

Unramified base change of simple supercuspidal L -packets of quasi-split classical groups

By

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Abstract

In this paper, we prove that the base change lifts of simple supercuspidal L -packets of quasi-split symplectic or special orthogonal groups with respect to an unramified extension are again simple supercuspidal L -packets, and determine the structures of them, under some assumption on the residual characteristic.

§ 1. Introduction

Let F be a p -adic field. We consider a general linear group GL_N over F . Then, according to the *local Langlands correspondence* for GL_N , which was established by Harris and Taylor ([HT01]), irreducible admissible representations of $\mathrm{GL}_N(F)$ can be classified in terms of N -dimensional representations of $W_F \times \mathrm{SL}_2(\mathbb{C})$. Here W_F is the Weil group of F . More precisely, if we denote the set of equivalence classes of irreducible admissible representations of $\mathrm{GL}_N(F)$ by $\Pi(\mathrm{GL}_{N,F})$, and the set of conjugacy classes of L -parameters of GL_N by $\Phi(\mathrm{GL}_{N,F})$, then we have a natural bijection $\mathrm{LLC}_{\mathrm{GL}_{N,F}}$ between these sets:

$$\mathrm{LLC}_{\mathrm{GL}_{N,F}} : \Pi(\mathrm{GL}_{N,F}) \xrightarrow{\cong} \Phi(\mathrm{GL}_{N,F}).$$

Now let us take a finite extension E of F . Then we can consider the restriction of representations of $W_F \times \mathrm{SL}_2(\mathbb{C})$ to $W_E \times \mathrm{SL}_2(\mathbb{C})$. In particular, from L -parameters of $\mathrm{GL}_{N,F}$, we obtain L -parameters of $\mathrm{GL}_{N,E}$. Therefore, by using the local Langlands

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correspondences for $\mathrm{GL}_{N,F}$ and $\mathrm{GL}_{N,E}$, we can define a natural map from $\Pi(\mathrm{GL}_{N,F})$ to $\Pi(\mathrm{GL}_{N,E})$:

$$\begin{array}{ccc} \Pi(\mathrm{GL}_{N,E}) & \xleftarrow{\mathrm{LLC}} & \Phi(\mathrm{GL}_{N,E}) \\ \uparrow \scriptstyle \mathrm{BC} & & \uparrow \scriptstyle \mathrm{Res} \\ \Pi(\mathrm{GL}_{N,F}) & \xrightarrow{\mathrm{LLC}} & \Phi(\mathrm{GL}_{N,F}) \end{array}$$

In fact, such an operation had been studied by Arthur and Clozel ([AC89]) before the works of Harris and Taylor, and is called the *base change*. In particular, when the extension E/F is cyclic, they showed the existence of the base change without appealing to the local Langlands correspondence. In this case, the base change can be characterized representation-theoretically in terms of the characters of representations, and the equality of characters characterizing the base change is called the *Shintani character relation* (see Section 2.4 for details). Therefore it is an interesting problem to determine the base change explicitly by computing the Shintani character relation.

In this paper, in the case where E is unramified over F , we consider this problem for some special representations, which are called *simple supercuspidal representations*. The simple supercuspidal representations are irreducible supercuspidal representations having the minimal positive depth. They were defined by Gross and Reeder in [GR10] (and also by Reeder and Yu in [RY14]), and have been studied in the context of finding an explicit description of the local Langlands correspondence by many people. By noting that the construction of them is very explicit, we can parametrize the set of equivalence classes of simple supercuspidal representations of $\mathrm{GL}_N(F)$ by the following set:

$$\mathrm{SSC}(\mathrm{GL}_{N,F}) := \widehat{k_F^\times} \times k_F^\times \times \mathbb{C}^\times,$$

where k_F is the residue field of F and $\widehat{k_F^\times}$ is the set of characters of $k_F^\times$. By using this parametrization, the first main result of this paper can be stated as follows:

Theorem 1.1 (Theorem 3.11). *Let l be the degree of the unramified extension E over F . We assume that $(p, N) = 1$. Let us consider a simple supercuspidal representation of $\mathrm{GL}_N(F)$ corresponding to an element $(\omega, a, \zeta) \in \mathrm{SSC}(\mathrm{GL}_{N,F})$. Then its base change to $\mathrm{GL}_N(E)$ is again simple supercuspidal, and corresponds to the data*

$$\begin{cases} (\omega \circ \mathrm{Nr}, a, -\zeta^l) \in \mathrm{SSC}(\mathrm{GL}_{N,E}) & \text{if } l \text{ and } N \text{ are even numbers,} \\ (\omega \circ \mathrm{Nr}, a, \zeta^l) \in \mathrm{SSC}(\mathrm{GL}_{N,E}) & \text{otherwise.} \end{cases}$$

As an application of this result, we can also determine the unramified base change of simple supercuspidal representations of *quasi-split classical groups*. To be more precise, let $\mathbf{G}$ be a quasi-split special orthogonal or symplectic group over a p -adic field F . For

these groups, recently Arthur established the local Langlands correspondence ([Art13]). Before we recall it, we define the sets $\Pi(\mathbf{G})$ and $\Phi(\mathbf{G})$ in a similar way to the case of general linear groups. We consider the action of outer automorphisms $\text{Out}(\mathbf{G})$ of $\mathbf{G}$ on $\Pi(\mathbf{G})$ (resp. $\text{Out}(\widehat{\mathbf{G}})$ on $\Phi(\mathbf{G})$, where $\widehat{\mathbf{G}}$ is the Langlands dual group of $\mathbf{G}$), and denote the sets of orbits via such actions by $\widetilde{\Pi}(\mathbf{G}) := \Pi(\mathbf{G})/\text{Out}(\mathbf{G})$ and $\widetilde{\Phi}(\mathbf{G}) := \Phi(\mathbf{G})/\text{Out}(\widehat{\mathbf{G}})$. Note that the outer automorphisms $\text{Out}(\mathbf{G})$ and $\text{Out}(\widehat{\mathbf{G}})$ are nontrivial only when $\mathbf{G}$ is an even special orthogonal group. Then, according to Arthur's results, we have a natural surjection $\text{LLC}_{\mathbf{G}}$ with finite fibers from the set $\widetilde{\Pi}(\mathbf{G})$ to the set $\widetilde{\Phi}(\mathbf{G})$. In other words, we have a natural partition of the set $\widetilde{\Pi}(\mathbf{G})$ into finite sets:

$$\widetilde{\Pi}(\mathbf{G}) = \bigsqcup_{\phi \in \widetilde{\Phi}(\mathbf{G})} \Pi_{\phi}^{\mathbf{G}}.$$

Here $\Pi_{\phi}^{\mathbf{G}}$ is the fiber of ϕ via the map $\text{LLC}_{\mathbf{G}}$ and called the L -packet of ϕ . Therefore, in the same way as in the general linear group case, we can define the base change of L -packets from $\mathbf{G}$ to $\mathbf{G}_E$ by considering the restriction of L -parameters.

On the other hand, in consecutive papers [Oi16] and [Oi18], we determined the structures of L -packets consisting of simple supercuspidal representations of quasi-split special orthogonal and symplectic groups, under the assumption that the residual characteristic is not equal to 2. Furthermore, we also determined their endoscopic lifts to general linear groups. More precisely, let GL_N be a general linear group such that $\mathbf{G}$ is an (twisted) endoscopic group of GL_N (the size N of such GL_N is uniquely determined by $\mathbf{G}$). Then, we have an embedding from the L -group of $\mathbf{G}$ to that of GL_N . In particular, we can regard an L -parameter of $\mathbf{G}$ as an L -parameter of GL_N (recall that an L -parameter is a homomorphism from $W_F \times \text{SL}_2(\mathbb{C})$ to the L -group). Thus, by using the local Langlands correspondences for $\mathbf{G}$ and GL_N , we get a map from the set of L -packets of $\mathbf{G}$ to that of GL_N . This operation is called the endoscopic lifting from $\mathbf{G}$ to GL_N .

$$\begin{array}{ccc} \Pi(\text{GL}_{N,F}) & \xleftarrow{\text{LLC}} & \Phi(\text{GL}_{N,F}) \\ \uparrow \text{Endo. lift} & & \uparrow \text{composing } L\text{-embedding} \\ \widetilde{\Pi}(\mathbf{G}) & \xrightarrow{\text{LLC}} & \widetilde{\Phi}(\mathbf{G}) \end{array}$$

The point here is that the endoscopic lifting is an injection from the set of L -packets of $\mathbf{G}$ to that of GL_N . Therefore, by combining these results with Theorem 1.1, we immediately get the following result:

Theorem 1.2 (Theorem 3.12). *We assume that $(p, 2N) = 1$. The base change lift of a simple supercuspidal L -packet of $\mathbf{G}$ to $\mathbf{G}_E$ is again a simple supercuspidal L -packet. Moreover, we can describe the relationship between these L -packets in terms of*

a parametrization of simple supercuspidal representations of $\mathbf{G}$ and $\mathbf{G}_E$ (see Section 2.2 for details).

$$\begin{array}{ccc} \tilde{\Pi}(\mathbf{G}_E) & \xrightarrow[\text{lift}]{\text{Endo.}} & \Pi(\text{GL}_{N,E}) \\ \uparrow \text{BC} & & \uparrow \text{BC} \\ \tilde{\Pi}(\mathbf{G}) & \xrightarrow[\text{lift}]{\text{Endo.}} & \Pi(\text{GL}_{N,F}) \end{array}$$

We explain the outline of the proof of Theorem 1.1. First, we note that the “simple supercuspidality” of irreducible admissible representations of general linear groups can be characterized in terms of the Swan conductor of the corresponding L -parameters. Since E is unramified over F , the restriction of the L -parameters from $W_F \times \text{SL}_2(\mathbb{C})$ to $W_E \times \text{SL}_2(\mathbb{C})$ does not change the Swan conductor. In particular, we can conclude that the base change lifts of simple supercuspidal representations of $\text{GL}_N(F)$ to $\text{GL}_N(E)$ are again simple supercuspidal. Then our task is to determine the relationship between the parametrizing data $\text{SSC}(\text{GL}_{N,F})$ and $\text{SSC}(\text{GL}_{N,E})$. To determine this, we use the Shintani character relation. Since simple supercuspidal representations are constructed very explicitly by using the compact induction, we can compute their (twisted) characters by the *(twisted) character formula*. By choosing some special elements carefully, we can write the character of a simple supercuspidal representation at such elements in terms of the *Kloosterman sum* containing an information of parametrizing data. By combining such a computation with the Shintani character relation and using some elementary property of the Kloosterman sums, we can describe the relation between the parametrizing data of simple supercuspidal representations. This completes the proof.

We comment on the organization of this paper. In Section 2, we first recall the definition and a parametrization of simple supercuspidal representations of general linear groups and quasi-split classical groups. Moreover, we summarize results in [Oi16, Oi18]. Second, we recall basic properties of the base change. Finally, we recall the character formula for supercuspidal representations. In Section 3, we compute the (twisted) characters of simple supercuspidal representations of general linear groups at some special elements. Then, by combining it with the Shintani character relation, we prove Theorem 1.1. Finally, by using the main results in [Oi16, Oi18], we get Theorem 1.2.

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§ 2. Preliminaries

§ 2.1. Notations

We fix a p -adic field F . We denote its ring of integers, its maximal ideal, and its residue field by $\mathcal{O}_F$, $\mathfrak{p}_F$, and k_F , respectively. We fix a uniformizer ϖ of F . Let q be the order of k_F . For $x \in \mathcal{O}_F$, $\bar{x}$ denotes the image of x in k_F . We regard an element of $k_F^\times$ as an element of $F^\times$ by the Teichmüller lift. We write W_F for the Weil group of F .

Let E be an unramified extension of F of degree l . For this field E , we use the similar notations such as $\mathcal{O}_E$, k_E , and so on. Let σ be a generator of $\text{Gal}(E/F)$. We write shortly Nr and Tr for Nr_{k_E/k_F} and Tr_{k_E/k_F} , respectively.

We fix a non-square element ϵ_F of $k_F^\times$ and ϵ_E of $k_E^\times$ (these will be used to parametrize simple supercuspidal representations of classical groups).

For a multiplicative character ω of $k_F^\times$ and a non-trivial additive character ψ of k_F , we define the Gauss sum with respect to (ω, ψ) by $G(k_F; \omega, \psi) := \sum_{x \in k_F^\times} \omega(x) \psi(x)$.

We write I_N for the identity matrix of size N .

§ 2.2. Simple supercuspidal representations

We first recall the definition of simple supercuspidal representations in the case of a general linear group briefly. See [RY14, Section 2.6] or [Oi18, Section 2.1] for the general construction of simple supercuspidal representations of other groups.

Let I_F be the standard Iwahori subgroup of $\text{GL}_N(F)$, I_F^+ the pro-unipotent radical of I_F , and I_F^{++} the subgroup of I_F^+ generated by $[I_F^+, I_F^+]$ and diagonal matrices all of whose entries belong to $1 + \mathfrak{p}_F$:

$$I_F = \begin{pmatrix} \mathcal{O}_F^\times & & \\ & \ddots & \\ \mathfrak{p}_F & & \mathcal{O}_F^\times \end{pmatrix}, \quad I_F^+ = \begin{pmatrix} 1 + \mathfrak{p}_F & & \mathcal{O}_F \\ & \ddots & \\ \mathfrak{p}_F & & 1 + \mathfrak{p}_F \end{pmatrix} \supset I_F^{++}.$$

Then we have the isomorphism $I_F^+/I_F^{++} \cong k_F^{\oplus N}$; $(x_{ij})_{ij} \mapsto (\overline{x_{12}}, \dots, \overline{x_{N-1,N}}, \overline{x_{N1}\varpi^{-1}})$.

For $a \in k_F^\times$, we set $\varphi_a := \begin{pmatrix} 0 & I_{N-1} \\ \varpi a & 0 \end{pmatrix} \in \text{GL}_N(F)$. Note that this element satisfies $\varphi_a^N = \varpi a I_N$.

To define simple supercuspidal representations, we first fix a non-trivial additive character ψ of the residue field k_F . Then, for an element $(\omega, a, \zeta) \in \widehat{k_F^\times} \times k_F^\times \times \mathbb{C}^\times$ (here $\widehat{k_F^\times}$ is the set of characters of $k_F^\times$), we define a character $\chi_{\omega, a, \zeta}$ of $Z_F I_F^+ \langle \varphi_{a^{-1}} \rangle$ by

$$\begin{aligned} \chi_{\omega, a, \zeta}(z) &:= \omega(\bar{z}) \quad \text{for } z \in \mathcal{O}_F^\times \subset Z_F, \\ \chi_{\omega, a, \zeta}(x) &:= \psi(\overline{x_{12}} + \dots + \overline{x_{N-1,N}} + \overline{a x_{N1} \varpi^{-1}}) \quad \text{for } x \in I_F^+, \text{ and} \\ \chi_{\omega, a, \zeta}(\varphi_{a^{-1}}) &:= \zeta. \end{aligned}$$

Here Z_F is the center of $\mathrm{GL}_N(F)$.

Then the representation $\pi_{\omega,a,\zeta} := \mathrm{c}\text{-Ind}_{Z_F I_F^+ \langle \varphi_{a-1} \rangle}^{\mathrm{GL}_N(F)} \chi_{\omega,a,\zeta}$ is irreducible, hence supercuspidal. We call representations obtained in this way *simple supercuspidal* representations of $\mathrm{GL}_N(F)$. In fact, we can check that $\pi_{\omega,a,\zeta}$ and $\pi_{\omega',a',\zeta'}$ are equivalent if and only if we have $(\omega, a, \zeta) = (\omega', a', \zeta')$. Namely, the set $\mathrm{SSC}(\mathrm{GL}_{N,F}) := \widehat{k_F^\times} \times k_F^\times \times \mathbb{C}^\times$ parametrizes the set of equivalence classes of simple supercuspidal representations of $\mathrm{GL}_N(F)$ (see [Oi18, Section 2.3] for the details).

For other reductive groups (with some mild conditions such as tamely-ramifiedness), we can define simple supercuspidal representations in a similar procedure. In particular, for the following groups, we can define simple supercuspidal representations under the assumption that p is not equal to 2:

- the split special orthogonal group SO_{2n+1} of size $2n+1$,
- the symplectic group Sp_{2n} of size $2n$,
- the split (resp. quasi-split non-split unramified) special orthogonal group SO_{2n} (resp. $\mathrm{SO}_{2n}^{\mathrm{ur}}$) of size $2n$, and
- the quasi-split special orthogonal group SO_{2n}^μ of size $2n$ associated to a ramified quadratic character μ of $F^\times$.

For these groups, parametrizing sets of equivalence classes of simple supercuspidal representations can be taken as follows (here $\mu_2 = \{\pm 1\}$ is the set of signs):

group $\mathbf{G}$	parametrizing set $\mathrm{SSC}(\mathbf{G})$
SO_{2n+1}	$k_F^\times \times \mu_2$
Sp_{2n}	$\mu_2 \times \{0, 1\} \times k_F^\times$
$\mathrm{SO}_{2n}^{(\mathrm{ur})}$	$\mu_2 \times \{0, 1\} \times k_F^\times \times \mu_2$
SO_{2n}^μ	$\mu_2 \times k_F^\times$

Here we do not describe each parametrization. See [Oi16, Section 2.4] (the case of an odd special orthogonal group) and [Oi18, Sections 2.4, 2.5, 2.6, and 2.7] (the cases of symplectic and even special orthogonal groups) for the details.

We note that the parametrization of simple supercuspidal representations depends on the choice of a uniformizer ϖ and a non-trivial additive character ψ . When we consider the above parametrizations for general linear groups or quasi-split classical groups over E , we choose a uniformizer and a non-trivial additive character to be ϖ and $\psi \circ \mathrm{Tr}$.

Finally, we investigate σ -stable simple supercuspidal representations of $\mathrm{GL}_{N,E}$.

Lemma 2.1. *Let $X = (\omega, a, \zeta) \in \text{SSC}(\text{GL}_N(E)) = \widehat{k_E^\times} \times k_E^\times \times \mathbb{C}^\times$ and π_X be the corresponding simple supercuspidal representation of $\text{GL}_N(E)$. Then π_X is σ -stable if and only if ω is σ -stable and a belongs to $k_F^\times$.*

Proof. By the Frobenius reciprocity, we have $\pi_X^\sigma \cong \text{c-Ind}_{Z_E I_E^+ \langle \varphi_{\sigma(a)}^{-1} \rangle}^{\text{GL}_N(E)} \chi_{\omega, a, \zeta}^\sigma$. As $\chi_{\omega, a, \zeta}^\sigma$ equals $\chi_{\omega^\sigma, \sigma(a), \zeta}$, the parameter of π_X^σ is given by $(\omega^\sigma, \sigma(a), \zeta)$. This equals the original data X if and only if the conditions in the assertion are satisfied. $\square$

§ 2.3. Simple supercuspidal L -packets of quasi-split classical groups

In this subsection, we summarize the results in [Oi16, Oi18]. Let $\mathbf{G}$ be a quasi-split special orthogonal or symplectic group over F . Then $\mathbf{G}$ can be regarded as an endoscopic group of a general linear group GL_N , and the size N of such a general linear group is given as follows:

$\mathbf{G}$	SO_{2n+1}	Sp_{2n}	SO_{2n}
GL_N	GL_{2n}	GL_{2n+1}	GL_{2n}

Therefore we can consider the *endoscopic lifting* from $\mathbf{G}$ to GL_N . In [Oi16, Oi18], we determined the structures of L -packets of $\mathbf{G}$ containing simple supercuspidal representations and their endoscopic lifts to GL_N . The main results of [Oi16, Oi18] are stated as follows in terms of the parametrizing sets in Section 2.2:

Theorem 2.2 ([Oi16, Oi18]). *We assume that p is not equal to 2. Let $\pi_X^\mathbf{G}$ be a simple supercuspidal representation of $\mathbf{G}(F)$ corresponding to an element X of $\text{SSC}(\mathbf{G})$. Let $\Pi_\phi^\mathbf{G}$ be the L -packet of $\mathbf{G}$ containing the $\text{Out}(\mathbf{G})$ -orbit of $\pi_X^\mathbf{G}$, and $\pi_\phi^{\text{GL}_N}$ its endoscopic lift to GL_N .*

The case where $\mathbf{G} = \text{SO}_{2n+1}$: *The L -packet $\Pi_\phi^\mathbf{G}$ is a singleton. Moreover, if we put $X = (a, \zeta)$, then $\pi_\phi^{\text{GL}_{2n}}$ is a simple supercuspidal representation corresponding to $(\mathbb{1}, 2a, \zeta) \in \text{SSC}(\text{GL}_{2n, F})$.*

The case where $\mathbf{G} = \text{Sp}_{2n}$: *The L -packet $\Pi_\phi^\mathbf{G}$ consists of $\pi_{X_0}^\mathbf{G}$ and $\pi_{X_1}^\mathbf{G}$, where $X_0 = (\xi, 0, a)$ and $X_1 = (\xi, 1, a\epsilon_F^{-1}) \in \text{SSC}(\mathbf{G})$ (X is equal to one of X_0 and X_1). Moreover $\pi_\phi^{\text{GL}_{2n+1}}$ is given by the parabolic induction of $\pi_{\omega_0, 4a, \zeta}^{\text{GL}_{2n}} \boxtimes \omega_{\omega_0, 4a, \zeta}^{\text{GL}_{2n}}$. Here ω_0 is the non-trivial quadratic character of $k_F^\times$, $\zeta = \xi \cdot q^{-\frac{1}{2}} \omega_0(-1)G(k_F; \omega_0, \psi)$, and $\omega_{\omega_0, 4a, \zeta}^{\text{GL}_{2n}}$ is the central character of $\pi_{\omega_0, 4a, \zeta}^{\text{GL}_{2n}}$.*

The case where $\mathbf{G} = \text{SO}_{2n}^\mu$: *The L -packet $\Pi_\phi^\mathbf{G}$ is a singleton and, if we put $X = (\xi, a)$, then the unique $\text{Out}(\mathbf{G})$ -orbit in this L -packet is given by $\{\pi_{\xi, a}^\mathbf{G}, \pi_{\xi, -a}^\mathbf{G}\}$. Moreover $\pi_\phi^{\text{GL}_{2n}}$ is a simple supercuspidal representation corresponding to $(\omega_0, 4a', \zeta) \in \text{SSC}(\text{GL}_{2n})$, where $\zeta = \xi \cdot q^{-\frac{1}{2}}G(k_F; \omega_0, \psi)$ and $a' = (-1)^{n-1}a^2\epsilon_F^{-\nu_\mu}$. Here $\nu_\mu = 0$ if μ corresponds to the quadratic extension $F(\sqrt{\varpi})$ of F , and $\nu_\mu = 1$ otherwise.*

The case where $\mathbf{G} = \mathrm{SO}_{2n+2}^{(\mathrm{ur})}$: The L -packet $\Pi_\phi^\mathbf{G}$ consists of $\pi_{X_0}^\mathbf{G}$ and $\pi_{X_1}^\mathbf{G}$, where $X_0 = (\xi, 0, a, \zeta)$ and $X_1 = (\xi, 1, a\epsilon_F^{-1}, \zeta) \in \mathrm{SSC}(\mathbf{G})$ (X is equal to one of X_0 and X_1). Moreover $\pi_\phi^{\mathrm{GL}_{2n+2}}$ is given by the parabolic induction of

$$\pi_{\omega_0, b, \zeta \eta}^{\mathrm{GL}_{2n}} \boxtimes (\omega_{\omega_0, b, \zeta \eta}^{\mathrm{GL}_{2n}} \otimes \mu_\mathbf{G} \otimes \mu_\mathrm{ur}^\iota) \boxtimes \mu_\mathrm{ur}^\iota, \quad \text{where } \iota := \frac{1 - \zeta}{2}.$$

Here $\eta = (-1)^{t_\mathbf{G}} q^{-\frac{1}{2}} G(k_F; \omega_0, \psi) \omega_0(-1)\xi$, $b = (-1)^{n-1} 2^{2+t_\mathbf{G}} a$, $\mu_\mathbf{G}$ (resp. μ_ur) is the quadratic character of $F^\times$ corresponding to $\mathbf{G}$ (resp. the nontrivial unramified quadratic character of $F^\times$), and we put $t_\mathbf{G}$ to be 0 if $\mathbf{G} = \mathrm{SO}_{2n+2}$, and 1 if $\mathbf{G} = \mathrm{SO}_{2n+2}^{(\mathrm{ur})}$.

§ 2.4. Arthur–Clozel’s base change

In this subsection, we recall the definition and some fundamental properties of the base change established by Arthur and Clozel in [AC89].

Definition 2.3 ([AC89, Chapter 1, Lemma 1.1]). Let g be an element of $\mathrm{GL}_N(E)$. Then there exists an element of $\mathrm{GL}_N(F)$ which is conjugate to $g\sigma(g) \cdots \sigma(g)^{l-1} \in \mathrm{GL}_N(E)$. Moreover, such an element is unique up to $\mathrm{GL}_N(F)$ -conjugacy. We denote the $G(F)$ -conjugacy class of such elements of $\mathrm{GL}_N(F)$ by $\mathcal{N}(g)$.

For an irreducible admissible representation π_F of $\mathrm{GL}_N(F)$, we denote its character by Θ_{π_F} . On the other hand, if an irreducible admissible representation π_E of $\mathrm{GL}_N(E)$ is σ -stable (i.e., we have $\pi_E^\sigma \cong \pi_E$), then its σ -twisted character is defined (see Section 2 of Chapter 1 in [AC89]). We denote it by $\Theta_{\pi_E, \sigma}$. Note that we adopt the Whittaker normalization of σ -twisted characters. See Section 2 of Chapter 1 in [AC89] for the details of general properties of σ -twisted characters of σ -stable representations. By using these characters and σ -twisted characters of representations, the notion of the base change of representations is defined as follows:

Definition 2.4 (base change, [AC89, Chapter 1, Definition 6.1]). Let π_E (resp. π_F) be an irreducible admissible representation of $\mathrm{GL}_N(E)$ (resp. $\mathrm{GL}_N(F)$). Assume that π_E is σ -stable. We say that π_E is a base change lift of π_F if, for every $g \in \mathrm{GL}_N(E)$ such that $\mathcal{N}(g)$ is regular semisimple, the following equality (the *Shintani character relation*) holds:

$$\Theta_{\pi_E, \sigma}(g) = \Theta_{\pi_F}(\mathcal{N}(g)).$$

Here note that the right-hand side is well-defined since the character of a representation is invariant under conjugation, and that also the left-hand side is well-defined because the regular semisimplicity of $\mathcal{N}(g)$ implies the σ -regular σ -semisimplicity of g (see the proof of Proposition 2.2 and the end of Section 1 of Chapter 1 in [AC89]).

For an irreducible admissible representation π_F of $\mathrm{GL}_N(F)$, its base change lift to $\mathrm{GL}_N(E)$ is at most unique by the linear independence of twisted characters of σ -stable irreducible admissible representations (see [AC89, Chapter 1, Lemma 6.3]). On the other hand, by Theorem 6.2 of Chapter 1 in [AC89], if π_F is tempered, then its base change lift always exists and is tempered. Thus we get a map from the set of equivalence classes of irreducible tempered representations of $\mathrm{GL}_N(F)$ to the set of equivalence classes of σ -stable irreducible tempered representations of $\mathrm{GL}_N(E)$. We denote it by BC:

$$\mathrm{BC}: \Pi_{\mathrm{temp}}(\mathrm{GL}_{N,F}) \rightarrow \Pi_{\mathrm{temp}}^{\sigma}(\mathrm{GL}_{N,E}).$$

We remark that this map is in fact surjective, and that we can describe the fibers of this map. See [AC89, Chapter 1, Theorem 6.2 and Proposition 6.7] for the details.

Finally, we note that the base change is compatible with the restriction of the L -parameters from $W_F \times \mathrm{SL}_2(\mathbb{C})$ to $W_E \times \mathrm{SL}_2(\mathbb{C})$ under the local Langlands correspondence for general linear groups (see Lemma VII.2.6 (5) in [HT01]):

$$\begin{array}{ccc} \Pi(\mathrm{GL}_{N,E}) & \xrightarrow{\mathrm{LLC}} & \Phi(\mathrm{GL}_{N,E}) \\ \uparrow \mathrm{BC} & & \uparrow \mathrm{Res} \\ \Pi(\mathrm{GL}_{N,F}) & \xrightarrow{\mathrm{LLC}} & \Phi(\mathrm{GL}_{N,F}) \end{array}$$

As a consequence of this compatibility, we get the following:

Proposition 2.5. *Let π_E be a σ -stable simple supercuspidal representation of $\mathrm{GL}_N(E)$. We take $\pi_F \in \Pi(\mathrm{GL}_{N,F})$ such that $\mathrm{BC}(\pi_F) = \pi_E$ (as noted above, we can always take such an element). Then π_F is simple supercuspidal.*

Proof. First, since π_E is supercuspidal, its L -parameter $\mathrm{LLC}_{\mathrm{GL}_{N,E}}(\pi_E)$ is trivial on $\mathrm{SL}_2(\mathbb{C})$ and irreducible as a representation of W_E (this follows from the construction of $\mathrm{LLC}_{\mathrm{GL}_N}$, see the paragraph after [HT01, Corollary VII.2.19]). Since $\mathrm{LLC}_{\mathrm{GL}_{N,E}}(\pi_E)$ equals $\mathrm{LLC}_{\mathrm{GL}_{N,F}}(\pi_F)|_{W_E \times \mathrm{SL}_2(\mathbb{C})}$ as noted above, $\mathrm{LLC}_{\mathrm{GL}_{N,F}}(\pi_F)$ is trivial on $\mathrm{SL}_2(\mathbb{C})$ and irreducible as a representation of W_F . Hence π_F is supercuspidal.

Next, we recall that, for an irreducible supercuspidal representation π of $\mathrm{GL}_N(F)$, the simple supercuspidality of π is characterized in terms of the Swan conductor of $\mathrm{LLC}_{\mathrm{GL}_{N,F}}(\pi)$ (see [IT18, Section 1]). Here note that the Swan conductor criterion in [IT18, Section 1] is stated by using the “Swan conductor of π ”. However, since the local Langlands correspondence for general linear groups preserves the local ε -factors, these two notions coincide. See, e.g., [IT18, Section 1] and [GR10, Section 2.2] for the precise definitions of the Swan conductors for these representations. Then, since we assumed that E is unramified over F , the Swan conductor of $\mathrm{LLC}_{\mathrm{GL}_{N,F}}(\pi_F)$ is equal to that of $\mathrm{LLC}_{\mathrm{GL}_{N,F}}(\pi)|_{W_E \times \mathrm{SL}_2(\mathbb{C})} = \mathrm{LLC}_{\mathrm{GL}_{N,E}}(\pi_E)$. Thus the simple supercuspidality of π_E implies that of π_F . $\square$

Corollary 2.6. *The base change lift of a simple supercuspidal representation of $\mathrm{GL}_N(F)$ to $\mathrm{GL}_N(E)$ is again simple supercuspidal.*

Proof. By the behavior of the central character under the base change ([AC89, Chapter 1, Theorem 6.2 (d)]) and Proposition 2.5, the fiber $\mathrm{BC}^{-1}(\mathrm{SSC}_{\omega_E}^{\sigma}(\mathrm{GL}_{N,E}))$ of the set $\mathrm{SSC}_{\omega_E}^{\sigma}(\mathrm{GL}_{N,E})$ of σ -stable simple supercuspidal representations of $\mathrm{GL}_N(E)$ with fixed central character ω_E under the base change map consists of simple supercuspidal representations of $\mathrm{GL}_N(F)$ with central character ω_F . Here we have $\omega_E = \omega_F \circ \mathrm{Nr}_{E/F}$.

We compare the cardinality of $\mathrm{BC}^{-1}(\mathrm{SSC}_{\omega_E}^{\sigma}(\mathrm{GL}_{N,E}))$ to that of $\mathrm{SSC}_{\omega_E}^{\sigma}(\mathrm{GL}_{N,E})$. For a simple supercuspidal representation $\pi_{X'}$ of $\mathrm{GL}_N(E)$ with data $X' = (\omega', a', \zeta') \in \widehat{\mathrm{SSC}(\mathrm{GL}_{N,E})} = \widehat{k_E^{\times}} \times k_E^{\times} \times \mathbb{C}^{\times}$ and central character ω_E , we have $\omega_E|_{k_E^{\times}} = \omega'$ and $\zeta'^N = \omega_E(\varpi a'^{-1})$. Note that $\varphi_{a'^{-1}}^N = \varpi a'^{-1} I_N$. Moreover, by Lemma 2.1, $\pi_{X'}$ is σ -stable if and only if ω' is σ -stable and a' belongs to $k_F^{\times}$. Hence the cardinality of the set $\mathrm{SSC}_{\omega_E}^{\sigma}(\mathrm{GL}_{N,E})$ is given by $(q-1) \cdot N$. On the other hand, by a similar argument, we can check that the cardinality of the set $\mathrm{SSC}_{\omega_F}(\mathrm{GL}_{N,F})$ of simple supercuspidal representations of $\mathrm{GL}_N(F)$ with fixed central character ω_F is given by $(q-1) \cdot N$.

The set $\mathrm{BC}^{-1}(\mathrm{SSC}_{\omega_E}^{\sigma}(\mathrm{GL}_{N,E}))$ is contained in $\mathrm{SSC}_{\omega_F}(\mathrm{GL}_{N,F})$ and the map BC induces a surjection from $\mathrm{BC}^{-1}(\mathrm{SSC}_{\omega_E}^{\sigma}(\mathrm{GL}_{N,E}))$ to $\mathrm{SSC}_{\omega_E}^{\sigma}(\mathrm{GL}_{N,E})$. Thus, by this observation and the above calculation of cardinalities, we can conclude that the map BC induces a bijection from $\mathrm{SSC}_{\omega_F}(\mathrm{GL}_{N,F})$ to $\mathrm{SSC}_{\omega_E}^{\sigma}(\mathrm{GL}_{N,E})$. This implies our assertion. $\square$

§ 2.5. Character formula for supercuspidal representations

We first recall the following character formula for supercuspidal representations obtained by the compact induction:

Theorem 2.7 (character formula, [Sal88]). *Let K be an open subgroup of $\mathrm{GL}_N(F)$ such that K contains Z_F and K/Z_F is compact. Let ρ be a finite-dimensional irreducible representation of K . We assume that the representation $\pi := \mathrm{c}\text{-Ind}_K^{\mathrm{GL}_N(F)} \rho$ is irreducible, hence supercuspidal. Then, for every regular semisimple element $g \in \mathrm{GL}_N(F)$, we have*

$$\Theta_{\pi}(g) = \sum_{y \in K \backslash \mathrm{GL}_N(F)/K} \sum_{\substack{x \in KyK \\ xgx^{-1} \in K}} \mathrm{tr} \rho(xgx^{-1}).$$

In particular, we have

$$\Theta_{\pi}(g) = \sum_{\substack{x \in K \backslash \mathrm{GL}_N(F) \\ xgx^{-1} \in K}} \mathrm{tr} \rho(xgx^{-1}),$$

provided that the sum is finite.

We next recall the twisted version of the previous character formula:

Theorem 2.8 (twisted character formula, [LH17, Partie 1, Théorème 6.2.1]). *Let K be a σ -stable open subgroup of $\mathrm{GL}_N(F)$ such that K contains Z_F and K/Z_F is compact. Let ρ be a σ -stable finite-dimensional irreducible representation of K , and we fix an isomorphism $I_\sigma: \rho \cong \rho^\sigma$. We assume that the representation $\pi := \mathrm{c}\text{-Ind}_K^{\mathrm{GL}_N(F)} \rho$ is irreducible, hence supercuspidal. Then, for every σ -regular σ -semisimple element $g \in \mathrm{GL}_N(F)$, we have*

$$\Theta_{\pi,\sigma}(g) = \sum_{y \in K \backslash \mathrm{GL}_N(F)/K} \sum_{\substack{x \in KyK \\ xg\sigma(x)^{-1} \in K}} \mathrm{tr}(\rho(xg\sigma(x)^{-1}) \circ I_\sigma).$$

In particular, we have

$$\Theta_{\pi,\sigma}(g) = \sum_{\substack{x \in K \backslash \mathrm{GL}_N(F) \\ xg\sigma(x)^{-1} \in K}} \mathrm{tr}(\rho(xg\sigma(x)^{-1}) \circ I_\sigma),$$

provided that the sum is finite. Here the left-hand side is the σ -twisted character of π normalized with respect to $\mathrm{c}\text{-Ind } I_\sigma: \pi \cong \pi^\sigma$.

We note that, for σ -stable simple supercuspidal representations, the Whittaker normalization of the σ -twisted characters coincides with the normalization compactly induced from the identity map $\mathrm{id}: \chi_{\omega,a,\zeta} \rightarrow \chi_{\omega,a,\zeta}^\sigma$ (see, e.g., [Oi16, Proposition 5.1]).

§ 3. Main theorem

§ 3.1. Computation of Shintani character relation and the unramified base change of simple supercuspidal representations of GL_N

We take $u \in k_F^\times$ and $z \in k_E^\times \subset Z_E$ (here Z_E is the center of $\mathrm{GL}_N(E)$). We assume that $\mathrm{Tr}(z) = 1$ (note that we can always take such an element from the surjectivity of the trace map on the residue fields). We consider an element $1 + z\varphi_u$ of $\mathrm{GL}_N(E)$. Then the element

$$\prod_{i=0}^{l-1} \sigma^i(1 + z\varphi_u) = \prod_{i=0}^{l-1} (1 + \sigma^i(z)\varphi_u)$$

is σ -invariant, hence belongs to $\mathrm{GL}_N(F)$. Namely, this element represents the $\mathrm{GL}_N(F)$ -conjugacy class $\mathcal{N}(1 + z\varphi_u)$. By abuse of notation, we denote this element again by $\mathcal{N}(1 + z\varphi_u)$.

Lemma 3.1. *The element $\mathcal{N}(1 + z\varphi_u) \in \mathrm{GL}_N(F)$ is a regular semisimple element belonging to I_F^+ . Moreover, the image of $\mathcal{N}(1 + z\varphi_u) \in \mathrm{GL}_N(F)$ in $I_F^+/I_F^{++} \cong k_F^{\oplus N}$ is given by $(1, \dots, 1, u)$.*

Proof. For every i , $1 + \sigma^i(z)\varphi_u$ belongs to I_E^+ and its image in $I_E^+/I_E^{++} \cong k_E^{\oplus N}$ is given by $(\sigma^i(z), \dots, \sigma^i(z), \sigma^i(z)u)$. Thus $\mathcal{N}(1 + z\varphi_u)$ belongs to $I_E^+ \cap \mathrm{GL}_N(F) = I_F^+$ and its image in $I_F^+/I_F^{++} \cong k_F^{\oplus N}$ is given by

$$\sum_{i=0}^{l-1} (\sigma^i(z), \dots, \sigma^i(z), \sigma^i(z)u) = (\mathrm{Tr}(z), \dots, \mathrm{Tr}(z), \mathrm{Tr}(z)u) = (1, \dots, 1, u).$$

Moreover, since u is not zero, the characteristic polynomial of $\mathcal{N}(1 + z\varphi_u) - I_N$ is an Eisenstein polynomial with F -coefficients, hence irreducible over F . Thus $\mathcal{N}(1 + z\varphi_u)$ is regular semisimple. $\square$

Lemma 3.2. *Let g be an element of $\mathrm{GL}_N(F)$ satisfying $g\mathcal{N}(1 + z\varphi_u)g^{-1} \in Z_F I_F^+ \langle \varphi_{a-1} \rangle$. Then g belongs to $Z_F I_F \langle \varphi_{a-1} \rangle$.*

Proof. This follows from that every entry of the image of $\mathcal{N}(1 + z\varphi_u)$ in I_F^+/I_F^{++} does not vanish (a consequence of Lemma 3.1). See, e.g., Lemma 3.5 in [Oi16]. $\square$

Before we start to compute the characters of simple supercuspidal representations, we define some notations. We set $Z_{k_F} := \{\mathrm{diag}(z, \dots, z) \mid z \in k_F^\times\}$, $T_{k_F} := \{\mathrm{diag}(t_1, \dots, t_N) \mid t_i \in k_F^\times\}$, and, for $x \in k_F^\times$, define the *Kloosterman sum* by

$$\mathrm{Kl}_x^N(\psi) := \sum_{\substack{x_1, \dots, x_N \in k_F^\times \\ x_1 \cdots x_N = x}} \psi(x_1 + \cdots + x_N).$$

Proposition 3.3. *Let $X = (\omega, a, \zeta) \in \mathrm{SSC}(\mathrm{GL}_{N,F})$. Then we have*

$$\Theta_{\pi_X}(\mathcal{N}(1 + z\varphi_u)) = \mathrm{Kl}_{au}^N(\psi).$$

Proof. By Lemma 3.2, the index set of the character formula (Theorem 2.7) is given by $Z_F I_F^+ \langle \varphi_{a-1} \rangle \backslash Z_F I_F \langle \varphi_{a-1} \rangle \cong Z_{k_F} \backslash T_{k_F}$. Since the set $Z_{k_F} \backslash T_{k_F}$ is bijective to the set $\{\mathrm{diag}(s_1, \dots, s_N) \mid s_i \in k_F^\times, s_1 \cdots s_N = 1\}$ via the map $\mathrm{diag}(t_1, \dots, t_N) \mapsto \mathrm{diag}(\frac{t_1}{t_2}, \dots, \frac{t_{N-1}}{t_N}, \frac{t_N}{t_1})$, by the character formula, we have

$$\Theta_{\pi_X}(\mathcal{N}(1 + z\varphi_u)) = \sum_{t \in Z_{k_F} \backslash T_{k_F}} \psi\left(\frac{t_1}{t_2} + \cdots + \frac{t_{N-1}}{t_N} + au \frac{t_N}{t_1}\right) = \mathrm{Kl}_{au}^N(\psi).$$

$\square$

Proposition 3.4. *Let $X' = (\omega', a', \zeta') \in \mathrm{SSC}(\mathrm{GL}_{N,E})$ be σ -stable data as described in Lemma 2.1. That is, ω' is σ -stable and a' belongs to $k_F^\times$. Then we have*

$$\Theta_{\pi_{X'}, \sigma}(1 + z\varphi_u) = \mathrm{Kl}_{a'u}^N(\psi).$$

Proof. We compute the index set of the twisted character formula (Theorem 2.8). Let g be an element of $\mathrm{GL}_N(E)$ satisfying $g(1 + z\varphi_u)\sigma(g)^{-1} \in Z_E I_E^+ \langle \varphi_{a'^{-1}} \rangle$. Then we have

$$g\mathcal{N}(1 + z\varphi_u)g^{-1} = \prod_{i=0}^{l-1} \sigma^i(g(1 + z\varphi_u)\sigma(g)^{-1}) \in Z_E I_E^+ \langle \varphi_{a'^{-1}} \rangle.$$

Thus, by Lemma 3.2 for $\mathrm{GL}_N(E)$, g belongs to $Z_E I_E \langle \varphi_{a'^{-1}} \rangle$. Therefore the index set of the twisted character formula can be represented by a subset of $Z_{k_E} \backslash T_{k_E}$.

Let $t = \mathrm{diag}(t_1, \dots, t_N)$ be an element of T_{k_E} . Let us consider the condition of t so that $t(1 + z\varphi_u)\sigma(t)^{-1}$ belongs to $Z_E I_E^+ \langle \varphi_{a'^{-1}} \rangle$. As we have

$$t(1 + z\varphi_u)\sigma(t)^{-1} = t(1 + z\varphi_u)t^{-1} \cdot t\sigma(t)^{-1}$$

and $t(1 + z\varphi_u)t^{-1}$ belongs to $Z_E I_E^+ \langle \varphi_{a'^{-1}} \rangle$, we have

$$t(1 + z\varphi_u)\sigma(t)^{-1} \in Z_E I_E^+ \langle \varphi_{a'^{-1}} \rangle \iff t\sigma(t)^{-1} \in Z_E I_E^+ \langle \varphi_{a'^{-1}} \rangle.$$

The latter condition is equivalent to the condition that $t_1\sigma(t_1)^{-1} = \dots = t_N\sigma(t_N)^{-1}$.

Since we have an isomorphism

$$\begin{aligned} Z_{k_F} \backslash T_{k_F} &= Z_{k_F} \backslash \{t \in T_{k_E} \mid t_1\sigma(t_1)^{-1} = \dots = t_N\sigma(t_N)^{-1} = 1\} \\ &\xrightarrow{\cong} Z_{k_E} \backslash \{t \in T_{k_E} \mid t_1\sigma(t_1)^{-1} = \dots = t_N\sigma(t_N)^{-1}\}, \end{aligned}$$

by using the twisted character formula, we get

$$\begin{aligned} \Theta_{\pi_{X'}, \sigma}(1 + z\varphi_u) &= \sum_{t \in Z_{k_F} \backslash T_{k_F}} \psi \circ \mathrm{Tr} \left(z \frac{t_1}{t_2} + \dots + z \frac{t_{N-1}}{t_N} + a'zu \frac{t_N}{t_1} \right) \\ &= \sum_{t \in Z_{k_F} \backslash T_{k_F}} \psi \left(\mathrm{Tr}(z) \frac{t_1}{t_2} + \dots + \mathrm{Tr}(z) \frac{t_{N-1}}{t_N} + \mathrm{Tr}(z) a'u \frac{t_N}{t_1} \right) \\ &= \mathrm{Kl}_{a'u}^N(\psi). \end{aligned}$$

Recall that we took z to satisfy $\mathrm{Tr}(z) = 1$ and that a' belongs to k_F . □

Corollary 3.5. *Let $X = (\omega, a, \zeta) \in \mathrm{SSC}(\mathrm{GL}_{N,F})$ and $X' = (\omega', a', \zeta') \in \mathrm{SSC}(\mathrm{GL}_{N,E})$ and we assume that $\mathrm{BC}(\pi_X) = \pi_{X'}$. Then we have $a' = a$.*

Proof. By applying the Shintani character relation (Definition 2.4) for elements $1 + z\varphi_u$ and $\mathcal{N}(1 + z\varphi_u)$, we get an equality $\Theta_{\pi_{X'}, \sigma}(1 + z\varphi_u) = \Theta_{\pi_X}(\mathcal{N}(1 + z\varphi_u))$. By Propositions 3.3 and 3.4, we have $\mathrm{Kl}_{a'u}^N(\psi) = \mathrm{Kl}_{au}^N(\psi)$. Since this equality holds for every $u \in k_F^\times$, we get $a' = a$ (see, e.g., Lemma 3.4 in [IT18]). □

Let $X = (\omega, a, \zeta) \in \mathrm{SSC}(\mathrm{GL}_{N,F})$ and $X' = (\omega', a', \zeta') \in \mathrm{SSC}(\mathrm{GL}_{N,E})$ and we assume that $\mathrm{BC}(\pi_X) = \pi_{X'}$. Then, by Corollary 3.5 and the proof of Corollary 2.6, we

have $a' = a$ and $\omega' = \omega \circ \text{Nr}$. We finally compare the third parameters of X and X' . From now on, we assume that

$$(p, N) = 1.$$

To determine the relationship between ζ and ζ' , we may assume that $a = a' = 1$ by replacing the fixed uniformizer ϖ . We first consider the case where l is a prime number. For $u \in k_F^\times$ and $z \in k_E^\times$ satisfying $\text{Tr}(z) = 1$, we consider the following element of $\text{GL}_N(E)$:

$$\varphi_u(1 + z\varphi_u) \in \text{GL}_N(E).$$

Then we have

$$\prod_{i=0}^{l-1} \sigma^i(\varphi_u(1 + z\varphi_u)) = \varphi_u^l \prod_{i=0}^{l-1} (1 + \sigma^i(z)\varphi_u) = \varphi_u^l \mathcal{N}(1 + z\varphi_u),$$

and this element belongs to $\text{GL}_N(F)$.

Lemma 3.6. *The element $\varphi_u^l \mathcal{N}(1 + z\varphi_u)$ is regular semisimple.*

Proof. The N -th power of $\varphi_u^l \mathcal{N}(1 + z\varphi_u)$ is given by $(\varpi u)^l \mathcal{N}(1 + z\varphi_u)^N$. The characteristic polynomial of $\mathcal{N}(1 + z\varphi_u)^N$ is irreducible by the same argument as in the proof of Lemma 3.1 (note that here we use the assumption that $(p, N) = 1$). Thus $\mathcal{N}(1 + z\varphi_u)^N$ is regular semisimple, and so is $(\varpi u)^l \mathcal{N}(1 + z\varphi_u)^N$. Since its N -th power is regular semisimple, $\varphi_u^l \mathcal{N}(1 + z\varphi_u)$ is itself regular semisimple. $\square$

Lemma 3.7. *Let g be an element of $\text{GL}_N(F)$ satisfying $g\varphi_u^l \mathcal{N}(1 + z\varphi_u)g^{-1} \in Z_F I_F^+ \langle \varphi_1 \rangle$. Then g belongs to $Z_F I_F \langle \varphi_1 \rangle$.*

Proof. By taking the N -th power of the condition in the assertion, we can conclude that $g\mathcal{N}(1 + z\varphi_u)^N g^{-1} \in Z_F I_F^+ \langle \varphi_1 \rangle$. Then, by the same argument as in the proof of Lemma 3.2 (note that here we use the assumption that $(p, N) = 1$), we get the assertion. $\square$

Let l_0 be the integer which satisfies $0 < l_0 \leq N$ and is congruent to l modulo N . Namely, for some non-negative integer α , we have $l = \alpha N + l_0$.

Proposition 3.8. *We have*

$$\Theta_{\pi_X}(\varphi_u^l \mathcal{N}(1 + z\varphi_u)) = \zeta^l \cdot \begin{cases} \sum_{\xi \in \mu_{N,u}(k_F)} \omega(\xi)^l \psi_N(\xi) & \text{if } l \text{ does not divide } N, \\ \sum_{\xi \in \mu_{m,u}(k_F)} \omega(\xi) \text{Kl}_\xi^l(\psi_m) & \text{if } l \text{ divides } N \text{ } (N = ml). \end{cases}$$

Here $\mu_{N,u}(k_F)$ is the set of N -th roots of u in k_F , and $\psi_m := \psi \circ m$ is the composition of ψ and the multiplication by m .

Proof. By Lemma 3.7, the index set of the character formula (Theorem 2.7) is represented by a subset of $Z_{k_F} \backslash T_{k_F}$. Let us determine this subset. Let $t = \text{diag}(t_1, \dots, t_N)$ be an element of T_{k_F} . Then $t\mathcal{N}(1 + z\varphi_u)t^{-1}$ always belongs to $Z_F I_F^+ \langle \varphi_1 \rangle$. Thus we have $t\varphi_u^l \mathcal{N}(1 + z\varphi_u)t^{-1} \in Z_F I_F^+ \langle \varphi_1 \rangle$ if and only if we have $t\varphi_u^l t^{-1} \in Z_F I_F^+ \langle \varphi_1 \rangle$. By comparing the valuation of the determinant, the latter condition is equivalent to the condition that $\varphi_1^{-l} t \varphi_u^l t^{-1} \in Z_{k_F} I_F^+$. As we have

$$\begin{aligned} \varphi_1^{-l} t \varphi_u^l t^{-1} &= u^\alpha \varphi_1^{-l_0} t \varphi_u^{l_0} t^{-1} \\ &= u^\alpha \text{diag}\left(u \frac{t_{N-l_0+1}}{t_1}, \dots, u \frac{t_N}{t_{l_0}}, \frac{t_1}{t_{l_0+1}}, \frac{t_2}{t_{l_0+2}}, \dots, \frac{t_{N-l_0}}{t_N}\right), \end{aligned}$$

the condition that $\varphi_1^{-l} t \varphi_u^l t^{-1} \in Z_{k_F} I_F^+$ is equivalent to the condition that

$$(*) \quad u^{\alpha+1} \frac{t_{N-l_0+1}}{t_1} = \dots = u^{\alpha+1} \frac{t_N}{t_{l_0}} = u^\alpha \frac{t_1}{t_{l_0+1}} = \dots = u^\alpha \frac{t_{N-l_0}}{t_N}.$$

We denote this index set $\{t \in Z_{k_F} \backslash T_{k_F} \mid t \text{ satisfies } (*)\}$ by T' .

We first consider the case where l does not divide N . In this case, if we put

$$\xi_t := u^{\alpha+1} \frac{t_{N-l_0+1}}{t_1} = \dots = u^{\alpha+1} \frac{t_N}{t_{l_0}} = u^\alpha \frac{t_1}{t_{l_0+1}} = \dots = u^\alpha \frac{t_{N-l_0}}{t_N},$$

then we have

$$\xi_t^N = u^{\alpha+1} \frac{t_{N-l_0+1}}{t_1} \dots u^{\alpha+1} \frac{t_N}{t_{l_0}} \cdot u^\alpha \frac{t_1}{t_{l_0+1}} \dots u^\alpha \frac{t_{N-l_0}}{t_N} = u^{\alpha N + l_0} = u^l.$$

Thus ξ_t belongs to $\mu_{N,u^l}(k_F)$. Conversely, for every element $\xi \in \mu_{N,u^l}(k_F)$, there exists a unique $t \in T'$ satisfying $\xi_t = \xi$. On the other hand, by the condition $(*)$, we have $u^\alpha \frac{t_1}{t_{l_0+1}} = u^\alpha \frac{t_2}{t_{l_0+2}}$. Thus we get $\frac{t_1}{t_2} = \frac{t_{l_0+1}}{t_{l_0+2}}$. By noting that l_0 is coprime to N , we finally get $\frac{t_1}{t_2} = \dots = \frac{t_{N-1}}{t_N} = u \frac{t_N}{t_1}$. We denote these elements by γ_t . Then we have $\gamma_t^N = u$ and

$$\gamma_t^l = \gamma_t^{l_0} \cdot u^\alpha = \frac{t_1}{t_2} \dots \frac{t_{l_0}}{t_{l_0+1}} \cdot u^\alpha = \xi_t.$$

If we let M be the smallest non-negative number satisfying $Ml_0 - 1 = rN$ for some integer r , then we have a bijection between the two sets $\mu_{N,u^l}(k_F)$ and $\mu_{N,u}(k_F)$ as follows:

$$\begin{aligned} \mu_{N,u^l}(k_F) &\rightarrow \mu_{N,u}(k_F); & \xi &\mapsto \xi^M u^{-\alpha M - r}, \\ \mu_{N,u^l}(k_F) &\leftarrow \mu_{N,u}(k_F); & \gamma^l &\leftarrow \gamma. \end{aligned}$$

Hence we can compute the character as follows:

$$\begin{aligned}
\Theta_{\pi_X}(\varphi_u^l \mathcal{N}(1 + z\varphi_u)) &= \sum_{t \in T'} \chi_X(t\varphi_u^l \mathcal{N}(1 + z\varphi_u)t^{-1}) \\
&= \sum_{t \in T'} \chi_X(\varphi_1^l) \cdot \chi_X(\varphi_1^{-l} t\varphi_u^l t^{-1}) \cdot \chi_X(t\mathcal{N}(1 + z\varphi_u)t^{-1}) \\
&= \zeta^l \sum_{t \in T'} \omega(\xi_t) \psi\left(\frac{t_1}{t_2} + \cdots + \frac{t_{N-1}}{t_N} + u \frac{t_N}{t_1}\right) \\
&= \zeta^l \sum_{t \in T'} \omega(\xi_t) \psi(N\gamma_t) = \zeta^l \sum_{\gamma \in \mu_{N,u}(k_F)} \omega(\gamma)^l \psi_N(\gamma).
\end{aligned}$$

We next consider the case where l divides N . As in the assertion, we put $N = lm$. In this case, we have $\alpha = 0$ and the condition $(*)$ is equivalent to

$$\begin{aligned}
\frac{t_1}{t_{l+1}} &= \frac{t_{l+1}}{t_{2l+1}} = \cdots = \frac{t_{(m-2)l+1}}{t_{(m-1)l+1}} = u \frac{t_{(m-1)l+1}}{t_1} \\
&= \frac{t_2}{t_{l+2}} = \frac{t_{l+2}}{t_{2l+2}} = \cdots = \frac{t_{(m-2)l+2}}{t_{(m-1)l+2}} = u \frac{t_{(m-1)l+2}}{t_2} \\
&= \cdots \\
&= \frac{t_l}{t_{2l}} = \frac{t_{2l}}{t_{3l}} = \cdots = \frac{t_{(m-1)l}}{t_{ml}} = u \frac{t_{ml}}{t_l}.
\end{aligned}$$

If we put

$$\xi_t := \frac{t_1}{t_{l+1}} = \cdots = u \frac{t_{ml}}{t_l},$$

then we have

$$\xi_t^m = \frac{t_1}{t_{l+1}} \cdot \frac{t_{l+1}}{t_{2l+1}} \cdots \frac{t_{(m-2)l+1}}{t_{(m-1)l+1}} \cdot u \frac{t_{(m-1)l+1}}{t_1} = u.$$

Thus we get a bijection

$$\begin{aligned}
Z_{k_F} \setminus \{t \in T_{k_F} \mid t \text{ satisfies } (*)\} &\xrightarrow{\cong} \mu_{m,u}(k_F) \times \{(s_1, \dots, s_l) \in (k_F^\times)^l \mid s_1 \cdots s_l = 1\} \\
\text{diag}(t_1, \dots, t_N) &\mapsto \left(\xi_t, \left(\frac{t_1}{t_2}, \dots, \frac{t_{l-1}}{t_l}, \frac{t_l}{t_1} \right) \right).
\end{aligned}$$

Thus, by the character formula, we have

$$\begin{aligned}
\Theta_{\pi_X}(\varphi_u^l \mathcal{N}(1 + z\varphi_u)) &= \sum_{t \in T'} \chi_X(\varphi_1^l) \cdot \chi_X(\varphi_1^{-l} t\varphi_u^l t^{-1}) \cdot \chi_X(t\mathcal{N}(1 + z\varphi_u)t^{-1}) \\
&= \zeta^l \sum_{t \in T'} \omega(\xi_t) \psi\left(\frac{t_1}{t_2} + \cdots + \frac{t_{N-1}}{t_N} + u \frac{t_N}{t_1}\right) \\
&= \zeta^l \sum_{t \in T'} \omega(\xi_t) \psi\left(m \left(\frac{t_1}{t_2} + \cdots + \frac{t_{l-1}}{t_l} + \frac{t_l}{t_1} \xi_t \right)\right) \\
&= \zeta^l \sum_{\xi \in \mu_{m,u}(k_F)} \omega(\xi) \text{Kl}_\xi^l(\psi_m).
\end{aligned}$$

□

Proposition 3.9. *We have*

$$\Theta_{\pi_{X'}, \sigma}(\varphi_u(1+z\varphi_u)) = \begin{cases} \zeta' \sum_{\xi \in \mu_{N,u}(k_F)} \omega^l(\xi) \psi_N(\xi) & \text{if } l \nmid N, \\ \zeta' \sum_{\xi \in \mu_{m,u}(k_F)} \sum_{\substack{x \in k_E^\times \\ \text{Nr}(x) = \xi}} \omega'(x) \psi_m \circ \text{Tr}(x) & \text{if } l \mid N \ (N = ml). \end{cases}$$

Proof. We compute the index set of the twisted character formula (Theorem 2.8). First, by the same argument as in the proof of Proposition 3.4 and Proposition 3.8, the index set can be represented by a subset of $T' := Z_{k_E} \setminus \{t = \text{diag}(t_1, \dots, t_N) \in T_{k_E} \mid (*)\}$. For an element $t = \text{diag}(t_1, \dots, t_N)$ of T_{k_E} , let us consider the condition of t to satisfy $(*)$ and that $t\varphi_u(1+z\varphi_u)\sigma(t)^{-1} \in Z_E I_E^+ \langle \varphi_1 \rangle$. Again by the same argument as in the proof of Proposition 3.4, the latter condition is equivalent to $\varphi_1^{-1} t \varphi_u \sigma(t)^{-1} \in Z_{k_E} I_E^+$. Since we have

$$\varphi_1^{-1} t \varphi_u \sigma(t)^{-1} = \text{diag}\left(u \frac{t_N}{\sigma(t_1)}, \frac{t_1}{\sigma(t_2)}, \dots, \frac{t_{N-1}}{\sigma(t_N)}\right),$$

the condition that $\varphi_1^{-1} t \varphi_u \sigma(t)^{-1} \in Z_{k_E} I_E^+$ is equivalent to the condition that

$$(*) \quad \frac{t_1}{\sigma(t_2)} = \dots = \frac{t_{N-1}}{\sigma(t_N)} = u \frac{t_N}{\sigma(t_1)} =: x_t.$$

Here we note that, as we have

$$\text{Nr}(x_t) = x_t \sigma(x_t) \cdots \sigma^{l-1}(x_t) = u^\alpha \frac{t_1}{t_{l_0+1}} = u^\alpha \frac{t_2}{t_{l_0+2}} = \dots = u^{\alpha+1} \frac{t_N}{t_{l_0}} (= \xi_t),$$

the condition $(*)$ follows from the condition $(\star)$.

We first consider the case where l does not divide N . In this case, as we see in the proof of Proposition 3.8, we can take an element $\gamma_t \in \mu_{N,u}(k_F)$ satisfying $\gamma_t^l = \xi_t$ uniquely. If we assume $t_1 = 1$, then, by the condition $(\star)$, we have $x_t \sigma(x_t) \cdots \sigma^{l-1}(x_t) = \xi_t$ and $x_t \sigma(x_t) \cdots \sigma^{N-1}(x_t) = u$. Thus, if we put $x_t = x_0 \gamma_t$, then we have $x_0 \cdots \sigma^{l-1}(x_0) = 1$ and $x_0 \cdots \sigma^{N-1}(x_0) = 1$. Hence, by noting that $(l, N) = 1$, we get $x_0 = 1$. Therefore the index set of the twisted character formula is represented by

$$T'' := \{\text{diag}(1, \zeta^{-1}, \zeta^{-2}, \dots, \zeta^{-(N-1)}) \mid \zeta \in \mu_{N,u}(k_F)\}.$$

By using the twisted character formula, we get

$$\begin{aligned}
\Theta_{\pi_{X'}, \sigma}(\varphi_u(1 + z\varphi_u)) &= \sum_{t \in T''} \chi_{X'}(t\varphi_u(1 + z\varphi_u)\sigma(t)^{-1}) \\
&= \sum_{t \in T''} \chi_{X'}(\varphi_1) \cdot \chi_{X'}(\varphi_1^{-1}t\varphi_u\sigma(t)^{-1}) \cdot \chi_{X'}(\sigma(t)(1 + z\varphi_u)\sigma(t)^{-1}) \\
&= \zeta' \sum_{t \in T''} \omega'(x_t)\psi \circ \text{Tr}\left(z \frac{\sigma(t_1)}{\sigma(t_2)} + \cdots + z \frac{\sigma(t_{N-1})}{\sigma(t_N)} + zu \frac{\sigma(t_N)}{\sigma(t_1)}\right) \\
&= \zeta' \sum_{\gamma \in \mu_{N,u}(k_F)} \omega'(\gamma)\psi \circ \text{Tr}(Nz\gamma) \\
&= \zeta' \sum_{\gamma \in \mu_{N,u}(k_F)} \omega^l(\gamma)\psi_N(\gamma).
\end{aligned}$$

Note that we have $\omega'(\gamma) = \omega \circ \text{Nr}(\gamma) = \omega(\gamma^l)$.

We next consider the case where l divides N . Recall we put $N = lm$. In this case, for every $x \in k_E^\times$ satisfying $\text{Nr}(x) = \xi_t$, if we define an element $t \in T$ by $t_N := u^{-1}x$, $t_{N-1} := u^{-1}x\sigma(x)$, $\dots$, $t_2 := u^{-1}x\sigma(x) \cdots \sigma^{N-2}(x)$, $t_1 := u^{-1}x\sigma(x) \cdots \sigma^{N-1}(x) = 1$, then t satisfies $(\star)$ and $x_t = x$. Namely, the index set of the twisted character formula is represented by $T'' := \{\text{diag}(t_1, \dots, t_N) \mid \xi \in \mu_{m,u}(k_F), x \in k_E^\times, \text{Nr}(x) = \xi, t_i: \text{ as above}\}$. Thus we get

$$\begin{aligned}
\Theta_{\pi_{X'}, \sigma}(\varphi_u(1 + z\varphi_u)) &= \zeta' \sum_{t \in T''} \omega'(x_t)\psi \circ \text{Tr}\left(z \frac{\sigma(t_1)}{\sigma(t_2)} + \cdots + z \frac{\sigma(t_{N-1})}{\sigma(t_N)} + zu \frac{\sigma(t_N)}{\sigma(t_1)}\right) \\
&= \zeta' \sum_{\xi \in \mu_{m,u}(k_F)} \sum_{\substack{x \in k_E^\times \\ \text{Nr}(x) = \xi}} \omega'(x)\psi \circ \text{Tr}(z\sigma^N(x) + \cdots + z\sigma^2(x) + z\sigma(x)) \\
&= \zeta' \sum_{\xi \in \mu_{m,u}(k_F)} \sum_{\substack{x \in k_E^\times \\ \text{Nr}(x) = \xi}} \omega'(x)\psi \circ \text{Tr}(zm \text{Tr}(x)) \\
&= \zeta' \sum_{\xi \in \mu_{m,u}(k_F)} \sum_{\substack{x \in k_E^\times \\ \text{Nr}(x) = \xi}} \omega'(x)\psi_m \circ \text{Tr}(x).
\end{aligned}$$

□

Corollary 3.10. *The sign ζ' is equal to $-\zeta^l$ if $l = 2$ and N is even, and ζ^l otherwise.*

Proof. By applying the Shintani character relation (Definition 2.4) for elements $\varphi_u(1 + z\varphi_u)$ and $\varphi_u^l \mathcal{N}(1 + z\varphi_u)$, we get an equality

$$\Theta_{\pi_{X'}, \sigma}(\varphi_u(1 + z\varphi_u)) = \Theta_{\pi_X}(\varphi_u^l \mathcal{N}(1 + z\varphi_u)).$$

We consider the sum of this equality over $u \in k_F^\times$. Then, by Proposition 3.9, the left-hand side becomes

$$\begin{cases} \zeta' \sum_{\xi \in k_F^\times} \omega^l(\xi) \psi_N(\xi) = \zeta' G(k_F; \omega^l, \psi_N) & \text{if } l \text{ does not divide } N, \\ \zeta' \sum_{x \in k_E^\times} \omega'(x) \psi_m \circ \text{Tr}(x) = \zeta' G(k_E; \omega', \psi_m \circ \text{Tr}) & \text{if } l \text{ divides } N. \end{cases}$$

On the other hand, by Proposition 3.8, the right-hand side becomes

$$\begin{cases} \zeta^l G(k_F; \omega^l, \psi_N) & \text{if } l \text{ does not divide } N, \\ \zeta^l G(k_F; \omega, \psi_m)^l & \text{if } l \text{ divides } N. \end{cases}$$

By the Hasse–Davenport relation $G(k_E; \omega \circ \text{Nr}, \psi \circ \text{Tr}) = (-1)^{l-1} G(k_F; \omega, \psi)^l$ (see, e.g., [Wei49, 503 page]) and the non-vanishing of the Gauss sum, we get the assertion. $\square$

Finally, we treat the general case where l is a positive integer. By considering the prime factorization of l , we get the following:

Theorem 3.11. *We assume that $(p, N) = 1$. The unramified base change for simple supercuspidal representations of GL_N is described as follows:*

$$\text{SSC}(\text{GL}_{N,F}) \rightarrow \text{SSC}(\text{GL}_{N,E}); \quad (\omega, a, \zeta) \mapsto (\omega \circ \text{Nr}, a, \delta_{l,N} \zeta^l),$$

where $\delta_{l,N} = -1$ if both of l and N are even integers, and $\delta_{l,N} = 1$ otherwise.

§ 3.2. Unramified base change of simple supercuspidal representations of quasi-split classical groups

Let $\mathbf{G}$ be a quasi-split special orthogonal or symplectic group over F .

Theorem 3.12. *We assume that $(p, 2n) = 1$ and use the same notations as in Theorems 2.2 and 3.11. Then the unramified base change $\Pi_\phi^{\mathbf{G}_E}$ with respect to E/F of $\Pi_\phi^{\mathbf{G}}$ is a simple supercuspidal L -packet of $\mathbf{G}_E$, and described as follows:*

The case where $\mathbf{G} = \text{SO}_{2n+1}$: *Let $\Pi_\phi^{\mathbf{G}}$ be $\{\pi_{a,\zeta}^{\mathbf{G}}\}$, where $(a, \zeta) \in \text{SSC}(\text{SO}_{2n+1,F})$. Then we have $\Pi_\phi^{\mathbf{G}_E} = \{\pi_{a,\delta_{l,2n}\zeta^l}^{\mathbf{G}_E}\}$.*

The case where $\mathbf{G} = \text{Sp}_{2n}$: *Let $\Pi_\phi^{\mathbf{G}}$ be $\{\pi_{X_0}^{\mathbf{G}}, \pi_{X_1}^{\mathbf{G}}\}$, where $X_\kappa = (\xi, \kappa, a\epsilon_F^{-\kappa}) \in \text{SSC}(\text{Sp}_{2n,F})$. Then we have $\Pi_\phi^{\mathbf{G}_E} = \{\pi_{X_{E,0}}^{\mathbf{G}_E}, \pi_{X_{E,1}}^{\mathbf{G}_E}\}$, where $X_{E,\kappa} = (\xi^l, \kappa, a\epsilon_E^{-\kappa})$.*

The case where $\mathbf{G} = \text{SO}_{2n}^\mu$: *Let $\Pi_\phi^{\mathbf{G}}$ be $\{\tilde{\pi}_{\xi,\pm a}^{\mathbf{G}}\}$, where $\tilde{\pi}_{\xi,\pm a}^{\mathbf{G}} = \{\pi_{\xi,a}^{\mathbf{G}}, \pi_{\xi,-a}^{\mathbf{G}}\}$ for some $(\xi, a) \in \text{SSC}(\text{SO}_{2n,F}^\mu)$. Then we have $\Pi_\phi^{\mathbf{G}_E} = \{\tilde{\pi}_{\xi^l,\pm a'}^{\mathbf{G}_E}\}$, where $\tilde{\pi}_{\xi^l,\pm a'}^{\mathbf{G}_E} = \{\pi_{\xi^l,a'}^{\mathbf{G}_E}, \pi_{\xi^l,-a'}^{\mathbf{G}_E}\}$. Here a' is an element of $k_E^\times$ satisfying $a'^2 = a^2 \epsilon_F^{-\nu_\mu} \epsilon_E^{\nu_\mu}$.*

The case where $\mathbf{G} = \mathrm{SO}_{2n+2}^{(\mathrm{ur})}$: Let $\Pi_\phi^{\mathbf{G}}$ be $\{\pi_{X_0}^{\mathbf{G}}, \pi_{X_1}^{\mathbf{G}}\}$, where $X_\kappa = (\xi, \kappa, a\epsilon_F^{-\kappa}, \zeta) \in \mathrm{SSC}(\mathrm{SO}_{2n+2, F}^{(\mathrm{ur})})$. Then we have $\Pi_\phi^{\mathbf{G}^E} = \{\pi_{X_{E,0}}^{\mathbf{G}^E}, \pi_{X_{E,1}}^{\mathbf{G}^E}\}$, where $X_{E,\kappa}$ is given by $(\xi^l, \kappa, 2^{t_{\mathbf{G}} - t_{\mathbf{G}^E}} a\epsilon_E^{-\kappa}, \zeta^l)$.

Proof. We check the assertions by a case-by-case computation.

The case where $\mathbf{G} = \mathrm{SO}_{2n+1}$: By Theorem 2.2, the endoscopic lift of the simple supercuspidal representation of $\mathrm{SO}_{2n+1, F}$ corresponding to (a, ζ) to $\mathrm{GL}_{2n, F}$ is again simple supercuspidal and corresponds to $(1, 2a, \zeta)$. By Theorem 3.11, its base change to $\mathrm{GL}_{2n, E}$ is given by $(1, 2a, \delta_{l, 2n}\zeta^l)$. Thus, again by Theorem 2.2, it is the endoscopic lift of a simple supercuspidal representation of $\mathrm{SO}_{2n+1, E}$ corresponding to $(a, \delta_{l, 2n}\zeta^l)$.

The case where $\mathbf{G} = \mathrm{Sp}_{2n}$: By Theorem 2.2, the endoscopic lift of the simple supercuspidal representation of $\mathrm{Sp}_{2n, F}$ corresponding to $(\xi, \kappa, a\epsilon_F^{-\kappa})$ to $\mathrm{GL}_{2n+1, F}$ is given by the parabolic induction of $\pi_{\omega_0, 4a, \zeta}^{\mathrm{GL}_{2n, F}} \boxtimes \omega_{\omega_0, 4a, \zeta}^{\mathrm{GL}_{2n, F}}$. By Theorem 3.11 and the compatibility of the base change and the parabolic induction, its base change to $\mathrm{GL}_{2n+1, E}$ is given by the parabolic induction of $\pi_{\omega_0 \circ \mathrm{Nr}, 4a, \delta_{l, 2n}\zeta^l}^{\mathrm{GL}_{2n, E}} \boxtimes \omega_{\omega_0 \circ \mathrm{Nr}, 4a, \delta_{l, 2n}\zeta^l}^{\mathrm{GL}_{2n, E}}$. By noting that the character $\omega_0 \circ \mathrm{Nr}$ is the non-trivial quadratic character of $k_E^\times$, again by Theorem 2.2, it is the endoscopic lift of $\pi_{\xi', \kappa, a\epsilon_E^{-\kappa}}^{\mathrm{Sp}_{2n, E}}$ for some $(\xi', \kappa, a\epsilon_E^{-\kappa}) \in \mathrm{SSC}(\mathrm{Sp}_{2n, E})$. We determine ξ' . By Theorem 2.2, we have

$$\delta_{l, 2n}\zeta^l = \xi' \cdot q^{-\frac{1}{2}} (\omega_0 \circ \mathrm{Nr})(-1)G(k_E; \omega_0 \circ \mathrm{Nr}, \psi \circ \mathrm{Tr}).$$

On the other hand, we have $\zeta = \xi \cdot q^{-\frac{1}{2}} \omega_0(-1)G(k_F; \omega_0, \psi)$. Thus, by the Hasse–Davenport relation, we get $\delta_{l, 2n}\xi^l(-1)^{l-1} = \xi'$. As we have $\delta_{l, 2n}(-1)^{l-1} = 1$, we get the assertion.

The case where $\mathbf{G} = \mathrm{SO}_{2n}^\mu$: By Theorem 2.2, the endoscopic lift of the simple supercuspidal representation of $\mathrm{SO}_{2n, F}^\mu$ corresponding to (ξ', a') to $\mathrm{GL}_{2n, F}$ is again simple supercuspidal and corresponds to $(\omega_0, 4a, \zeta)$. By Theorem 3.11, its base change to $\mathrm{GL}_{2n, E}$ is given by $(\omega_0 \circ \mathrm{Nr}, 4a, \delta_{l, 2n}\zeta^l)$. Thus, by a similar, but simpler computation as in the previous case, we get the assertion. Here we note that the parity of ν_μ does not change with respect to any unramified extension E/F .

The case where $\mathbf{G} = \mathrm{SO}_{2n+2}^{(\mathrm{ur})}$: By Theorem 2.2, the endoscopic lift of the simple supercuspidal representation of $\mathrm{SO}_{2n+2, F}^{(\mathrm{ur})}$ corresponding to $(\xi, \kappa, a\epsilon_F^{-\kappa}, \zeta)$ to $\mathrm{GL}_{2n+2, F}$ is given by the parabolic induction of

$$\pi_{\omega_0, b, \zeta\eta}^{\mathrm{GL}_{2n, F}} \boxtimes (\omega_{\omega_0, b, \zeta\eta}^{\mathrm{GL}_{2n, F}} \otimes \mu_{\mathbf{G}} \otimes \mu_{\mathrm{ur}}^\iota) \boxtimes \mu_{\mathrm{ur}}^\iota, \quad \text{where } \iota := \frac{1 - \zeta}{2}.$$

By Theorem 3.11 and the compatibility of the base change and the parabolic induction, its base change to $\mathrm{GL}_{2n+2,E}$ is given by the parabolic induction of

$$\pi_{\omega_0 \circ \mathrm{Nr}, b, \delta_{l, 2n} \zeta^l \eta^l}^{\mathrm{GL}_{2n, E}} \boxtimes (\omega_{\omega_0 \circ \mathrm{Nr}, b, \delta_{l, 2n} \zeta^l \eta^l}^{\mathrm{GL}_{2n, E}} \otimes \mu_{\mathbf{G}}^l \otimes \mu_{\mathrm{ur}}^l) \boxtimes \mu_{\mathrm{ur}}^l.$$

Note that the restrictions of quadratic characters $\mu_{\mathbf{G}}$ and μ_{ur} to W_E are trivial when l is even. Therefore, again by Theorem 2.2, it is the endoscopic lift of a simple supercuspidal L -packet of $\mathrm{SO}_{2n+2,E}^{(\mathrm{ur})}$ consisting of simple supercuspidal representations corresponding to some data $(\xi', \kappa, a' \epsilon_E^{-\kappa}, \zeta')$. Here ζ' is given by ζ^l . As we have $(-1)^{n-1} 2^{2+t_{\mathbf{G}_E}} a' = (-1)^{n-1} 2^{2+t_{\mathbf{G}}} a$, we get $a' = 2^{t_{\mathbf{G}} - t_{\mathbf{G}_E}} a$. On the other hand, by noting the third parameter of $\pi_{\omega_0 \circ \mathrm{Nr}, b, \delta_{l, 2n} \zeta^l \eta^l}^{\mathrm{GL}_{2n, E}}$, we get

$$\zeta'(-1)^{t_{\mathbf{G}_E}} q^{-\frac{1}{2}} G(k_E; \omega_0 \circ \mathrm{Nr}, \psi \circ \mathrm{Tr})(\omega_0 \circ \mathrm{Nr})(-1)^{\xi'} = \delta_{l, 2n} \zeta^l \eta^l.$$

As we have $\eta = (-1)^{t_{\mathbf{G}}} q^{-\frac{1}{2}} G(k_F; \omega_0, \psi) \omega_0(-1) \xi$, we get $\xi' = \xi^l$.

□

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