

Selberg trace formula and Jacobi forms

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1 Introduction

In this note we present a calculation of the traces of Hecke operators acting on the spaces of Jacobi forms via the general Selberg trace formula. We can represent those traces in a closed form with the use of some arithmetic quantities and the residues at poles of certain Selberg type zeta functions. The calculation of those traces has been done by Skoruppa-Zagier ([S-Z1, 2]) in some cases in a different manner. They have employed the Bergman kernel functions for the spaces of Jacobi forms and also some results of Shimura [Sh] concerning modular forms of half integral weight. Here we use the general Selberg trace formula due originally to Selberg [Se] and to Hejhal [He], Fischer [Fi]. For our calculation we exclusively follow Fischer [Fi].

In this short survey we exhibit only the results which is a generalization of our previous work [Ar] and we shall give a proof in another occasion in details.

2 Jacobi forms and Hecke operators

We use the symbol $e(\alpha)$ as an abbreviation of $\exp(2\pi i\alpha)$. Let l be a positive integer. Let $G_{\mathbb{Q}}^J$ be the Jacobi group defined over $\mathbb{Q}$:

$$G_{\mathbb{Q}}^J = \{(g, (\lambda, \mu), \rho) \mid g \in \mathbb{Q}^l, \lambda, \mu \in \mathbb{Q}^l, \rho \in \text{Sym}_l(\mathbb{Q})\},$$

where $\mathbb{Q}^l$ (resp. $\text{Sym}_l(\mathbb{Q})$) denotes the space of rational column vectors (resp. rational symmetric matrices) of size l . The composition law of $G_{\mathbb{Q}}^J$ is given by

$$g_1 g_2 = (M_1 M_2, (\lambda_1, \mu_1) M_2 + (\lambda_2, \mu_2), \rho_1 + \rho_2 - \mu_1^t \lambda_1 + \mu^{*t} \lambda^* + \lambda^{*t} \mu_2 + \mu_2^t \lambda^*)$$

$$(g_j = (M_j, (\lambda_j, \mu_j), \rho_j) \in G_{\mathbb{Q}}^J, j = 1, 2)$$

with $(\lambda^*, \mu^*) = (\lambda_1, \mu_1) M_2$. Denote by $G_{\mathbb{R}}^J$ the group of real points of $G_{\mathbb{Q}}^J$. Denote by $\mathcal{D}$ the product of the upper half plane $\mathfrak{H}$ and $\mathbb{C}^l$, the space of complex column vectors of size l : $\mathcal{D} = \mathfrak{H} \times \mathbb{C}^l$. The Jacobi group $G_{\mathbb{R}}^J$ acts on $\mathcal{D}$ in the following manner:

$$g(\tau, z) = \left(M\tau, \frac{z + \lambda\tau + \mu}{J(M, \tau)} \right)$$

$$(g = (M, (\lambda, \mu), \rho) \in G_{\mathbf{R}}^J, (\tau, z) \in \mathcal{D}),$$

where $J(M, \tau) = c\tau + d$ for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Let S be a positive definite half integral symmetric matrix of size l . We define a factor of automorphy $J_{k,S}(g, (\tau, z))$ associated to S and a half integer k by

$$J_{k,S}(g, (\tau, z)) = J(M, \tau)^k e \left(-\text{tr}(S\rho) - \tau S[\lambda] - 2S(\lambda, z) + \frac{c}{J(M, \tau)} S[z + \lambda\tau + \mu] \right)$$

$$\left(g = (M, (\lambda, \mu), \rho) \in G_{\mathbf{R}}^J, M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\tau, z) \in \mathcal{D} \right),$$

where the branch of $J(M, \tau)^k = \exp(k \log J(M, \tau))$ is chosen so that $-\pi < \arg J(M, \tau) \leq \pi$. Let Γ be a congruence subgroup of $SL_2(\mathbf{Z})$ having the element -1_2 and Γ^J the subgroup of $G_{\mathbf{Q}}^J$ given by

$$\Gamma^J = \{(M, (\lambda, \mu), \rho) \mid M \in \Gamma, \lambda, \mu \in \mathbf{Z}^l, \rho \in \text{Sym}_l(\mathbf{Z})\},$$

where $\mathbf{Z}^l$ (resp. $\text{Sym}_l(\mathbf{Z})$) denotes the $\mathbf{Z}$ -lattice consisting of integral column vectors (resp. integral symmetric matrices) in $\mathbf{Q}^l$ (resp. $\text{Sym}_l(\mathbf{Q})$). For any function $\phi : \mathcal{D} \rightarrow \mathbf{C}$ and $g = (M, (\lambda, \mu), \rho) \in G_{\mathbf{R}}^J$, we set

$$(\phi|_{k,S}g)(\tau, z) = J_{k,S}(g, (\tau, z))^{-1} \phi(g(\tau, z)),$$

$$(\phi|_{k,S}^*g)(\tau, z) = J_{0,S}(g, (\tau, z))^{-1} (\overline{J(M, \tau)})^{-k+l} |J(M, \tau)|^{-l} \phi(g(\tau, z)).$$

In the definition of the latter $(\phi|_{k,S}^*g)$, we may assume that k is an integer, since only such cases can occur in the discussion later on. If k is an integer, then these operations satisfy

$$\phi|_{k,S}g_1g_2 = \phi|_{k,S}g_1|_{k,S}g_2$$

and

$$\phi|_{k,S}^*g_1g_2 = \phi|_{k,S}^*g_1|_{k,S}^*g_2.$$

Note that $\mathfrak{H} \cup \{\infty\} \cup \mathbf{Q}$ is the total set of cusps of Γ . For each element M of Γ , put $M\infty = \zeta$. Denote by Γ_{ζ} the stabilizer group of ζ in Γ : $\Gamma_{\zeta} = \{\sigma \in \Gamma \mid \sigma\zeta = \zeta\}$. There exists a unique positive integer N such that the group $M^{-1}\Gamma_{\zeta}M$ of $SL_2(\mathbf{Z})$ is generated by -1_2 and $\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}$. Let k be a positive integer. Now we define the space $J_{k,S}(\Gamma)$ (resp. $J_{k,S}^*(\Gamma)$) of holomorphic (resp. skew-holomorphic) Jacobi forms of index S and weight k with respect to Γ^J . We define $J_{k,S}(\Gamma)$ (resp. $J_{k,S}^*(\Gamma)$) to be the space consisting of all functions $\phi : \mathcal{D} \rightarrow \mathbf{C}$ which satisfy the following three conditions:

- (i) $\phi(\tau, z)$ is holomorphic in τ and z
 (resp. $\phi(\tau, z)$ is a smooth function in τ and holomorphic in z)
- (ii) $\phi(\tau, z)$ satisfies the identity

$$\phi|_{k,S}\gamma = \phi \quad (\text{resp. } \phi|_{k,S}^*\gamma = \phi) \quad \text{for } \forall \gamma \in \Gamma^J$$

- (iii) The function $\phi|_{k,S}M$ (resp. $\phi|_{k,S}^*M$) for any $M \in SL_2(\mathbb{Z})$ has a Fourier Jacobi expansion of the form

$$\begin{aligned} (\phi|_{k,S}M)(\tau, z) &= \sum_{\substack{n \in \mathbb{Z}, r \in \mathbb{Z}^l \\ 4n - N^t r S^{-1} r \geq 0}} c(n, r) e\left(\frac{n\tau}{N} + {}^t r z\right) \\ \left(\text{resp. } (\phi|_{k,S}^*M)(\tau, z) &= \sum_{\substack{n \in \mathbb{Z}, r \in \mathbb{Z}^l \\ 4n - N^t r S^{-1} r \leq 0}} c(n, r) e\left(\frac{n\bar{\tau}}{N} + \frac{i\eta}{2}({}^t r S^{-1} r) + {}^t r z\right) \right), \end{aligned}$$

where $\eta = \text{Im}\tau$ and a positive integer N is chosen for each M in the above manner. In the above (iii), $M \in SL_2(\mathbb{Z})$ is identified with the element $(M, (0, 0), 0)$ in $G_{\mathbb{R}}^J$.

Denote by $J_{k,S}^{cusp}(\Gamma)$ (resp. $J_{k,S}^{*cusp}(\Gamma)$) the subspace of cusp forms of $J_{k,S}(\Gamma)$ (resp. $J_{k,S}^*(\Gamma)$) consisting of all Jacobi forms $\phi \in J_{k,S}(\Gamma)$ (resp. all skew-holomorphic Jacobi forms $\phi \in J_{k,S}^*(\Gamma)$) whose Fourier coefficients $c(n, r)$ in the above (iii) equals zero if $4n - N^t r S^{-1} r = 0$.

Let $\Delta \subseteq G_{\mathbb{Q}}^J$ be a finite union of double cosets with respect to Γ^J : $\Delta = \sum_j \Gamma^J \sigma_j \Gamma^J$ ($\sigma_j \in G_{\mathbb{Q}}^J$). Following Skoruppa-Zagier [S-Z2], we define an operator $H_{k,S,\Gamma}(\Delta)$ (resp. $H_{k,S,\Gamma}^{skew}(\Delta)$) acting on $J_{k,S}(\Gamma)$ (resp. $J_{k,S}^*(\Gamma)$) by

$$\begin{aligned} \phi|_{H_{k,S,\Gamma}(\Delta)} &= \sum_{\xi \in \Gamma^J \backslash \Delta} \phi|_{k,S}\xi \\ \left(\text{resp. } \phi|_{H_{k,S,\Gamma}^{skew}(\Delta)} &= \sum_{\xi \in \Gamma^J \backslash \Delta} \phi|_{k,S}^* \phi \right), \end{aligned}$$

where the summation is taken over a complete set of representatives ξ for the left Γ^J -cosets of Δ . The operator $H_{k,S,\Gamma}(\Delta)$ (resp. $H_{k,S,\Gamma}^{skew}(\Delta)$) is well-defined and maps $J_{k,S}(\Gamma)$ (resp. $J_{k,S}^*(\Gamma)$) to $J_{k,S}(\Gamma)$ (resp. $J_{k,S}^*(\Gamma)$) and cusp forms to cusp forms (see Proposition 1.1 of [S-Z2]). For L -functions associated with common eigen Jacobi forms in this situation we refer the reader to Sugano [Su].

3 An operator acting on the space of theta series

Let S be a positive definite half-integral symmetric matrix of size l as before and R_S denote the $\mathbf{Z}$ -module $(2S)^{-1}\mathbf{Z}^l/\mathbf{Z}^l$. Set

$$d = \det(2S) = \#(R_S).$$

We write, for simplicity,

$$S(u, v) = {}^t u S v \quad \text{and} \quad S[u] = {}^t u S u \quad \text{for } u, v \in \mathbf{C}^l.$$

Denote by $V = \mathbf{C}^d$ the $\mathbf{C}$ -vector space consisting of column vectors $(x_r)_{r \in R_S}$ ($x_r \in \mathbf{C}$). Let $\langle x, y \rangle_S$ be the positive definite hermitian scalar product given by

$$\langle x, y \rangle_S = \sum_{r \in R_S} x_r \overline{y_r} \quad (x = (x_r)_{r \in R_S}, y = (y_r)_{r \in R_S} \in V).$$

For each $r \in (2S)^{-1}\mathbf{Z}^l$, we define a theta series $\theta_r(\tau, z)$ to be the sum

$$\sum_{q \in \mathbf{Z}^l} e(\tau S[q + r] + 2S(q + r, z)) \quad ((\tau, z) \in \mathcal{D}).$$

Since $\theta_{r+\mu}(\tau, z) = \theta_r(\tau, z)$ for any $\mu \in \mathbf{Z}^l$, one can define $\theta_r(\tau, z)$ for each $r \in R_S$. For each $\tau \in \mathfrak{H}$, let $\Theta_{S, \tau}$ denote the space of holomorphic functions $\theta : \mathbf{C}^l \rightarrow \mathbf{C}$ with the property

$$\theta(z + \lambda\tau + \mu) = e(-\tau S[\lambda] - 2S(\lambda, z))\theta(z).$$

It is known that $\{\theta_r(\tau, z)\}_{r \in R_S}$ forms a basis of the space $\Theta_{S, \tau}$. For each element $X = (\lambda, \mu) \in \mathbf{Q}^l \times \mathbf{Q}^l$, we denote by $[X]$ the element $(1_2, X, 0)$ of $G_{\mathbf{Q}}^J$. We set

$$\begin{aligned} L &= \mathbf{Z}^l \times \mathbf{Z}^l, \\ H_{\mathbf{Z}} &= \{(1_2, X, \rho) \mid X \in L, \rho \in \text{Sym}_l(\mathbf{Z})\}. \end{aligned}$$

Then, $H_{\mathbf{Z}}$ is a subgroup of $G_{\mathbf{Q}}^J$. For each $\xi \in G_{\mathbf{Q}}^J$, denote by L_{ξ} the sublattice $\{X \in L \mid \xi[X]\xi^{-1} \in H_{\mathbf{Z}}\}$ of L . Following Skoruppa-Zagier [S-Z2], we define an operator $U_S(\xi)$ acting on $\Theta_{S, \tau}$ as follows:

$$\theta|_{U_S(\xi)} = \left(\sum_{X \in L_{\xi} \setminus L} \theta|_{l/2, S} \xi[X] \right) \times \frac{1}{[L : L_{\xi}]}.$$

For this operator Skoruppa-Zagier (Proposition 4.1 of [S-Z2]) proved the following.

Theorem .1 (Skoruppa-Zagier) (i) For each $\theta \in \Theta_{S,\tau}$ and $\xi \in G_{\mathbf{Q}}^J$, $\theta|U_S(\xi) \in \Theta_{S,\tau}$
(ii) We arrange $\theta_r, \theta_r|U_S(\xi)$, ($r \in R_S$) as column vectors of $\mathbb{C}^d$. Then there exists a matrix $U_S(\xi)$ of size d (or a linear transformation of $V = \mathbb{C}^d$) such that

$$(2.1) \quad (\theta_r|U_S(\xi))_{r \in R_S} = U_S(\xi)(\theta_r)_{r \in R_S},$$

where $U_S(\xi)$ is independent of the choice of $\tau \in \mathfrak{H}$.

Remark. (1) For the matrix $U_S(\xi)$, we have used the same notation as for the operator $U_S(\xi)$ by abuse of notation.

(2) If $\xi = (M, 0, 0)$ and $M \in SL_2(\mathbb{Z})$, then the identity (2.1) is nothing but the theta transformation formula:

$$(\theta_r(M(\tau, z)))_{r \in R_S} = J_{1/2,S}(M, (\tau, z))U_S(M)(\theta_r(\tau, z))_{r \in R_S} \quad (\forall M \in SL_2(\mathbb{Z})),$$

where $M(\tau, z) = (M\tau, \frac{z}{c\tau+d})$ and $U_S(M) = U_S((M, 0, 0))$ in this case is a unitary matrix with respect to the inner product $\langle, \rangle_S$.

4 Where does $U_S(\xi)$ come from?

Let k be a positive integer and put $\kappa = (k - 1/2)/2$. We define a factor of automorphy $j_M(\tau)$ by

$$j_M(\tau) = \exp(2i\kappa \arg J(M, \tau)).$$

Denote by $\mathcal{M}_{S,k-1/2}(\Gamma)$ the space of all functions $f : \mathfrak{H} \rightarrow V$ satisfying the following conditions

- (i) $\eta^{-\kappa} f(\tau)$ is holomorphic on $\mathfrak{H}$ and also finite at any cusps of Γ
- (ii) $f(M\tau) = \overline{U_S(M)} j_M(\tau) f(\tau)$ for any $M \in \Gamma$.

Since each Jacobi form $\phi(\tau, z)$ of $J_{k,S}(\Gamma)$ is an element of $\Theta_{S,\tau}$ as a function of z , $\phi(\tau, z)$ has an expression as a linear combination of θ_r 's:

$$\phi(\tau, z) = \sum_{r \in R_S} \eta^{-\kappa} f_r(\tau) \theta_r(\tau, z).$$

Then the collection $f(\tau) = (f_r(\tau))_{r \in R_S}$ is a modular form of $\mathcal{M}_{S,k-1/2}(\Gamma)$. It is well-known that $J_{k,S}(\Gamma)$ is isomorphic to $\mathcal{M}_{S,k-1/2}(\Gamma)$ as $\mathbb{C}$ -linear spaces via the correspondence $\iota : \phi \rightarrow f = (f_r)_{r \in R_S}$. Let $\Delta \subseteq G_{\mathbf{Q}}^J$ be a finite union of Γ^J -double cosets. Let $p : G_{\mathbf{Q}}^J \rightarrow SL_2(\mathbb{Q})$ denote the natural projection map. For each A of $p(\Delta)$ we put

$$V_{\Delta}(A) = \sum_{\xi \in H_{\mathbf{Z}} \backslash p^{-1}(A) \cap \Delta / H_{\mathbf{Z}}} [L : L_{\xi}] U_S(\xi),$$

where the summation is over a complete set of representatives ξ of the double cosets of $p^{-1}(A) \cap \Delta$ with respect to $H_{\mathbf{Z}}$ (this is a finite sum). Then this quantity $V_{\Delta}(A)$ is well-defined. If $\Delta = \Gamma^J$, then $V_{\Delta}(A)$ equals the linear operator $U_S(A) = U_S((A, 0, 0))$. The action of Hecke operators $H_{k,S,\Gamma}(\Delta)$ on $J_{k,S}(\Gamma)$ is transferred in terms of modular forms of $\mathcal{M}_{S,k-1/2}(\Gamma)$. There exists a linear operator $\widetilde{H}_{k,S,\Gamma}(\Delta)$ acting on $\mathcal{M}_{S,k-1/2}(\Gamma)$ such that $\iota \circ H_{k,S,\Gamma}(\Delta) = \widetilde{H}_{k,S,\Gamma}(\Delta) \circ \iota$. Then we easily have

$$(f|\widetilde{H}_{k,S,\Gamma}(\Delta))(\tau) = \sum_{A \in \Gamma \backslash p(\Delta)} {}^t V_{\Delta}(A) j_A(\tau)^{-1} f(A\tau) \quad (f \in \mathcal{M}_{S,k-1/2}(\Gamma)),$$

where A runs over a complete set of representatives of the left Γ -cosets of $p(\Delta)$ and the sum is well-defined.

In this manner the operator $U_S(\xi)$ is coming in our sight. It seems that $U_S(\xi)$ is a very attractive arithmetic object.

5 Selberg type zeta functions

For $M \in SL_2(\mathbf{Z})$, we write $U_S(M)$ instead of $U_S((M, 0, 0))$ in (2.1). We set

$$R_S^0 = \{r \in R_S \mid r \equiv -r \pmod{\mathbf{Z}'}\}.$$

Since $U_S(-1_2)$ has eigen values $\pm e^{-\pi i l/2}$ (see (1.6) of [Ar]), it has the block decomposition

$$(4.1) \quad U_S(-1_2) = e^{-\pi i l/2} Q \begin{pmatrix} 1_{d(+)} & 0 \\ 0 & -1_{d(-)} \end{pmatrix} Q^{-1},$$

where Q is a certain unitary matrix of size d and $d(+) = (d + d_0)/2$ (resp. $d(-) = (d - d_0)/2$). We easily have

$$V_{\Delta}(A)U_S(-1_2) = U_S(-1_2)V_{\Delta}(A) \quad \text{for any } A \in p(\Delta).$$

Therefore, $V_{\Delta}(A)$ has the block decomposition similar to (4.1):

$$(4.2) \quad V_{\Delta}(A) = Q \begin{pmatrix} V_{\Delta}^+(A) & 0 \\ 0 & V_{\Delta}^-(A) \end{pmatrix} Q^{-1}$$

with $V_{\Delta}^+(A)$ (resp. $V_{\Delta}^-(A)$) a matrix of size $d(+)$ (resp. $d(-)$). For $A \in SL_2(\mathbf{Q})$, let $Z_{\Gamma}(A)$ denote the centralizer of A in Γ . Denote by $Hyp^+(\Delta)$ the set of hyperbolic elements P of $p(\Delta)$ with $\text{tr} P > 2$ which do not fix any cusps of Γ . We set, for $\varepsilon = \pm$,

$$\zeta_{\Delta,S,\varepsilon}(s) = \sum_{P \in Hyp^+(\Delta) \backslash \Gamma} \text{tr} V_{\Delta}^{\varepsilon}(P) \log N(P_0) \times \frac{N(P)^{-s}}{1 - N(P)^{-1}},$$

where $Hyp^+(\Delta)/\Gamma$ denote a complete set of representatives of the Γ -conjugacy classes of elements of $Hyp^+(\Delta)$, and where, for each $P \in Hyp^+(\Delta)$, P_0 together with the element -1_2 is the generator of the centralizer $Z_\Gamma(P)$. It can be shown that $\zeta_{\Delta,S,\varepsilon}(s)$ is absolutely convergent for $\text{Re}(s) > 1$. If $\Delta = \Gamma^J$, then, $\zeta_{\Delta,S,\varepsilon}(s)$ coincides with the logarithmic derivative of the Selberg zeta function associated with Γ , S :

$$\zeta_{\Delta,S,\varepsilon}(s) = (Z'_{\Gamma,S,\varepsilon}/Z_{\Gamma,S,\varepsilon})(s),$$

where $\varepsilon = \pm$ and

$$Z_{\Gamma,S,\varepsilon}(s) = \prod_{\{P_0\}_\Gamma, \text{tr} P_0 > 2} \prod_{n=0}^{\infty} \det(1_{d(\varepsilon)} - U_S^\varepsilon(P_0)N(P_0)^{-s-n}),$$

P_0 running over the Γ -conjugacy classes of primitive hyperbolic elements of Γ with $\text{tr} P_0 > 2$. Here, $U_S^\pm(A)$ ($A \in SL_2(\mathbb{Z})$) is defined similarly as in (4.2) from $U_S(A)$. For details concerning the Selberg zeta functions $Z_{\Gamma,S,\varepsilon}(s)$ we refer to [Ar]. Via the theory of general Selberg trace formula the Selberg type zeta functions $\zeta_{\Delta,S,\varepsilon}(s)$ can be analytically continued to a meromorphic function of s in the whole complex plane. This analytic continuation is crucial to the calculation of the traces of Hecke operators.

6 Traces of Hecke operators

Let Δ be as before. Each elliptic element R of $SL_2(\mathbb{R})$ is $SL_2(\mathbb{R})$ -conjugate to some $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ with $0 < \theta < 2\pi$, where θ is uniquely determined by R . We often write $\theta(R)$ for this θ . Denote by $Ell^+(\Delta)$ the set of all elliptic elements R of $p(\Delta)$ with $0 < \theta(R) < \pi$. Denote by $Ell^+(\Delta)/\Gamma$ a complete set of representatives of the Γ -conjugacy classes of all elements of $Ell^+(\Delta)$. Let $\zeta_1, \zeta_2, \dots, \zeta_h$ be a complete set of representatives of the Γ -equivalence classes of cusps of Γ . For each j ($1 \leq j \leq h$), one can choose an element $A_j \in SL_2(\mathbb{R})$ such that -1_2 and $T_j := A_j^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} A_j$ generate the stabilizer group Γ_{ζ_j} of the cusp ζ_j . For each j ($1 \leq j \leq h$), denote by $Hyp_j^+(\Delta)$ the set of all hyperbolic elements P of $p(\Delta)$ with $\text{tr} P > 2$ and $P\zeta_j = \zeta_j$. The set $Hyp_j^+(\Delta)$ is stable under the conjugation by any element of Γ_{ζ_j} . Denote by $Hyp_j^+(\Delta)/\Gamma_{\zeta_j}$ a complete set of representatives of the Γ_{ζ_j} -conjugacy classes of all elements of $Hyp_j^+(\Delta)$. Moreover for each j ($1 \leq j \leq h$), we denote by $Par_j^+(\Delta)$ the set of all parabolic elements P of $p(\Delta)$ satisfying the conditions $\text{tr} P = 2$, $P\zeta_j = \zeta_j$ and $P \neq 1_2$. Let $N = 4\det(2S) = 4d$ and $\Gamma(N)$ the principal congruence subgroup of $SL_2(\mathbb{Z})$ with level N . Set, for each j ($1 \leq j \leq h$),

$$\Gamma_j^+ = \Gamma_{\zeta_j} \cap \Gamma(N).$$

Then the group Γ_j^+ is generated by $T_j^{n_j}$ with a positive integer n_j . This integer n_j is uniquely determined. We call two elements A, B of $Par_j^+(\Delta)$ Γ_j^+ -equivalent, if there exists an element M of Γ_j^+ with $B = MA$. Denote by $Par_j^+(\Delta)/\Gamma_j^+$ a complete set of representatives of the Γ_j^+ -equivalence classes of all elements of $Par_j^+(\Delta)$. Each element P of $Par_j^+(\Delta)$ has an expression

$$P = A_j^{-1} \begin{pmatrix} 1 & r(P) \\ 0 & 1 \end{pmatrix} A_j$$

with a uniquely determined rational number $r(P) \in \mathbb{Q}$, $r(P) \neq 0$. If P is a representative of $Par_j^+(\Delta)/\Gamma_j^+$, $r(P)$ is uniquely determined modulo n_j .

Let k be an integer and set

$$\kappa = (k - l/2)/2.$$

Denote by $\varepsilon(k)$ the sign $+$ or $-$ according as k is even or not. We set

$$\begin{aligned} C_{\Gamma, \Delta}(k) &= \frac{1}{4\pi} \text{vol}(\Gamma \backslash \mathfrak{H}) \text{tr}(V_{\Delta}^{\varepsilon(k)}(1_2))(2\kappa - 1) \\ &+ \sum_{R \in Ell^+(\Delta) // \Gamma} \text{tr}(V_{\Delta}^{\varepsilon(k)}(R)) \times \frac{e^{-2i\kappa\theta(R) + i\theta(R)}}{\nu(e^{i\theta(R)} - e^{-i\theta(R)})} \\ &- \frac{1}{2} \sum_{j=1}^h \sum_{P \in HyP_j^+(\Delta) // \Gamma_{\zeta_j}} \text{tr}(V_{\Delta}^{\varepsilon(k)}(P)) \times \frac{N(P)^{-\kappa}}{1 - N(P)^{-1}} \\ &- \sum_{j=1}^h \sum_{P \in Par_j^+(\Delta) / \Gamma_j^+} \frac{1}{2n_j} \text{tr}(V_{\Delta}^{\varepsilon(k)}(P)) \times \begin{cases} 1 - i \cot \frac{\pi r(P)}{n_j} & \dots r(P) \not\equiv 0 \pmod{n_j} \\ 1 & \dots r(P) \equiv 0 \pmod{n_j} \end{cases} \end{aligned}$$

and

$$\begin{aligned} C_{\Gamma, \Delta}^*(k) &= \frac{1}{4\pi} \text{vol}(\Gamma \backslash \mathfrak{H}) \text{tr}(V_{\Delta}^{\varepsilon(k)}(1_2))(-2\kappa - 1) \\ &+ \sum_{R \in Ell^+(\Delta) // \Gamma} \text{tr}(V_{\Delta}^{\varepsilon(k)}(R)) \times \frac{e^{-2i\kappa\theta(R) - i\theta(R)}}{\nu(e^{i\theta(R)} - e^{-i\theta(R)})} \\ &- \frac{1}{2} \sum_{j=1}^h \sum_{P \in HyP_j^+(\Delta) // \Gamma_{\zeta_j}} \text{tr}(V_{\Delta}^{\varepsilon(k)}(P)) \times \frac{N(P)^{\kappa}}{1 - N(P)^{-1}} \\ &- \sum_{j=1}^h \sum_{P \in Par_j^+(\Delta) / \Gamma_j^+} \frac{1}{2n_j} \text{tr}(V_{\Delta}^{\varepsilon(k)}(P)) \times \begin{cases} 1 + i \cot \frac{\pi r(P)}{n_j} & \dots r(P) \not\equiv 0 \pmod{n_j} \\ 1 & \dots r(P) \equiv 0 \pmod{n_j} \end{cases} \end{aligned}$$

where

$$\text{vol}(\Gamma \backslash \mathfrak{H}) = \int_{\Gamma \backslash \mathfrak{H}} \eta^{-2} d\xi d\eta \quad (\xi = \text{Re}\tau, \eta = \text{Im}\tau).$$

We denote by $\text{tr}(H_{k,S,\Gamma}(\Delta), J_{k,S}(\Gamma))$ the trace of the action of $H_{k,S,\Gamma}(\Delta)$ on $J_{k,S}(\Gamma)$ and so on. Let $\Theta_{S,\Gamma}$ denote the space of theta functions $\theta(\tau, z)$ satisfying the following conditions:

- (i) $\theta(\tau, z)$, as a function of z , is an element of $\Theta_{S,\tau}$
- (ii) $\theta|_{l/2,S}M = \theta$ for any $M \in \Gamma$.

Then, $\Theta_{S,\Gamma}$ is isomorphic to the space $J_{l/2,S}(\Gamma)$ of Jacobi forms of weight $l/2$ with respect to Γ^J . The Hecke operator $H_{l/2,S,\Gamma}(\Delta)$ operates on $\Theta_{S,\Gamma}$. We have the following theorem.

Theorem .2 Assume that Γ is a congruence subgroup of $SL_2(\mathbb{Z})$ having the element -1_2 . Let k be an integer and $\Delta \subseteq G_{\mathbb{Q}}^J$ a finite union of Γ^J -double cosets.

(i) If $k > l/2 + 2$, then,

$$\text{tr}(H_{k,S,\Gamma}(\Delta), J_{k,S}^{cusp}(\Gamma)) = C_{\Gamma,\Delta}(k).$$

If $k < l/2 - 2$, then,

$$\text{tr}(H_{l-k,S,\Gamma}^{skew}(\Delta), J_{l-k,S}^{*cusp}(\Gamma)) = C_{\Gamma,\Delta}^*(k).$$

(ii) Assume that l is odd. Denote by ε the sign $+$ or $-$ according as l is congruent to 1 or 3 modulo 4 (i.e., $\varepsilon = \varepsilon((l-1)/2)$).

If $k = (l+3)/2$, then,

$$\text{tr}(H_{k,S,\Gamma}(\Delta), J_{k,S}^{cusp}(\Gamma)) = \text{Res}_{s=3/4}\zeta_{\Delta,S,\varepsilon}(s) + C_{\Gamma,\Delta}(k).$$

If $k = (l+1)/2$, then,

$$\text{tr}(H_{k,S,\Gamma}(\Delta), J_{k,S}(\Gamma)) = \text{Res}_{s=3/4}\zeta_{\Delta,S,-\varepsilon}(s).$$

If $k = (l-1)/2$, then,

$$\text{tr}(H_{l-k,S,\Gamma}^{skew}(\Delta), J_{l-k,S}^*(\Gamma)) = \text{Res}_{s=3/4}\zeta_{\Delta,S,\varepsilon}(s).$$

If $k = (l-3)/2$, then,

$$\text{tr}(H_{l-k,S,\Gamma}^{skew}(\Delta), J_{l-k,S}^{*cusp}(\Gamma)) = \text{Res}_{s=3/4}\zeta_{\Delta,S,-\varepsilon}(s) + C_{\Gamma,\Delta}^*(k).$$

(iii) Assume that l is even. Let $\varepsilon = \varepsilon(l/2)$.

If $k = l/2 + 2$, then,

$$\text{tr}(H_{k,S,\Gamma}(\Delta), J_{k,S}^{cusp}(\Gamma)) = \text{Res}_{s=1}\zeta_{\Delta,S,\varepsilon}(s) + C_{\Gamma,\Delta}(k).$$

If $k = l/2 - 2$, then,

$$\text{tr}(H_{l-k,S,\Gamma}^{skew}(\Delta), J_{l-k,S}^{*cusp}(\Gamma)) = \text{Res}_{s=1}\zeta_{\Delta,S,\varepsilon}(s) + C_{\Gamma,\Delta}^*(k).$$

If $k = l/2$, then,

$$\text{tr}(H_{l/2,S,\Gamma}(\Delta), \Theta_{S,\Gamma}) = \text{Res}_{s=1}\zeta_{\Delta,S,\varepsilon}(s).$$

For the proof we use Fischer's resolvent trace formula [Fi] and the method of Skoruppa-Zagier [S-Z2]. We can deduce the following corollary from (ii), (iii) of the above theorem.

Corollary .3 (i) *Assume that l is odd. Then,*

$$\mathrm{tr}(H_{(l+3)/2,S,\Gamma}(\Delta), J_{(l+3)/2,S}^{cusp}(\Gamma)) = \mathrm{tr}(H_{(l+1)/2,S,\Gamma}^{skew}(\Delta), J_{(l+1)/2,S}^*(\Gamma)) + C_{\Gamma,\Delta} \left(\frac{l+3}{2} \right),$$

$$\mathrm{tr}(H_{(l+3)/2,S,\Gamma}^{skew}(\Delta), J_{(l+3)/2,S}^{*cusp}(\Gamma)) = \mathrm{tr}(H_{(l+1)/2,S,\Gamma}(\Delta), J_{(l+1)/2,S}(\Gamma)) + C_{\Gamma,\Delta}^* \left(\frac{l-3}{2} \right).$$

(ii) *Assume that l is even. Then,*

$$\mathrm{tr}(H_{l/2+2,S,\Gamma}(\Delta), J_{l/2+2,S}^{cusp}(\Gamma)) = \mathrm{tr}(H_{l/2,S,\Gamma}(\Delta), \Theta_{S,\Gamma}) + C_{\Gamma,\Delta} \left(\frac{l}{2} + 2 \right),$$

$$\mathrm{tr}(H_{l/2+2,S,\Gamma}^{skew}(\Delta), J_{l/2+2,S}^{*cusp}(\Gamma)) = \mathrm{tr}(H_{l/2,S,\Gamma}(\Delta), \Theta_{S,\Gamma}) + C_{\Gamma,\Delta}^* \left(\frac{l}{2} - 2 \right).$$

Remark. In the case of $l = 1$ the first identity of the above (i) has been already obtained by Skoruppa-Zagier[S-Z1]. The results in Theorem 2 and Corollary 3 are consistent with those of [S-Z1, 2].

References

- [Ar] Arakawa, T., *Selberg zeta functions and Jacobi forms*, preprint
- [Fi] Fischer, J., *An approach to the Selberg trace formula via the Selberg zeta-function*. Lecture Notes in Math. 1253. Springer 1987.
- [He] Hejhal, D. A., *The Selberg trace formula for $PSL(2, \mathbf{R})$* , Vol.2. Lecture Notes in Math. 1001. Springer 1983.
- [Sh] Shimura, G., *On the trace formula for Hecke operators*. Acta Math. **132** (1974)245-281.
- [Su] Sugano, T., *Jacobi forms and the theta lifting*, preprint
- [S-Z1] Skoruppa, N-P. and Zagier, D., *Jacobi forms and a certain space of modular forms*. Invent. math. **94** (1988), 113-146.
- [S-Z2] Skoruppa, N-P. and Zagier, D., *A trace formula for Jacobi forms*. J. reine angew. Math. **393** (1989), 168-198.