On the Global Solution of the Cauchy Problem for some Non-Linear Hyperbolic Equations

By

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This work gives one proof for the existence of the solution of the global Cauchy problem on the equation which appears in the research of the non-linear field theory and in the theory of analogue circuits for neural mechanisms.

1. About the non-linear hyperbolic equation, there are many works for the Cauchy problem and the mixed boundary value problem[1),2)]. But research has been limited for the local solution of these problems. In fact, for many Cauchy problems of the non-linear hyperbolic equations for example, for the equation of wave propagation in compressible flow, the existence of the global (in time) solution is impossible. But we can prove the existence for some classes of equations. Now we shall note in this report the existence of the global solution of this problem for an equation of the following type:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + \varphi(u)\frac{\partial u}{\partial t} + f(u) \tag{1}$$

where, functions $\varphi(u)$, $f(u)$ satisfy the following conditions:

(i) $\varphi(u) \geqq 0$, $sgn\, u\Phi(u) \geqq 0$ (for $|u| \geqq K$)

$\left[\Phi(u) = \int_0^u \varphi(u)du, \ K \text{ is a positive number}\right]$

(ii) $sgn\, uf(u) > 0$ for all $u \neq 0$

$sgn\, uf(u) > a > 0$ for $|u| > K$

$|\Phi(u)f(u)| \leqq cF(u)$ where $F(u) = \int_0^u f(u)du$ $\begin{pmatrix} c \text{ is constant} \\ |u| \text{ is small} \end{pmatrix}$

(iii) $\varphi''(u)$, $f''(u)$ are continous

We are going to consider the Cauchy problem for the initial data:

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) \tag{2}$$

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$u_0(x)$ has first and second derivatives and it is square integrables with its first and second derivatives. $u_1(x)$ has a first derivative and is square integrable with its first derivative. We shall take as the initial data the finite energy data in the sense of § 3. We shall show the global (in time) Cauchy problem is possible for this initial data.

At first, we remark that the condition is sufficiently general and that they contain as special cases two important examples.

One example is the equation of non-linear meson fields presented by Schiff treated recently by Konrad Jorgens[3]. This example is a special case of the equation (1) under the condition i) iii) for $\varphi(u)\equiv 0$, $f(u)=|u|^2u$.

Another example is an equation presented by J. Nagumo[4] for the analogue circuit of a neural mechanism. It is a special case in which

$$\varphi(u)\equiv 1-u+\varepsilon u^2\,,\quad f(u)\equiv u\,.$$

Theorem 1. The equation (1) has a solution $u(x, t)$ with the continuous first and second derivatives which is finite energy in x for all t and $u(x,0)=u_0(x)$, $u(x,0)=u_1(x)$, where $u_0(x)$, $u_1(x)$ are finite energy.

2. Associated systems of equations.

We introduce two new unknown functions.

$$\begin{aligned} &\frac{\partial u}{\partial t}+\Phi(u) = w \\ &\frac{\partial u}{\partial x} = v\,. \end{aligned} \tag{3}$$

Using these new unknowns, we consider the following system of equations and the system of initial data.

$$\begin{aligned} \frac{\partial u}{\partial t} &= w-\Phi(u) \\ \frac{\partial v}{\partial t} &= \frac{\partial w}{\partial x}-\varphi(u)v \\ \frac{\partial w}{\partial t} &= \frac{\partial v}{\partial x}-f(u)\,, \end{aligned} \tag{4}$$

$$\begin{aligned} u(x,0) &= u_0(x) \\ v(x,0) &= \frac{\partial u_0}{\partial x}(x) \\ w(x,0) &= u_1(x)+\Phi(u_0(x))\,. \end{aligned} \tag{5}$$

The equivalence of the Cauchy problem (1)-(2) and (4)-(5) is the result of classical general theory[5].

3. Generalized energy of the solution

At first we consider the case in which the initial data $u_0(x)$, $u_1(x)$ have compact carrier, that is to say, functions $u_0(x)$, $u_1(x)$ vanish except at the finite icterval of the x-axis. Assume now $u(x, t)$ is a solution of (1) and (2) for $0 \leq t \leq h$ (where h is a positive constant).

Multiplying both sides of equations, $f(u)$, v and w respectively and adding them, we have

$$f(u)\frac{\partial u}{\partial t} = f(u)w + \Phi(u)f(u)$$

$$v\frac{\partial v}{\partial t} = v\frac{\partial w}{\partial x} - \varphi(u)v^2$$

$$w\frac{\partial w}{\partial t} = w\frac{\partial v}{\partial x} - f(u)w$$

$$f(u)\frac{\partial u}{\partial t} + v\frac{\partial v}{\partial t} + w\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}(vw) - \Phi(u)f(u) - \varphi(u)v^2 .$$

We obtain

$$\frac{\partial}{\partial t}\left(F(u) + \frac{v^2}{2} + \frac{w^2}{2}\right) = \frac{\partial}{\partial x}(vw) - \varphi(u)v^2 - \Phi(u)f(u) . \tag{6}$$

The integration in x yields

$$\frac{d}{dt}E(t) = -\int_{-\infty}^{+\infty}[\varphi(u)v^2 + \Phi(u)f(u)]\,dx . \tag{7}$$

Where we denote by $E(t)$ the integral $\int_{-\infty}^{+\infty}\left(F(u) + \frac{v^2}{2} + \frac{w^2}{2}\right)dx$, and we say $E(t)$ is the generalized energy of the solution $u(x, t)$ at t. Since all functions under the integral sign have compact carriers, which is assured by the uniqueness theorem, the integrals are finites.

Hence, remembering conditions i) ii), we majorize the second member of (7)

$$\frac{dE}{dt} = -\int[\varphi(u)v^2 + \Phi(u)f(u)]\,dx$$

$$= -\int_{\varepsilon_0(t)}[\varphi(u)v^2 + \Phi(u)f(u)]\,dx - \int_{\varepsilon_1(t)}[\varphi(u)v^2 + \Phi(u)f(u)]\,dx$$

where $\varepsilon_0(t)$ is $E\{x; |u(x, t)| \geq K\}$ and $\varepsilon_1(t)$ is $E\{x; |u(x, t)[\leq K\}$. By the condition i) and ii) we can omit the second integral. Then

$$\frac{dE}{dt} \leq -\int_{\varepsilon_1(t)}[\varphi(u)v_2 + \Phi(u)f(u)]\,dx . \tag{8}$$

Using conditions ii) and iii), we have

$$\frac{dE}{dt} \leq M_\varphi \int_{-\infty}^{+\infty} v^2 dx + \int_{-\infty}^{+\infty} |\Phi(u)f(u)|\,dx , \quad (M_\varphi = \operatorname*{Max}_{|u| \leq K} |\varphi(u)|)$$

$$= M_\varphi \int_{-\infty}^{+\infty}\left[\frac{v^2}{2} + F(u)\right]dx + M_\varphi E(t) .$$

Thus, writing

$$\frac{dE}{dt} \leqq M_\varphi E(t) \tag{9}$$

we obtain

$$E(t) \leqq E(0)e^{M_\varphi t}. \tag{10}$$

We call this inequality by "Energy Inequality".

4. A priori bound for the solution.

We proceed to get the following estimate for the bound of solution for t, $0 \leqq t \leqq h$ by the initial energy $E(0)$. Because of the properties of $F(u)$, we have

$$F(K) \operatorname{mes} E\{x;\, |u(x,t)| \geqq K\} \leqq \int_{E\{x\,;\,|u|>K\}} \left[\frac{v^2}{2} + \frac{w^2}{2} + F(u)\right] dx \leqq E(t).$$

Combining with (10), we get

$$F(K) \operatorname{mes} E\{x;\ |u(x,t)| \leqq K\} \leqq E(0)e^{M_\varphi h} \qquad (\text{for } t,\, 0 \leqq t \leqq h). \tag{11}$$

It is easy to see,

$$|u(x_1, t) - u(x_2, t)| < |x_1 - x_2|^{1/2} \left[\int_{-\infty}^{+\infty} v^2 dx\right]^{1/2} \leqq |x_1 - x_2|^{1/2} E(t)^{1/2}.$$

Therefore, it holds that

$$|u(x,t)| \leqq K + \frac{E(0)e^{M_\varphi h}}{F(K)} = C \qquad (0 \leqq t \leqq h). \tag{12}$$

5. Haar's Inequality.

Let us consider two equations of (4);

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{\partial w}{\partial x} - \varphi(u)v \\ \frac{\partial w}{\partial t} &= \frac{\partial v}{\partial x} - f(u). \end{aligned} \tag{13}$$

This system is considered as a linear system for unknown v and w.

If $u(x, t)$ is a twice continously differentiable solution of (1), we can regard (13) as a system of linear equation with bounded coefficients for $0 \leqq t \leqq h$. Then we can apply Haar's inequality[6)] for this system, hence there exist a constant C for which we have

$$\begin{aligned} |v(x,t)| &\leqq C' \\ |w(x,t)| &\leqq C'' \end{aligned} \qquad (\text{for } 0 \leqq t \leqq h) \tag{14}$$

where C', C'' depend only on the initial values and φ, f.

More generally, if we differentiate (13) by x (If it is possible), we get

$$\frac{\partial v_x}{\partial t} = \frac{\partial w_x}{\partial x} - \varphi'(u)v^2 - \varphi(u)v_x \tag{15}$$
$$\frac{\partial w_x}{\partial t} = \frac{\partial v_x}{\partial x} - f'(u)v\,.$$

It is obvious that we have also appriori bounds for v, w:

$$|v_x(x, t)| \leqq c_1', \quad |w_x(x, t)| \leqq c_1'' \qquad (\text{for } t,\ 0 \leqq t \leqq h)\,. \tag{16}$$

By the same reason, we obtain

$$|v_{xx}(x, t)| \leqq c_2', \quad |w_{xx}(x, t)| \leqq c_2'' \quad (\text{for } t,\ 0 \leqq t \leqq h)\,. \tag{17}$$

6. The local existence theorem.

We show here the result obtained by R. Courant[2),7)] for the local Cauchy problem of the semilinear hyperbolic system of equation.

We consider the following system of equations:

$$U_t + AU_x + B = 0\,. \tag{18}$$

Where $U(x,t)$ is a vector with k components $u_1, \cdots, u_k$ and the $k\times k$ coefficient matrices $A(x, t)$ and vector $B(x, t, u)$ have continuous derivatives in the domain to be specified. We may assume the x-axis, $t=0$, as the initial line and prescribe on an interval F of this line, initial values:

$$U(x, 0) = U_0(x)\,. \tag{19}$$

There $U(x, t)$ is a vector with continous derivatives. Assuming the system is hyperbolic, we can prove the following theorem.

Theorem 2. If we assume for first derivatives of the coefficients of $A(x, t)$, $B(x, t, u)$, and also for an eigen element of matrix $A(x, t)$, Lipschitz continuity, The Cauchy problem (18), (19) has a solution $U(x, t)$ with continous derivatives in the strip G for sufficiently small t, and the width of G depend only on the maximum norm of the initial value (19).

In this x, t plane we consider a closed domain G so that all charactesistics C, followed from a point $P(\xi, \tau)$ in G backward towards decreasing values of t, meet the given section I in which the initial values are prescribed: Then I contains the domain of dependence of all points P of G. G_h is the strip $0 \leqq t \leqq h$ within G.

It is obvious that the system (4), (5) is the special case of the system (18), (19) $k=3$ with the smoothness conditions.

Therefore, we conclude that (4), (5) has a solution $u(x, t)$, $v(x, t)$, $w(x, t)$ in the strip G for sufficiently small h, and that the width of G, i.e. h depends only on the maximum norm of initial values:

$$u_0(x)\,,\quad \frac{\partial u_0}{\partial x}\,,\quad u_1(x)+\Phi(u_0(x))\,.$$

Whose carriers are supposed to be in I.

7. Global existence theorem.

We proved the boundedness of functions, u, v, w in §4. That means the existence of the solution of the system (4) for the initial data with the compact carrier because of the result of §6, for all t.

For the initial data with the finite energy in the sense of §4, we can approximate this initial data by functions with compact carriers for the topology of the norm

$$E\{U\} = \int_{-\infty}^{+\infty}\left[\frac{w^2}{2}+\frac{v^2}{2}+F(u)\right]dx\,,\quad U=(u,\,v,\,w)\,.$$

If we apply tne energy inequality for the difference of two solutions of system (4), we conclude the existence of the finite energy solution of (4) for all t.

In order to obtain the result for the equation (1) and initial data (2), we will suppose that the initial data $u_0(x)$, $u_1(x)$ is finite energy in the usual sense, that is to say, $\int_{-\infty}^{+\infty}\left[u_0(x)^2+u_1(x)^2+\left(\frac{\partial u_0}{\partial x}\right)\right]dx$ is finite. This energy majorizes the generalized energy of the initial data of the system (4) because of the property of φ and f in ii) and by the lemma of Sovolev. Thus we finish the proof of the theorem 1.

Remark. We can replace condition ii) by weaker conditions, but we do not here discuss the most general conditions. Also it is possible that for every initial data with continous first and second derivatives, we prove the existence of the global (in time and in space) solution for the above equation.

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