

MAHLER MEASURE OF THE COLORED JONES POLYNOMIAL AND THE VOLUME CONJECTURE

東京工業大学大学院理工学研究科 村上 奇 (HITOSHI MURAKAMI)

ABSTRACT. In this note, I will discuss a possible relation between the Mahler measure of the colored Jones polynomial and the volume conjecture. In particular, I will study the colored Jones polynomial of the figure-eight knot on the unit circle. I will also propose a method to prove the volume conjecture for satellites of the figure-eight knot.

1. MAHLER MEASURE

Let $f(t)$ be a (non-zero) Laurent polynomial in t with coefficient in $\mathbb{Z}$. The Mahler measure $M(f)$ of f [5, 6, 14] is defined to be

$$M(f) := \exp \left(\int_0^1 \log |f(\exp(2\pi\sqrt{-1}x))| dx \right)$$

It is known that $M(f)$ is the product of the absolute values of the leading coefficient and all the roots that are greater than one. It is convenient to define its logarithmic version:

$$m(f) := \int_0^1 \log |f(\exp(2\pi\sqrt{-1}x))| dx.$$

Then the logarithmic Mahler measure can be regarded as a sort of ‘mean’ of the logarithms of the values on the unit circle. Visit the web pages

<http://mathworld.wolfram.com/MahlerMeasure.html>

for more about the Mahler measure and also

<http://math.ucr.edu/~xl/knotprob/knotprob.html>

for problems on the Mahler measure of the Jones polynomial.

2. MAHLER MEASURE OF THE ALEXANDER POLYNOMIAL

Let K be a knot in the three-sphere S^3 and $M_N(K)$ be the N -fold cyclic branched covering over S^3 branched along K . Then it is well known that the order of the first homology group of $M_N(K)$ can be obtained in terms of the Alexander polynomial $\Delta(K; t)$ of K (see for example [4, Corollary 9.8]).

Theorem 2.1.

$$(2.1) \quad |H_1(M_N(K); \mathbb{Z})| = \prod_{d=1}^{N-1} \Delta(K, \exp(2d\pi\sqrt{-1}/N)),$$

where $|A|$ denotes the cardinality of a set A if A is a finite set and 0 if it is infinite.

Date: 23rd June, 2002.

2000 Mathematics Subject Classification. Primary 57M27; Secondary 57M25, 57M50, 17B37, 81R50.

This research is partially supported by Grant-in-Aid for Scientific Research (B).

If we take the logarithm of the both side of Equation (2.1) and divide by N , we have

$$\frac{\log |H_1(M_N(K); \mathbb{Z})|}{N} = \frac{\sum_{d=1}^{N-1} \log |\Delta(K; \exp(2\pi d\sqrt{-1}/N))|}{N}$$

When N grows, the right hand side approaches to the ‘mean’ of the logarithms of the values of $\Delta(K; t)$ on the unit circle, the logarithmic Mahler measure. In fact the following theorem is known to be true.

Theorem 2.2 (D. Silver and S. Williams [15]).

$$\lim_{N \rightarrow \infty} \frac{\log |H_1(M_N(K); \mathbb{Z})|}{N} = \mathbf{m}(\Delta(K; t))$$

See [2, 1, 13] for other topics of the homology of the branched cyclic cover over a knot. See also [16] for the Mahler measure of the Alexander polynomial of a link.

3. MAHLER MEASURE OF THE COLORED JONES POLYNOMIALS

Let $J_N(K; t)$ be the N -dimensional colored Jones polynomial of a knot K normalized so that $J_N(O; t) = 1$ for the unknot O . We want to know the asymptotic behavior of $J_N(K; t)$ for large N .

Since

$$\begin{aligned} \mathbf{m}(J_N(K; t)) &= \int_0^1 \log |J_N(K; \exp(2\pi\sqrt{-1}x))| dx \\ &= \int_0^N \frac{\log |J_N(K; \exp(2r\pi\sqrt{-1}/N))|}{N} dr, \end{aligned}$$

it is helpful to study the asymptotic behavior of $\log |J_N(K; \exp(2\pi\sqrt{-1}r/N))|$ for a fixed r . Note that for $r = 1$, this problem is nothing but the volume conjecture [11, 8, 10, 9, 19, 18, 17, 12].

In the following sections I will discuss the colored Jones polynomials of the figure-eight knot evaluated on the unit circle.

4. SOME CALCULATIONS ABOUT THE FIGURE-EIGHT KNOT

Let E denote the figure-eight knot 4_1 . Due to K. Habiro and T. Le, the following formula is known.

$$(4.1) \quad J_N(E; t) = \sum_{k=0}^{N-1} \prod_{j=1}^k \left(t^{(N+j)/2} - t^{-(N+j)/2} \right) \left(t^{(N-j)/2} - t^{-(N-j)/2} \right).$$

Using this formula we can prove the following result.

Theorem 4.1. *Let r be a positive integer or a real number satisfying $5/6 < r < 7/6$. Then*

$$\lim_{N \rightarrow \infty} 2\pi \frac{\log |J_N(E; \exp(2r\pi\sqrt{-1}))|}{N} = \frac{2\Lambda(r\pi + \theta(r)/2) - 2\Lambda(r\pi - \theta(r)/2)}{r},$$

where $\Lambda(z) := -\int_0^z \log |\sin x| dx$ is the Lobachevski function and $\theta(r)$ is the smallest positive number satisfying $\cos \theta(r) = \cos(2r\pi) - 1/2$.

In particular, if r is a positive integer then

$$2\pi \lim_{N \rightarrow \infty} \frac{\log |J_N(E; \exp(2r\pi\sqrt{-1}/N))|}{N} = \frac{\text{Vol}(S^3 \setminus E)}{r}.$$

Proof of Theorem 4.1 when r is a positive integer. Replacing t with $\exp(2r\pi\sqrt{-1}/N)$ in Equation (4.1), we have

$$J_N(E; \exp(2r\pi\sqrt{-1}/N)) = \sum_{k=0}^{N-1} \prod_{j=1}^k \{2 \sin(jr\pi/N)\}^2$$

If we put $f(k) := \prod_{j=1}^k \{2 \sin(jr\pi/N)\}^2$, then f takes its maximum at $kr\pi/N = 5\pi/6$ if N is large. Therefore

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{\log |J_N(E; \exp(2r\pi\sqrt{-1}/N))|}{N} &= 2 \lim_{N \rightarrow \infty} \sum_{j=1}^{5N/6r} \frac{\log(2 \sin(jr\pi/N))}{N} \\ &= \frac{2}{r\pi} \lim_{N \rightarrow \infty} \int_0^{5\pi/6} \log(2 \sin x) dx \\ &= -\frac{2}{r\pi} \Lambda(5\pi/6) \\ &= \frac{\text{Vol}(S^3 \setminus E)}{2r\pi}. \end{aligned}$$

See [8, Theorem 4.2] for details. □

Remark 4.2. The case where $r = 1$ is due to R. Kashaev [3] and T. Ekholm [8].

Proof of Theorem 4.1 when $5/6 < r < 1$. We will assume N is sufficiently large so that j/N can behave as if it is a continuous parameter.

Put $\omega := \exp(2\pi\sqrt{-1}/N)$. Since

$$\omega^{r(N+j)/2} - \omega^{-r(N+j)/2} = 2\sqrt{-1} \sin(r(N+j)\pi/N)$$

and

$$\omega^{r(N-j)/2} - \omega^{-r(N-j)/2} = 2\sqrt{-1} \sin(r(N-j)\pi/N),$$

we have

$$\begin{aligned} \prod_{j=1}^k (\omega^{r(N+j)/2} - \omega^{-r(N+j)/2}) (\omega^{r(N-j)/2} - \omega^{-r(N-j)/2}) \\ = \prod_{j=1}^k 4 \sin(rj\pi/N + r\pi) \sin(rj\pi/N - r\pi). \end{aligned}$$

Put

$$\begin{aligned} g(j) &:= 4 \sin(rj\pi/N + r\pi) \sin(rj\pi/N - r\pi) \\ &= 2 \cos(2r\pi) - 2 \cos(2rj\pi/N) \end{aligned}$$

and

$$f(k) := \prod_{j=1}^k g(j)$$

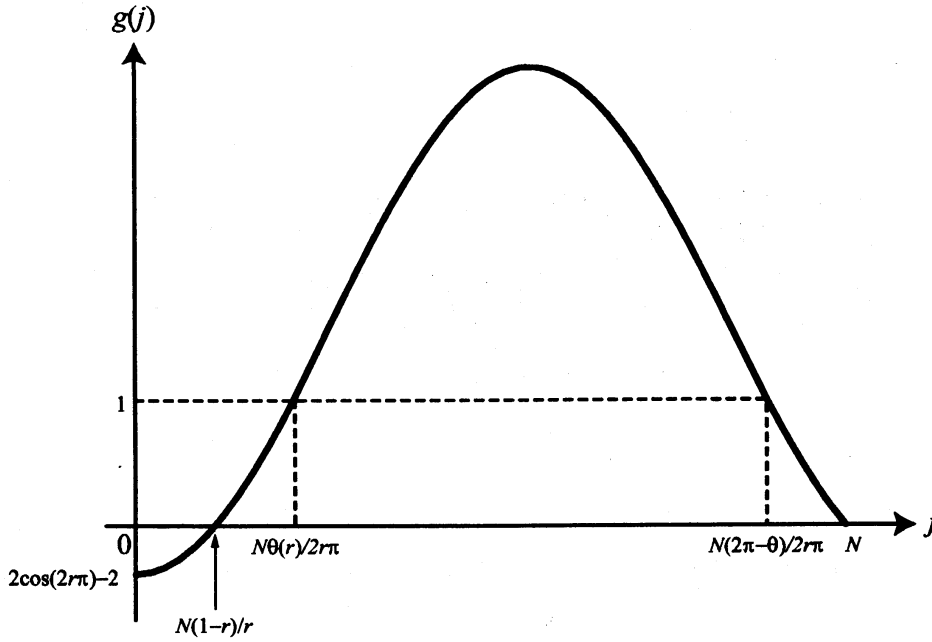
so that $J_N(E; \omega^r) = \sum_{k=0}^{N-1} f(k)$. We also put

$$A := \frac{N(1-r)}{r}, \quad B := \frac{N\theta(r)}{2r\pi}, \quad \text{and} \quad C := \frac{N(2\pi - \theta(r))}{2r\pi}$$

where $\theta(r)$ is the smallest positive number satisfying $\cos \theta(r) = \cos(2r\pi) - 1/2$ as before. Note that since $5/6 < r < 1$, $1/2 < \cos(2r\pi) < 1$ and so the equation $\cos \theta(r) = \cos(2r\pi) - 1/2$ has a solution.

Note that $0 < A < B < C < N$ (see Figure 1).

HITOSHI MURAKAMI

FIGURE 1. Graph of $g(j)$ when $5/6 < r < 1$

Since we have

- (1) $g(j) < 0$ for $j < A$, and $g(j) > 0$ for $j > A$, and
- (2) $f_j > 1$ for $B < j < C$,

we see

- (3) If $j < A$ then the signs of $f(j)$ alternate, that is, $f(j-1)f(j) < 0$, and if $j > A$ then the signs of $f(j)$ are constant, and
- (4) $|f(0)| > |f(1)| > \dots > |f(B)|$ and $|f(B+1)| < \dots < |f(C)|$.

Let f_{MAX}^1 be the maximum of $\{|f_j|\}$ for $0 \leq j \leq N-1$. Note that $f_{\text{MAX}} = f(C)$. We can show the following inequality.

Claim 4.3.

$$0 < f_{\text{MAX}} - 1 \leq |J_N(E; \omega^r)| \leq N f_{\text{MAX}}$$

Proof of the Claim 4.3. We only show the second inequality for the case where A is even. In this case since $f(0) = 1$, $f(2j-1) + f(2j) < 0$ for $2j < A$, and $f(j) < 0$ for $j \geq A-1$, we have

$$\begin{aligned} & |J_N(E; \omega^r)| \\ &= |f(0) + \{f(1) + f(2)\} + \{f(3) + f(4)\} + \dots + \{f(A-3) + f(A-2)\} \\ &\quad + f(A-1) + f(A) + f(A+1) + \dots + f(N-1)| \\ &= |f(1) + f(2)| + |f(3) + f(4)| + \dots + |f(A-3) + f(A-2)| \\ &\quad + |f(A-1)| + |f(A)| + \dots + |f(N-1)| \\ &\quad - 1 \\ &> f_{\text{MAX}} - 1 \end{aligned}$$

and the second equality follows. $\square$

¹MAX are temporarily Nana, Reina, and Lina.

Therefore we have

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \frac{\log |J_N(E; \omega^r)|}{N} \\
&= \lim_{N \rightarrow \infty} \frac{\log(f_{\text{MAX}})}{N} \\
&= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=0}^C \left\{ \log \left(2 \sin \left(\frac{rj\pi}{N} + r\pi - \pi \right) \right) + \log \left(2 \sin \left(-\frac{rj\pi}{N} + r\pi \right) \right) \right\} \\
&= \frac{1}{r\pi} \int_{r\pi-\pi}^{r\pi-\theta(r)/2} \log(2 \sin x) dx + \frac{1}{r\pi} \int_{r\pi-\pi+\theta(r)/2}^{r\pi} \log(2 \sin x) dx \\
&= \frac{1}{r\pi} (\Lambda(r\pi - \pi) - \Lambda(r\pi - \theta(r)/2) + \Lambda(r\pi - \pi + \theta(r)/2) - \Lambda(r\pi)) \\
&= \frac{1}{r\pi} (\Lambda(r\pi + \theta(r)/2) - \Lambda(r\pi - \theta(r)/2)).
\end{aligned}$$

Here we use the π -periodicity of the Lobachevski function. (See [7].) $\square$

Proof when $1 < r < 7/6$. The proof is similar to the case where $5/6 < r < 1$. See Figure 2. $\square$

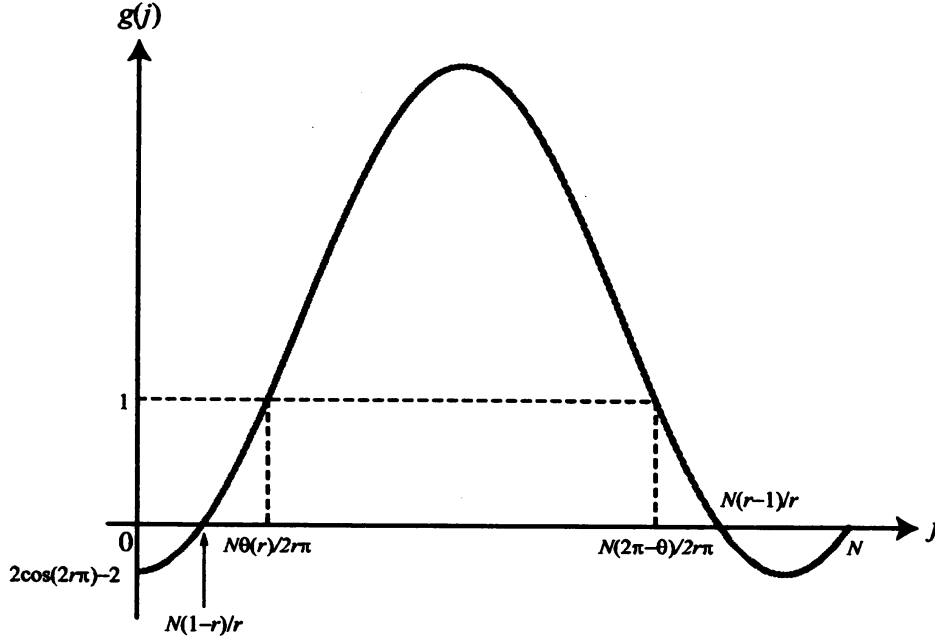


FIGURE 2. Graph of $g(j)$ when $1 < r < 7/6$

As a corollary we have

Corollary 4.4.

$$\lim_{r \rightarrow 1} \left\{ \lim_{N \rightarrow \infty} \frac{\log |J_N(E; \exp(2r\pi\sqrt{-1}))|}{N} \right\} = \lim_{N \rightarrow \infty} \frac{\log |J_N(E; \exp(2\pi\sqrt{-1}))|}{N}$$

HITOSHI MURAKAMI

By some calculation using PARI-GP ² and MAPLE V, it seems that the following equality holds.

$$(4.2) \quad 2r\pi \lim_{N \rightarrow \infty} \frac{\log |J_N(E; \omega^r)|}{N} = \begin{cases} V(r) & \text{if } 0 \leq r \leq 1, \\ W(r - [r]) & \text{if } r > 1, \end{cases}$$

where $[r]$ denotes the greatest integer which does not exceed r , and

$$V(x) := \begin{cases} 0 & \text{if } 0 \leq x < 1/6, \\ \Lambda(x\pi + \theta(x)/2 - \pi/2) - \Lambda(x\pi - \theta(x)/2 - \pi/2) & \text{if } 1/6 \leq x < 3/4, \\ \Lambda(x\pi + \theta(x)/2) - \Lambda(x\pi - \theta(x)/2) & \text{if } 3/4 \leq x \leq 1, \end{cases}$$

and

$$W(x) := \begin{cases} \Lambda(x\pi) + \theta(x)/2 - \Lambda(x\pi - \theta(x)/2) & \text{if } 0 \leq x < 1/4, \\ \Lambda(x\pi) + \theta(x)/2 - \pi/2 - \Lambda(x\pi - \theta(x)/2 - \pi/2) & \text{if } 1/4 \leq x < 3/4, \\ \Lambda(x\pi + \theta(x)/2) - \Lambda(x\pi - \theta(x)/2) & \text{if } 3/4 \leq x \leq 1. \end{cases}$$

See Figures 3 and 4 for graphs of V and W . See also Figures 6, 7, 8, 9, 10, and 11

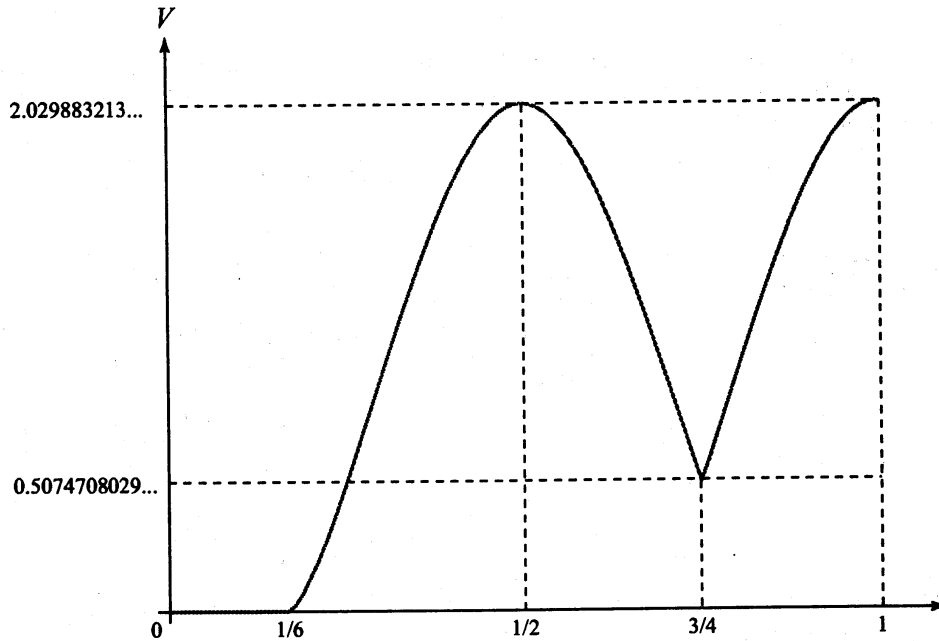


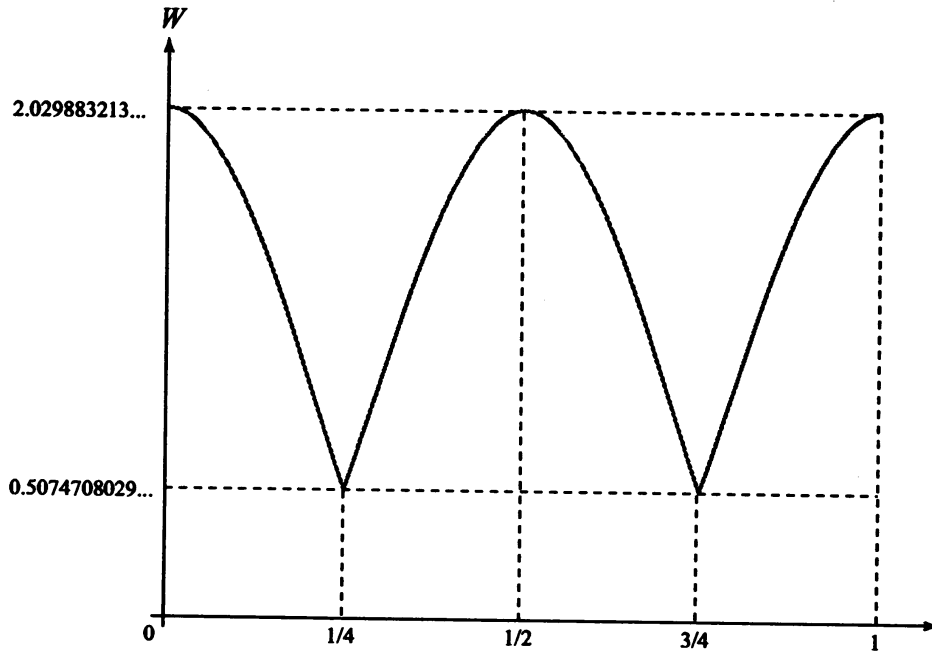
FIGURE 3. Graph of V , where 2.029883213... is the volume of the figure-eight knot complement.

for some results of calculations supporting Equation 4.2.

² GP/PARI CALCULATOR Version 2.0.20 (beta)
i586 running cygwin_98-4.10 (ix86 kernel) 32-bit version
(readline v1.0 enabled, extended help not available)

Copyright (C) by 1989-1999 by
C. Batut, K. Belabas, D. Bernardi, H. Cohen and M. Olivier.

The program is available at <http://www.pari-gp-home.de/>

FIGURE 4. Graph of W .

If Equation (4.2) is true, one could have the following result on the asymptotic behavior of the logarithmic Mahler measure of the colored Jones polynomials of the figure-eight knot.

Remark 4.5. Caution! There are *fake* calculations in the following.

$$\begin{aligned}
 \lim_{N \rightarrow \infty} \frac{\mathbf{m}(J_N(E; t))}{\log N} &= \lim_{N \rightarrow \infty} \frac{1}{\log N} \int_0^1 \log |J_N(E; \exp(2\pi\sqrt{-1}x))| dx \\
 &\stackrel{?}{=} \lim_{N \rightarrow \infty} \frac{1}{\log N} \int_0^N \frac{\log |J_N(E; \exp(2\pi\sqrt{-1}r/N))|}{N} dr \\
 &\stackrel{?}{=} \lim_{N \rightarrow \infty} \frac{1}{2\pi \log N} \left\{ \int_0^1 \frac{V(r)}{r} dr + \sum_{k=1}^{N-1} \int_k^{k+1} \frac{W(r - [r])}{r} dr \right\} \\
 &= \lim_{N \rightarrow \infty} \frac{1}{2\pi \log N} \left\{ \int_0^1 \frac{V(r)}{r} dr + \sum_{k=1}^{N-1} \int_0^1 \frac{W(r)}{r+k} dr \right\},
 \end{aligned}$$

where $\stackrel{?}{=}$ means that there is a doubt in the equality. At the first I use N in the integral, which should be independent of N , and at the second I assume (4.2).

Now since

$$\frac{1}{k+1} \leq \frac{1}{r+k} \leq \frac{1}{k}$$

for $0 \leq r \leq 1$, we have

$$\int_0^1 \frac{W(r)}{k+1} dr \leq \int_0^1 \frac{W(r)}{r+k} dr \leq \int_0^1 \frac{W(r)}{k} dr.$$

Therefore we have

$$\sum_{k=1}^{N-1} \int_0^1 \frac{W(r)}{k+1} dr \leq \sum_{k=1}^{N-1} \int_0^1 \frac{W(r)}{r+k} dr \leq \sum_{k=1}^{N-1} \int_0^1 \frac{W(r)}{k} dr.$$

Since

$$\lim_{N \rightarrow \infty} \frac{\sum_{k=1}^{N-1} \frac{1}{k+1}}{\log N} = \lim_{N \rightarrow \infty} \frac{\sum_{k=1}^{N-1} \frac{1}{k}}{\log N} = 1$$

and $V(r) = 0$ for $0 \leq r \leq 1/6$, we finally have

$$2\pi \lim_{N \rightarrow \infty} \frac{\mathbf{m}(J_N(E; t))}{\log N} \stackrel{?}{=} \int_0^1 W(r) dr = 1.450191516....$$

5. SATELLITES OF THE FIGURE-EIGHT KNOT

In this section, I would like to study the volume conjecture for the $(2, 1)$ -cable and the Whitehead double of the figure-eight knot. Linear skein method gives us formulas to describe the colored Jones polynomials of such knots but one of the difficulties is that the value of the unknot is not 1 but $(t^{N/2} - t^{-N/2})/(t^{1/2} - t^{-1/2})$ (see for example [4, Chapter 14]), and so they vanish if we evaluate them at the N -th root of unity. To avoid this I will use Corollary 4.4 to analyze the asymptotic behaviors of the colored Jones polynomials. Unfortunately, I cannot give a rigorous result here but I hope that this method gives an insight to solve the volume conjecture for satellite knots.

Remark 5.1. Caution! There are many *fake* arguments in this section.

Let E^2 be the $(2, 1)$ -cable of the figure-eight knot. By using techniques in [4, Chapter 14], we see

$$J_N(E^2; t)(t^{N/2} - t^{-N/2})/(t^{1/2} - t^{-1/2}) = \sum_{\substack{c: \text{odd} \\ 1 \leq c \leq 2N-1}} u(c; t^{1/4}) J_c(E; t),$$

where $u(c; t^{1/4})$ is a monomial in $t^{1/4}$. Replacing t with ω^r with $5/6 < r < 7/6$ ($r \neq 1$), we have

$$J_N(E^2; \omega^r) = \frac{\sin(r\pi/N)}{\sin(r\pi)} \sum_{\substack{c: \text{odd} \\ 1 \leq c \leq 2N-1}} u(c; \omega^{r/4}) J_c(E; \omega^r).$$

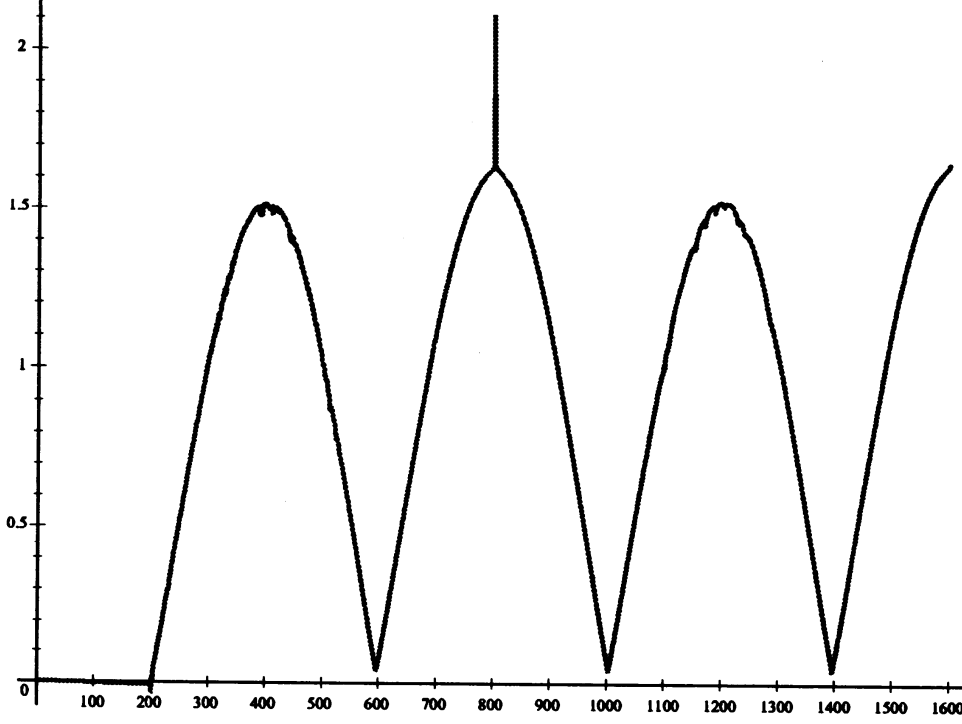
Note that $\sin(r\pi) \neq 0$. If one could show that the maximum of the terms in the summation dominates the limit, which is a kind of saddle point method, we could have

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{\log |J_N(E^2; \omega)|}{N} &\stackrel{?}{=} \lim_{r \rightarrow 1} \left\{ \lim_{N \rightarrow \infty} \frac{\log |J_N(E^2; \omega^r)|}{N} \right\} \\ &= \lim_{r \rightarrow 1} \left\{ \frac{\log \left| \max_{1 \leq c \leq 2N-1} J_c(E, \omega^r) \right|}{N} \right\} \\ &= \lim_{r \rightarrow 1} \left\{ \frac{\log |J_N(E, \omega^r)|}{N} \right\} \\ &= \frac{\log |J_N(E, \omega)|}{N}, \end{aligned}$$

proving the volume conjecture for the $(2, 1)$ -cable of the figure-eight knot. Here $\stackrel{?}{=}$ indicates that there is a doubt in the equality; at the first equality, I change the order of the limits, at the second, I assume the maximum dominates the limit, and at the third, I assume that $J_c(E, \omega^r)$ takes its maximum at $c = N$, which can be observed by calculation using PARI. See Figure 5.

I believe that the gaps here are not so big.

MAHLER MEASURE AND THE VOLUME CONJECTURE

FIGURE 5. Plot of $(c, 2\pi \log |J_c(E; \omega)|/N)$ with $N = 800$.

Let $D(E)$ be the Whitehead double of the figure-eight knot (with any framing). Then using similar techniques we have

$$J_N(D(E); t)(t^{N/2} - t^{-N/2})/(t^{1/2} - t^{-1/2}) = \sum_{\substack{c: \text{ odd} \\ c \leq 2N-1}} v(c; t) J_c(E; t),$$

with

$$v(c; t) = \sum_{\substack{d: \text{ odd} \\ d \leq 2N-1}} \frac{\Delta_d \theta(N-1, N-1, c-1)}{\Delta_c \theta(N-1, N-1, d-1)} \begin{Bmatrix} N-1 & N-1 & c \\ N-1 & N-1 & d \end{Bmatrix}$$

where $\Delta_x, \theta(x, y, z)$ and $\begin{Bmatrix} x & y & z \\ u & v & w \end{Bmatrix}$ are defined in [4, Chapter 14]. Similar calculation shows that for the Whitehead link W we have

$$J_N(W; t)(t^{N/2} - t^{-N/2})/(t^{1/2} - t^{-1/2}) = \sum_{\substack{c: \text{ odd} \\ c \leq 2N-1}} v(c; t) J_{N,c}(H; t),$$

where $J_{N,c}(H; t)$ is the colored Jones polynomial of the Hopf link H colored with N and c , which is equal to $\Delta_{(N-1)(c-1)}$.

Now we have the following *fake* calculations with doubtful equalities:

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{\log |J_N(D(E); \omega)|}{N} & \stackrel{?}{=} \lim_{r \rightarrow 1} \left\{ \lim_{N \rightarrow \infty} \frac{\log |J_N(D(E); \omega^r)|}{N} \right\} \\ & = \lim_{r \rightarrow 1} \left\{ \frac{\log \left| \max_{1 \leq c \leq 2N-1} v(c; \omega^r) J_c(E, \omega^r) \right|}{N} \right\} \end{aligned}$$

HITOSHI MURAKAMI

$$\begin{aligned}
&= \lim_{r \rightarrow 1} \left\{ \frac{\log |v(N; \omega^r) J_N(E; \omega^r)|}{N} \right\} \\
&= \lim_{r \rightarrow 1} \frac{\log |v(N; \omega^r)|}{N} + \lim_{r \rightarrow 1} \frac{\log |J_N(E; \omega^r)|}{N} \\
&= \lim_{r \rightarrow 1} \frac{\log |v(N; \omega^r)|}{N} + \frac{\log |J_N(E, \omega)|}{N},
\end{aligned}$$

On the other hand

$$\begin{aligned}
\lim_{N \rightarrow \infty} \frac{\log |J_N(W; \omega)|}{N} &= \lim_{r \rightarrow 1} \left\{ \lim_{N \rightarrow \infty} \frac{\log |J_N(W; \omega^r)|}{N} \right\} \\
&= \lim_{r \rightarrow 1} \left\{ \frac{\log |v(N; \omega^r) J_{N,N}(H; \omega^r)|}{N} \right\} \\
&= \lim_{r \rightarrow 1} \frac{\log |v(N; \omega^r)|}{N}
\end{aligned}$$

since $J_{N,N}(W, \omega^r)$ can be expressed in terms of sine of $1/N$. Therefore if we accept these calculations, we could prove

$$\lim_{n \rightarrow \infty} \frac{\log |J_N(D(E), \omega)|}{N} = \lim_{n \rightarrow \infty} \frac{\log |J_{N,N}(W, \omega)|}{N} + \lim_{n \rightarrow \infty} \frac{\log |J_N(E, \omega)|}{N}.$$

Noting that the complement of $D(E)$ is the union of those of the figure-eight knot and the Whitehead link, which is the volume conjecture for the Whitehead double of the figure-eight knot.

Acknowledgments. This article is prepared for the proceedings of the workshop ‘Volume Conjecture and Its Related Topics’ held at the International Institute for Advanced Studies from 5th to 8th March, 2002. It was also supported by the Research Institute for Mathematical Sciences, Kyoto University. I would like to thank the institutes for their hospitality. I also thank Kazuhiro Hikami for introducing the computer program PARI.

Part of this work was done when I was visiting Warwick University to attend the workshop ‘Quantum Topology’ from 18th to 22nd March, 2002, and Université du Québec à Montréal to attend the workshop ‘Knots in Montreal II’ from 20th to 21st April, 2002. Thanks are due to the universities and to the organizers of the workshops, Stavros Garoufalidis, Colin Rourke, Steven Boyer, and Adam Sikora.

REFERENCES

- [1] F. González-Acuña and H. Short, *Cyclic branched coverings of knots and homology spheres*, Rev. Mat. Univ. Complut. Madrid **4** (1991), no. 1, 97–120. MR **93g**:57004
- [2] C. McA. Gordon, *Knots whose branched cyclic coverings have periodic homology*, Trans. Amer. Math. Soc. **168** (1972), 357–370. MR **45** #4394
- [3] R. M. Kashaev, *The hyperbolic volume of knots from the quantum dilogarithm*, Lett. Math. Phys. **39** (1997), no. 3, 269–275. MR **98b**:57012
- [4] W. B. R. Lickorish, *An Introduction to Knot Theory*, Graduate Texts in Mathematics, vol. 175, Springer-Verlag, New York, 1997. MR **98f**:57015
- [5] K. Mahler, *An application of Jensen’s formula to polynomials*, Mathematika **7** (1960), 98–100. MR **23** #A1779
- [6] ———, *On some inequalities for polynomials in several variables*, J. London Math. Soc. **37** (1962), 341–344. MR **25** #2036
- [7] J. Milnor, *Hyperbolic geometry: the first 150 years*, Bull. Amer. Math. Soc. (N.S.) **6** (1982), no. 1, 9–24. MR **82m**:57005
- [8] H. Murakami, *The asymptotic behavior of the colored Jones function of a knot and its volume*, Proceedings of ‘Art of Low Dimensional Topology VI’ (T. Kohno, ed.), January 2000, arXiv:math.GT/0004036.
- [9] ———, *Optimistic calculations about the Witten-Reshetikhin-Turaev invariants of closed three-manifolds obtained from the figure-eight knot by integral Dehn surgeries*, Sūrikaiseikikenkyūsho Kōkyūroku (2000), no. 1172, 70–79. MR **1** 805 729

MAHLER MEASURE AND THE VOLUME CONJECTURE

- [10] ———, *Kashaev's invariant and the volume of a hyperbolic knot after Y. Yokota*, Physics and combinatorics 1999 (Nagoya), World Sci. Publishing, River Edge, NJ, 2001, pp. 244–272. MR 1 865 040
- [11] H. Murakami and J. Murakami, *The colored Jones polynomials and the simplicial volume of a knot*, Acta Math. **186** (2001), no. 1, 85–104. MR 2002b:57005
- [12] H. Murakami, J. Murakami, M. Okamoto, T. Takata, and Y. Yokota, *Kashaev's conjecture and the Chern-Simons invariants of knots and links*, arXiv:math.GT/0203119.
- [13] R. Riley, *Growth of order of homology of cyclic branched covers of knots*, Bull. London Math. Soc. **22** (1990), no. 3, 287–297. MR 92g:57017
- [14] K. Schmidt, *Dynamical systems of algebraic origin*, Birkhäuser Verlag, Basel, 1995. MR 97c:28041
- [15] D. S. Silver and S. G. Williams, *Mahler measure, links and homology growth*, arXiv:math.GT/0003127.
- [16] ———, *Mahler measure of Alexander polynomials*, arXiv:math.GT/0105234.
- [17] Y. Yokota, *On the volume conjecture for hyperbolic knots*, arXiv:math.QA/0009165.
- [18] ———, *On the volume conjecture for hyperbolic knots*, Proceedings of the 47th All Japan Topology Symposium (Inamori Hall, Kagoshima University), July 2000, pp. 38–44.
- [19] ———, *On the volume conjecture of hyperbolic knots*, Knot Theory – dedicated to Professor Kunio Murasugi for his 70th birthday (M. Sakuma, ed.), March 2000, pp. 362–367.

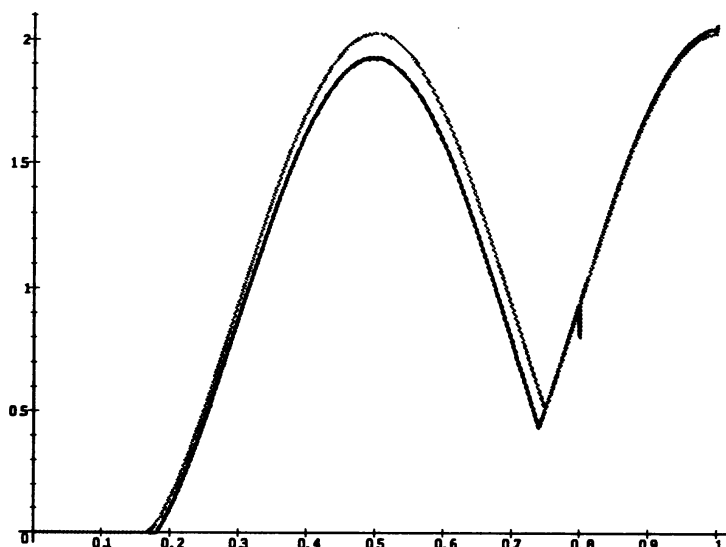


FIGURE 6. Graph of W (gray) and $2r\pi \log |J_N(E; \omega^r)|/N$ with $N = 2000$ (black) for $0 \leq r \leq 1$.

DEPARTMENT OF MATHEMATICS, TOKYO INSTITUTE OF TECHNOLOGY, OH-OKAYAMA, MEGURO,
TOKYO 152-8551, JAPAN
E-mail address: starshea@tky3.3web.ne.jp

HITOSHI MURAKAMI

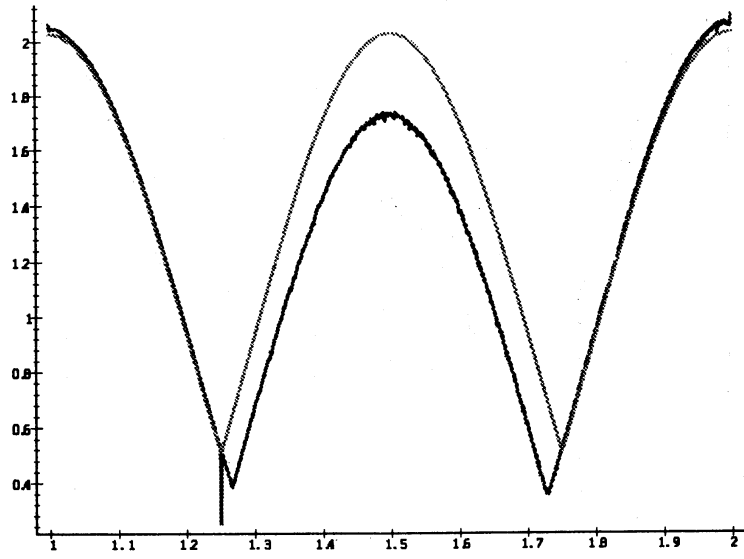


FIGURE 7. Graph of W (gray) and $2r\pi \log |J_N(E; \omega^r)|/N$ with $N = 2000$ (black) for $1 \leq r \leq 2$.

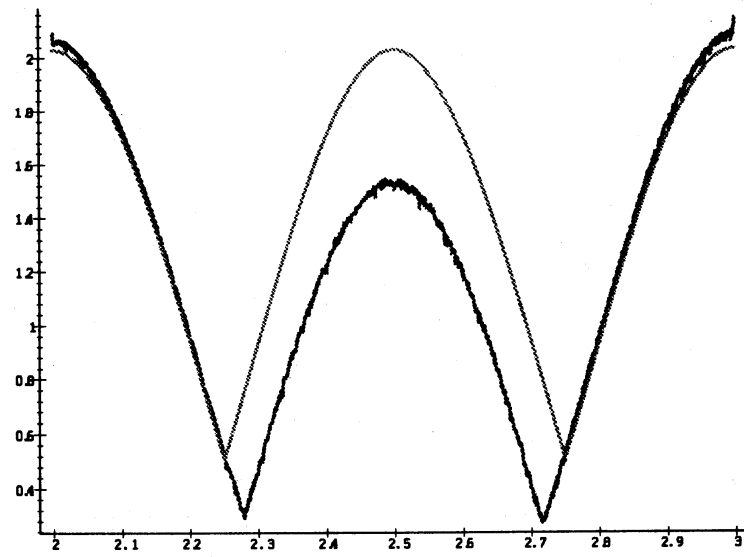


FIGURE 8. Graph of W (gray) and $2r\pi \log |J_N(E; \omega^r)|/N$ with $N = 2000$ (black) for $2 \leq r \leq 3$.

MAHLER MEASURE AND THE VOLUME CONJECTURE

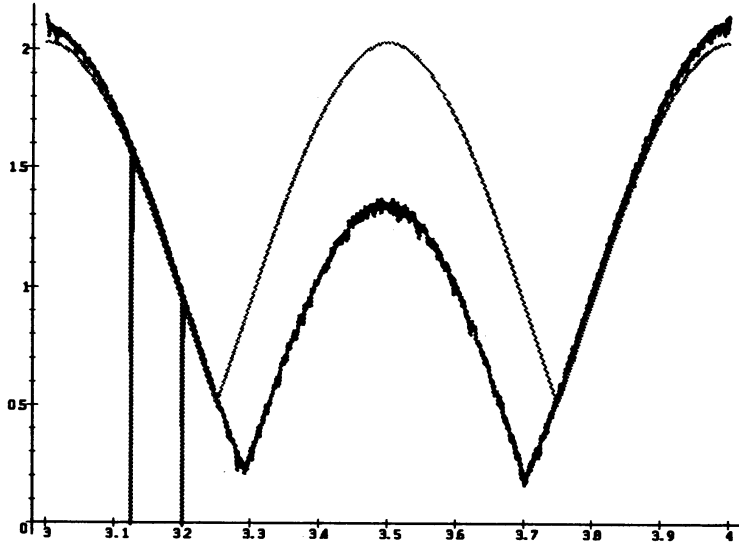


FIGURE 9. Graph of W (gray) and $2r\pi \log |J_N(E; \omega^r)|/N$ with $N = 2000$ (black) for $3 \leq r \leq 4$.

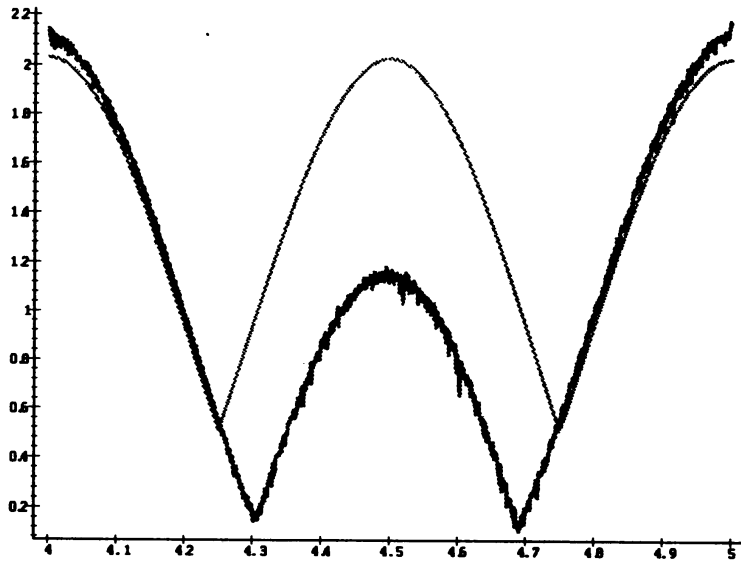


FIGURE 10. Graph of W (gray) and $2r\pi \log |J_N(E; \omega^r)|/N$ with $N = 2000$ (black) for $4 \leq r \leq 5$.

HITOSHI MURAKAMI

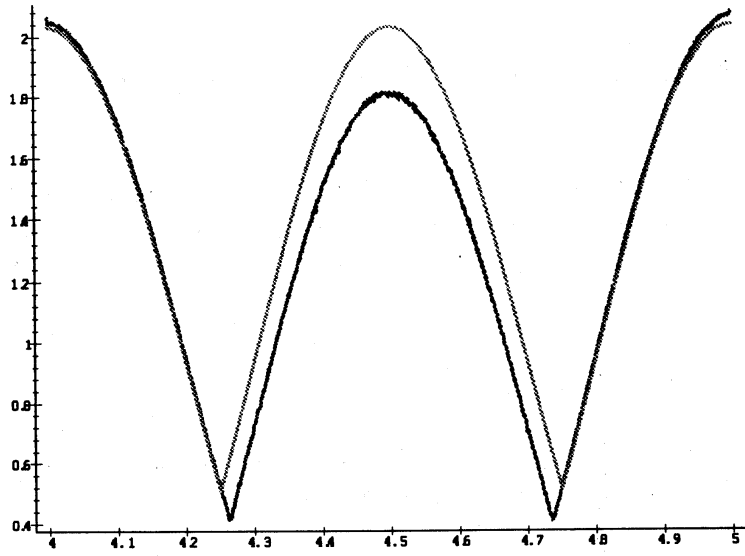


FIGURE 11. Graph of W (gray) and $2r\pi \log |J_N(E; \omega^r)|/N$ with $N = 8000$ (black) for $4 \leq r \leq 5$.