{G,N}^\alpha(X) \xrightarrow{\delta_N} \omega_G^\alpha(X) \rightarrow \text{colim}_{V \in \bar{A}_N} \{ \omega_G^{\alpha+V}(X), \chi_V \} \rightarrow \dots$$

Now we get

$$\text{Proposition 2.3.} \quad \text{colim}_{V \in \bar{A}_N} \{ \omega_G^{\alpha+V}(X), \chi_V \} \approx \omega_{G/N}^{\phi_N^\alpha(X^N)}.$$

Thus we get the desired exact sequence

$$(2.4) \quad \dots \rightarrow \lambda_{G,N}^\alpha(X) \xrightarrow{\delta_N} \omega_G^\alpha(X) \xrightarrow{\phi_N} \omega_{G/N}^{\phi_N^\alpha(X^N)} \rightarrow \dots,$$

which we call the fixed-point exact sequence.

3. Transfer-type morphisms.

Here we observe transfer-type morphisms for ψ_H and ϕ_H .
Let $G > H$. First we consider the transfer to ψ_H . There exists a finite dimensional complex G -module V and a G -embedding $i : G/H \hookrightarrow V$. Let $\mathcal{V}(i)$ be the G -tubular neighborhood of $i(G/H)$. Then $\mathcal{V}(i) \approx_G G \times_H B^V$. Collapsing the outside of $\mathcal{V}(i)$ we get a G -map

$$\Sigma^V \rightarrow (G \times_H B^V) / (G \times_H S^V) \approx_G G_+ \wedge_H \Sigma^V,$$

which induces a G -map

$$\Sigma^V X \rightarrow (G_+ \wedge_H \Sigma^V) \wedge X \approx_G G_+ \wedge_H (\Sigma^V \wedge X)$$

for any pointed G -complex X . Now we get the transfer to ψ_H :

$$(3.1) \quad \text{tr} = \text{tr}_H^G : \omega_H^{\psi_H^\alpha} (X) \rightarrow \omega_G^\alpha (X)$$

as the composition of the following:

$$\begin{aligned} \omega_H^{\psi_H^\alpha} (X) &\approx \omega_H^{\psi_H^{\alpha+V}} (\Sigma^V X) \stackrel{c}{\approx} \omega_G^{\alpha+V} (G_+ \wedge_H (\Sigma^V X)) \\ &\rightarrow \omega_G^{\alpha+V} (\Sigma^V X) \approx \omega_G^\alpha (X). \end{aligned}$$

We have

$$\text{Proposition 3.2.} \quad \text{tr}_H^G \circ \psi_H^G = [G/H],$$

the multiplication with $[G/H] \in A(G)$.

tr_H^G can be decomposed as the following composition. We may assume $i\{H\} \in S^V$. Decompose $\text{res}_H^G V = W \oplus \underline{R}$, $\underline{R} \ni i\{H\}$.

Then $W = \text{res}_H^G (V - R) \in R(H) + Z$, $T_{i_H} S^V = W$ and $i(G/H) \subset S^V$. As the Gysin homomorphism for this inclusion we get

$$(3.3) \quad j_V : \omega_H^{\gamma_H \alpha}(X) \rightarrow \omega_G^{\alpha+V-1}(S_+^V \wedge X).$$

Also, as the connecting morphism for the G -cofibration $S_+^V \rightarrow B_+^V \rightarrow \Sigma^V$ we get

$$(3.4) \quad \delta_V : \omega_G^{\alpha+V-1}(S_+^V \wedge X) \rightarrow \omega_G^{\alpha}(X).$$

Then we get

$$\text{Proposition 3.5.} \quad \text{tr}_V^G = \delta_V \circ j_V.$$

Let $G \triangleright N$, a normal subgroup. We define the transfer-type morphism for ϕ_N . Remark that B_N corresponds bijectively with $\text{Irr } G/N$. Thus we have the natural inclusion $R(G/N) + Z \subset R(G) + Z$. Any G/N -complex is naturally a G -complex. Thus we get a natural homomorphism

$$\theta = \theta_{G/N}^G : \omega_{G/N}^{\alpha}(X) \rightarrow \omega_G^{\alpha}(X)$$

for $\alpha \in R(G/N) + Z \subset R(G) + Z$ and G/N -complex X .

$$\text{Proposition 3.6.} \quad \phi_N \circ \theta_{G/N}^G = \text{id}.$$

In particular, the fixed-point exact sequence splits for $\alpha \in R(G/N) + Z$, i.e.,

Corollary 3.7. Let $G \triangleright N$, $\alpha \in R(G/N) + Z$ and X be a G/N -complex. Then

$$\omega_G^{\alpha}(X) \approx \lambda_{G,N}^{\alpha}(X) \oplus \omega_{G/N}^{\alpha}(X).$$

We obtain also

Proposition 3.8. Let $G = K \cdot L$, semi-direct product,
such that $G \triangleright N$. Then

$$\gamma_K \circ \theta_{G/N}^G = \text{id}.$$

for $\alpha \in R(K) + \mathbb{Z}$.

Let $G = K \cdot N$, semi-direct product, as above. There exist $V \in \bar{A}_N$ and a G -embedding $G/H \subset S^V$. Let

$$(3.9) \quad \kappa : \omega_G^{\alpha+V-1}(S_+^V \wedge X) \rightarrow \lambda_{G,N}^{\alpha}(X)$$

be the canonical map. Let X be a K -complex and $\alpha \in R(K) + \mathbb{Z}$. By composing (3.3) and (3.9) we get a natural homomorphism

$$(3.10) \quad k = \kappa \circ j_V : \omega_K^{\alpha}(X) \rightarrow \lambda_{G,N}^{\alpha}(X).$$

Proposition 3.11. Under this situation

$$\gamma_K \circ \delta_N \circ k = [G/K]_K : \omega_K^{\alpha}(X) \rightarrow \omega_K^{\alpha}(X),$$

the multiplication with $[G/K]_K = i^*[G/K]$, where $i : K \subset G$.

4. Power operations.

Let $G \triangleright H$, a subgroup, and V an H -module. Remark that

$$\text{ind}_H^G V = \prod (G \times_H V \rightarrow G/H),$$

the module of sections of the bundle $G \times_H V \rightarrow G/H$. Let X

be a pointed H -complex. In a parallel way we define

$$\text{ind}_H^G X = \prod (G \times_H X \rightarrow G/H),$$

the module of sections of the bundle $G \times_H X \rightarrow G/H$. $\text{ind}_H^G X$ is a G -complex which is topologically $X^{\wedge |G/H|}$. G -actions on $\text{ind}_H^G X$ preserves axes $\bigcup X \times \dots \times X \times \{\text{pt}\} \times X \times \dots \times X$. Thus, passing to quotients we get

$$\widetilde{\text{ind}}_H^G X = (\text{ind}_H^G X) / \{\text{axes}\},$$

which is topologically $X^{\wedge |G/H|}$.

Let $x \in \omega_H^\alpha(X)$, represented by an H -map $f : \Sigma^{U \oplus W} X \rightarrow \Sigma^{U \oplus V}$. Take

$$f^{\wedge |G/H|} : \widetilde{\text{ind}}_H^G(\Sigma^{U \oplus W} X) \rightarrow \widetilde{\text{ind}}_H^G(\Sigma^{U \oplus V}),$$

which is a G -map because the corresponding map of bundles is a G -map. We see that $f^{\wedge |G/H|}$ represents

$$(4.1) \quad \rho_{\text{ext}}(x) \in \omega_G^{\text{ind}_H^G \alpha}(\widetilde{\text{ind}}_H^G X),$$

called the external power of x . ρ_{ext} is not linear in general. However, let $\rho_{G,H}$ be the permutation representation of G/H . Put

$$(4.2) \quad \rho_{G,H} = \widetilde{\rho}_{G,H} \oplus 1.$$

Then

Proposition 4.3. $\chi_{\widetilde{\rho}} \circ \rho_{\text{ext}} : \omega_H^\alpha(X) \rightarrow \omega_G^{\text{ind}_H^G \alpha + \widetilde{\rho}}(\widetilde{\text{ind}}_H^G X)$ is a linear homomorphism, where $\widehat{\rho} = \widetilde{\rho}_{G,H}$.

Let V be a G -module. Then

$$\text{ind}_H^G \circ \text{res}_H^G V \approx \rho_{G,H} \otimes V.$$

Similarly, if X is a G -complex, then

$$G \times_H X \approx_G G/H \times X,$$

which induces a G -homeomorphism

$$\widetilde{\text{ind}}_H^G X \approx_G X^{\wedge |G/H|}$$

where G acts on the right hand side by the simultaneous actions of diagonal ones and permutations of factors by left actions on G/H . In particular, the diagonal map

$$\Delta_X : X \rightarrow X^{\wedge |G/H|}$$

is a G -map. Thus we get the internal power operation

$$\rho = \Delta_X^* \circ \rho_{\text{ext}} : \omega_H^\alpha(X) \rightarrow \omega_G^{\text{ind}_H^G \alpha}(X)$$

for a pointed G -complex X .

Proposition 4.4. Let $G \triangleright N$, normal, and $x \in \omega_N^\alpha(X)$ for a pointed G -complex X . Then

$$\psi_N \circ \rho(x) = \prod_{i=1}^{|G/N|} x^{g_i},$$

where $G = \coprod g_i N$ and $x \mapsto x^{g_i}$ is induced by conjugation with respect to g_i .

Proposition 4.5. Let $G \cdot N$, a semi-direct product such that $G \triangleright N$. Let X be a pointed K -complex (regarded as a G -complex through $G \rightarrow G/N = K)$. Let $\alpha \in R(K) + \mathbb{Z}$. Then the following diagram is commutative:

$$\begin{array}{ccc} \omega_K^\alpha(X) & \xrightarrow{\rho} & \omega_G^{\text{ind}_K^G \alpha}(X) \\ & \searrow \text{id.} \approx & \downarrow \phi_N \\ & & \omega_{G/N}^\alpha(X). \end{array}$$

Under the same situation as above, let V' be a finite dimensional complex K -module and $V = V' \oplus \mathbb{R}^k$, $k \geq 1$. Then $W = \text{ind}_K^G V - V$ contain a real $\tilde{\rho}_{G,K}$ as a summand. Remark that $W^N = \{0\}$ and $\phi_N \circ \chi_W = \phi_N$. We get two splittings of the fixed-point exact sequences for $\lambda = -V$:

$$0 \rightarrow \lambda_{G,N}^{-V}(X) \xrightarrow{\delta_N} \omega_G^{-V}(X) \xrightarrow{\phi_N} \omega_K^{-V}(X) \rightarrow 0.$$

Proposition 4.6. Under this situation $\psi_K \circ \chi_W \circ \rho = 0$.

Then the difference $\theta_{G/N}^G - \chi_W \circ \rho$ gives a homomorphism

$$(4.7) \quad \hat{\theta}_K : \omega_K^{-V}(X) \rightarrow \lambda_{G,N}^{-V}(X)$$

such that $\delta_N \circ \hat{\theta}_K = \theta_{G/N}^G - \chi_W \circ \rho$, $\psi_K \circ \delta_N \circ \hat{\theta}_K = \text{id}$.

Theorem 4.8. Under the above situation

$$\psi_K \circ \delta_N : \lambda_{G,N}^{-V}(X) \rightarrow \omega_K^{-V}(X)$$

is a split epimorphism with the splitting $\hat{\theta}_K$.

5. The equivariant Kahn-Priddy Theorem.

In Theorem 4.8 we put $N = \mathbb{Z}/2$ and $G = K \times \mathbb{Z}/2$. Using Clifford $C(W)$ -module (where W is a K -module) and equivariant S -duality we can prove the isomorphism

$$(5.1) \quad \lambda_{G,\mathbb{Z}/2}^{-V}(X) \simeq \omega_V^K(X; \mathbb{R}P_+^{\omega \rho})$$

which is natural with respect to X , where $\mathbb{R}P^{\omega \rho}$ is the real projective space in ω -regular representation of K , regarded as a K -complex.

A combination of Prop.3.11, Theorem 4.8 and (5. 1) implies

Theorem 5.2. There holds an epimorphism

$$\omega_{V(RP^{\infty})}^G(2) \rightarrow (\omega_V^G)(2)$$

at 2-primary components, where $V = V' \oplus R^k$, $k \geq 1$, and V' is a finite dimensional complex G-module.

The above theorem is the equivariant version of the Kahn-Priddy theorem for $p = 2$.