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Algebraic independence of the values of certain infinite products and their derivatives related to Fibonacci and Lucas numbers

東京理科大学理学部　黒沢　健 (Takeshi Kurosawa)
Faculty of Science, Tokyo University of Science
弘前大学大学院理工学研究科　立谷 洋平 (Yohei Tachiya)
Graduate School of Science and Technology, Hirosaki University
慶應義塾大学理工学部　田中 孝明 (Taka-aki Tanaka)
Faculty of Science and Technology, Keio University

1 Introduction

Some entire functions are known to have the property that their values as well as their successive derivatives at any distinct algebraic points other than their zeroes are algebraically independent. Let $f(z) = \sum_{k=1}^{\infty} \gamma^{k!} z^k$ and $g_d(z) = \sum_{k=1}^{\infty} \gamma^{d^k} z^k$, where γ is an algebraic number with $0 < |\gamma| < 1$ and d is an integer greater than 1. Nishioka [6, 8] showed, respectively, that the infinite set $\{f^{(l)}(\alpha) \mid l \in \mathbb{N}_0,\ \alpha \in \overline{\mathbb{Q}}^{\times}\}$ is algebraically independent and so is the infinite set $\{g_d^{(l)}(\alpha) \mid l \in \mathbb{N}_0,\ \alpha \in \overline{\mathbb{Q}}^{\times}\}$ for any fixed d, where $\mathbb{N}_0$ denotes the set of nonnegative integers and $F^{\times}$ denotes the multiplicative group of nonzero elements of any field F, and thus $\overline{\mathbb{Q}}^{\times}$ indicates the set of nonzero algebraic numbers.

Let $\{F_n\}_{n\geq 0}$ be the sequence of Fibonacci numbers defined by

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n \geq 0) \tag{1}$$

and define the function $G_d(z)$ by

$$G_d(z) = \prod_{k=0}^{\infty} \left(1 - \frac{z}{2^{d^k} F_{d^k}}\right) \qquad (d = 2, 3, 4, \ldots).$$

The authors [4] proved that the infinite set

$$\bigcup_{d=2}^{\infty} \left\{ G_d^{(l)}(\alpha) \;\middle|\; l \in \mathbb{N}_0,\ \alpha \in \overline{\mathbb{Q}}^{\times} \setminus \{2^{d^k} F_{d^k}\}_{k\geq 0} \right\}$$

is algebraically independent.

In contrast with the functions $f(z)$ and $g_d(z)$ above, the function $G_d(z)$ is interesting in view of the following two points:

- The algebraic independency of the values of the functions $g_d(z)$ above for varying d is open, namely it has not been known whether the infinite set

$$\bigcup_{d=2}^{\infty}\left\{ g_d^{(l)}(\alpha) \;\middle|\; l \in \mathbb{N}_0,\ \alpha \in \overline{\mathbb{Q}}^{\times} \right\}$$

  is algebraically independent.

- No information on the zeros of $f(z)$ and $g_d(z)$ above has been known so far and hence we cannot express them as the infinite products explicitly.

Restricting the complex variable z to a real variable x, we can replace the sequence $\{2^n F_n\}_{n\geq 0}$ appearing in the $G_d(z)$ by the usual Fibonacci and Lucas numbers defined, respectively, by (1) and by

$$L_0 = 2, \quad L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n \quad (n \geq 0). \tag{2}$$

In what follows, we consider a somewhat general sequence than those of the Fibonacci and Lucas numbers. Let $\{R_n\}_{n\geq 0}$ be a sequence of integers satisfying the binary linear recurrence relation

$$R_{n+2} = A_1 R_{n+1} + A_2 R_n \quad (n \geq 0), \tag{3}$$

where A_1 and A_2 are integers with $A_1 A_2 \neq 0$, $\Delta = A_1^2 + 4A_2 > 0$, and $R_1^2 \neq R_0 R_2$. We denote by $\mathbb{L}$ the set of nonzero real algebraic numbers.

The following main theorem of this paper is proved in Section 3 by using Lemma 2 in Section 2.

Theorem 1. *Let $\{R_n\}_{n\geq 0}$ be the sequence of integers defined by (3) and S the set of odd integers ≥ 3. Suppose either $A_2 = 1$, $R_0 \neq 0$ or $A_2 = -1$, $A_1 R_0 \neq 2R_1$. Then the numbers*

$$\prod_{k=0}^{\infty}\left(1 - \frac{a}{R_{d_1^k}}\right) \quad (d_1 \in S,\ a \in \mathbb{L} \setminus \{R_{d_1^k}\}_{k\geq 0})$$

and

$$\sum_{k=0}^{\infty} \frac{1}{(R_{d_2^k} - b)^l} \quad (d_2 \in S,\ l \in \mathbb{N},\ b \in \mathbb{L} \setminus \{R_{d_2^k}\}_{k\geq 0})$$

are algebraically independent.

In what follows, x denotes a real variable. Theorem 1 implies the following:

Theorem 2. *Let $\{R_n\}_{n\geq 0}$ and S be as in Theorem 1. Define*

$$f_d(x) = \prod_{k=0}^{\infty}\left(1 - \frac{x}{R_{d^k}}\right) \qquad (d \in S).$$

Then the values

$$f_d^{(l)}(\alpha) \quad (d \in S,\ l \in \mathbb{N}_0,\ \alpha \in \mathbb{L} \setminus \{R_{d^k}\}_{k\geq 0})$$

are algebraically independent.

Proof. Let $T_{dl}(x) = -\sum_{k=0}^{\infty} 1/(R_{d^k} - x)^l$ $(d \in S,\ l \in \mathbb{N})$. By Theorem 1, the infinite set of the numbers

$$\{f_d(\alpha) \mid d \in S,\ \alpha \in \mathbb{L} \setminus \{R_{d^k}\}_{k\geq 0}\} \cup \{T_{dl}(\alpha) \mid d \in S,\ l \in \mathbb{N},\ \alpha \in \mathbb{L} \setminus \{R_{d^k}\}_{k\geq 0}\}$$

is algebraically independent.

Since

$$f_d'(x) = f_d(x)T_{d1}(x)$$

and since $T_{dl}'(x) = lT_{dl+1}(x)$ $(l \geq 1)$, we see inductively that, for any $l \geq 2$,

$$f_d^{(l)}(x) = f_d(x)P_{dl}\left(T_{d1}(x), \ldots, T_{dl-1}(x)\right) + (l-1)!f_d(x)T_{dl}(x),$$

where $P_{dl}(Y_1, \ldots, Y_{l-1}) \in \mathbb{Z}[Y_1, \ldots, Y_{l-1}]$. Assume on the contrary that there exist a $D \in \mathbb{N} \setminus \{1\}$, an $L \in \mathbb{N}_0$, and distinct $\alpha_1^{(d)}, \ldots, \alpha_{n_d}^{(d)} \in \mathbb{L} \setminus \{R_{d^k}\}_{k\geq 0}$ $(d \in S,\ d \leq D)$ such that the values

$$f_d^{(l)}(\alpha_i^{(d)}) \quad (d \in S,\ d \leq D,\ 0 \leq l \leq L,\ 1 \leq i \leq n_d)$$

are algebraically dependent. Since $f_d'(\alpha_i^{(d)}) = f_d(\alpha_i^{(d)})T_{d1}(\alpha_i^{(d)})$ and since $f_d^{(l)}(\alpha_i^{(d)}) = f_d(\alpha_i^{(d)})P_{dl}\left(T_{d1}(\alpha_i^{(d)}), \ldots, T_{dl-1}(\alpha_i^{(d)})\right) + (l-1)!f_d(\alpha_i^{(d)})T_{dl}(\alpha_i^{(d)})$ $(l \geq 2)$, noting that for each d and i the number $T_{dl}(\alpha_i^{(d)})$ does not appear in the expression for the numbers $f_d^{(j)}(\alpha_i^{(d)})$ $(0 \leq j \leq l-1)$, we see that the values

$$f_d(\alpha_i^{(d)}), T_{d1}(\alpha_i^{(d)}), \ldots, T_{dL}(\alpha_i^{(d)}) \quad (d \in S,\ d \leq D,\ 1 \leq i \leq n_d)$$

are algebraically dependent, which is a contradiction and so the infinite set of the values $\bigcup_{d\in S}\{f_d^{(l)}(\alpha) \mid l \in \mathbb{N}_0,\ \alpha \in \mathbb{L} \setminus \{R_{d^k}\}_{k\geq 0}\}$ is algebraically independent. □

Example 1. Let $\{F_n\}_{n\geq 0}$ be the sequence of Fibonacci numbers defined by (1) and S the set of odd integers ≥ 3. Define

$$g_d(x) = \prod_{k=0}^{\infty}\left(1 - \frac{x}{F_{2d^k}}\right) \qquad (d \in S).$$

Then by Theorem 2 with $A_2 = -1$ the values

$$g_d^{(l)}(\alpha) \quad (d \in S,\ l \in \mathbb{N}_0,\ \alpha \in \mathbb{L} \setminus \{F_{2d^k}\}_{k\geq 0})$$

are algebraically independent, since $R_n := F_{2n}$ satisfies $R_0 = 0$, $R_1 = 1$, $R_{n+2} = 3R_{n+1} - R_n$ $(n \geq 0)$. In particular, the numbers

$$\prod_{k=0}^{\infty}\left(1 + \frac{1}{F_{2\cdot 3^k}}\right), \quad \prod_{k=0}^{\infty}\left(1 + \frac{1}{F_{2\cdot 5^k}}\right), \quad \sum_{k=0}^{\infty}\frac{1}{F_{2\cdot 3^k}+1}, \quad \text{and} \quad \sum_{k=0}^{\infty}\frac{1}{F_{2\cdot 5^k}+1}$$

are algebraically independent.

Example 2. Let $\{L_n\}_{n\geq 0}$ be the sequence of Lucas numbers defined by (2) and S the set of odd integers ≥ 3. Define

$$h_d(x) = \prod_{k=0}^{\infty}\left(1 - \frac{x}{L_{d^k}}\right) \qquad (d \in S). \tag{4}$$

Then by Theorem 2 with $A_2 = 1$ the values

$$h_d^{(l)}(\alpha) \quad (d \in S,\ l \in \mathbb{N}_0,\ \alpha \in \mathbb{L} \setminus \{L_{d^k}\}_{k\geq 0})$$

are algebraically independent. In particular, the numbers

$$\prod_{k=0}^{\infty}\left(1 + \frac{1}{L_{d^k}}\right), \qquad \sum_{k=0}^{\infty}\frac{1}{L_{d^k}+1} \qquad (d = 3, 5, 7, \ldots)$$

are algebraically independet.

Remark 1. If S is the set of even integers instead, some of the infinite products such as (4) can be algebraically dependent. For example, the authors [2, 3] showed that the transcendental numbers

$$\rho_1 = \prod_{k=1}^{\infty}\left(1 - \frac{5}{L_{2^k}}\right), \quad \rho_2 = \prod_{k=1}^{\infty}\left(1 + \frac{5}{L_{2^k}}\right), \quad \rho_3 = \prod_{k=1}^{\infty}\left(1 - \frac{23}{L_{2^k}}\right)$$

satisfy

$$4\sqrt{5}\rho_1\rho_2 + \rho_3 = 0,$$

while $\mathrm{trans.deg}_{\mathbb{Q}}\mathbb{Q}(\rho_1, \rho_2, \rho_3) = 2$; moreover, the transcendental numbers

$$\sigma_1 = \prod_{k=1}^{\infty}\left(1 - \frac{\sqrt{3}}{L_{4^k}}\right), \quad \sigma_2 = \prod_{k=1}^{\infty}\left(1 + \frac{\sqrt{3}}{L_{4^k}}\right), \quad \sigma_3 = \prod_{k=1}^{\infty}\left(1 - \frac{1}{L_{4^k}}\right), \quad \sigma_4 = \prod_{k=1}^{\infty}\left(1 + \frac{2}{L_{4^k}}\right)$$

satisfy

$$\sigma_1\sigma_2\sigma_3\sigma_4^{-1} = \frac{5}{8},$$

while $\mathrm{trans.deg}_{\mathbb{Q}}\,\mathbb{Q}(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = 3$; furthermore, the transcendental numbers

$$\tau_1 = \prod_{k=1}^{\infty}\left(1 - \frac{1}{L_{6^k}}\right), \quad \tau_2 = \prod_{k=1}^{\infty}\left(1 + \frac{1}{L_{6^k}}\right), \quad \tau_3 = \prod_{k=1}^{\infty}\left(1 + \frac{2}{L_{6^k}}\right),$$
$$\tau_4 = \prod_{k=1}^{\infty}\left(1 + \frac{\sqrt{3}}{L_{6^k}}\right), \quad \tau_5 = \prod_{k=1}^{\infty}\left(1 - \frac{\sqrt{3}}{L_{6^k}}\right)$$

satisfy

$$\tau_1\tau_2\tau_3\tau_4^{-1}\tau_5^{-1} = \frac{\sqrt{5}}{2},$$

while $\mathrm{trans.deg}_{\mathbb{Q}}\,\mathbb{Q}(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5) = 4$.

2 Lemmas

Lemma 1 (A special case of Theorem 1 in Nishioka [7]). *Let $d_1, \dots, d_t \geq 2$ be integers such that $\log d_i / \log d_j \notin \mathbb{Q}$ for any i, j with $i \neq j$. Let K be an algebraic number field. Suppose that $f_{ij}(z) \in K[[z]]$ $(i = 1, \dots, t;\ j = 1, \dots, M_i)$ satisfy the functional equations*

$$f_{ij}(z) = a_{ij}(z) f_{ij}(z^{d_i}) \qquad (1 \leq i \leq t,\ 1 \leq j \leq N_i)$$

and

$$f_{ij}(z) = f_{ij}(z^{d_i}) + b_{ij}(z) \qquad (1 \leq i \leq t,\ N_i + 1 \leq j \leq M_i),$$

where $a_{ij}(z), b_{ij}(z) \in K(z)$ with $a_{ij}(0) = 1$. Assume that for each i $(1 \leq i \leq t)$, $f_{ij}(z)$ $(1 \leq j \leq M_i)$ are algebraically independent over $\mathbb{C}(z)$. If α is an algebraic number with $0 < |\alpha| < 1$ such that all the $f_{ij}(z)$ converge at α and $a_{ij}(\alpha^{d_i^k})$ $(1 \leq i \leq t,\ 1 \leq j \leq N_i)$ are defined and nonzero for all $k \geq 0$, then the values

$$f_{ij}(\alpha) \quad (1 \leq i \leq t,\ 1 \leq j \leq M_i)$$

are algebraically independent.

Remark 2. It is not necessary in Lemma 1 to assume that $b_{ij}(\alpha^{d_i^k})$ $(1 \leq i \leq t,\ N_i + 1 \leq j \leq M_i)$ are defined for all $k \geq 0$, since $b_{ij}(\alpha^{d_i^k}) = f_{ij}(\alpha^{d_i^k}) - f_{ij}(\alpha^{d_i^{k+1}})$ and $f_{ij}(\alpha^{d_i^k})$ $(1 \leq i \leq t,\ N_i + 1 \leq j \leq M_i)$ are defined for all $k \geq 0$ by $|\alpha^{d_i^k}| \leq |\alpha|$.

Lemma 2. *Let $d_1, \dots, d_t \geq 2$ be integers such that $\log d_i / \log d_j \notin \mathbb{Q}$ for any i, j with $i \neq j$. Let K be an algebraic number field. Suppose that $f_{ij\lambda}(z) \in K[[z]]$ $(i = 1, \dots, t;\ j = 1, \dots, m;\ \lambda = 1, \dots, s)$ satisfy the functional equations*

$$f_{ij\lambda}(z) = a_{ij\lambda}(z) f_{ij\lambda}(z^{d_i^j}) \qquad (1 \leq i \leq t,\ 1 \leq j \leq m,\ 1 \leq \lambda \leq r) \tag{5}$$

with $f_{ij\lambda}(0) \neq 0$ and

$$f_{ij\lambda}(z) = f_{ij\lambda}(z^{d_i^j}) + b_{ij\lambda}(z) \qquad (1 \leq i \leq t,\ 1 \leq j \leq m,\ r + 1 \leq \lambda \leq s), \tag{6}$$

where $a_{ij\lambda}(z), b_{ij\lambda}(z) \in K(z)$ with $a_{ij\lambda}(0) = 1$. Assume that for each i $(1 \leq i \leq t)$, $f_{ij\lambda}(z)$ $(1 \leq j \leq m,\ 1 \leq \lambda \leq r)$ are multiplicatively independent modulo $\mathbb{C}(z)^\times$ and $f_{ij\lambda}(z)$ $(1 \leq j \leq m,\ r + 1 \leq \lambda \leq s)$ are linearly independent over $\mathbb{C}$ modulo $\mathbb{C}(z)$. If α is an algebraic number with $0 < |\alpha| < 1$ such that all the $f_{ij\lambda}(z)$ converge at α and $a_{ij\lambda}(\alpha^{d_i^{jk}})$ $(1 \leq i \leq t,\ 1 \leq j \leq m,\ 1 \leq \lambda \leq r)$ are defined and nonzero for all $k \geq 0$, then the values

$$f_{ij\lambda}(\alpha) \quad (1 \leq i \leq t,\ 1 \leq j \leq m,\ 1 \leq \lambda \leq s)$$

are algebraically independent.

Lemma 2 is proved by using Lemma 1 above and Lemma 3 below. In what follows, C denotes a field of characteristic 0. We define an endomorphism $\tau : C((z)) \to C((z))$ by

$$f^\tau(z) = f(z^d) \qquad (f(z) \in C((z))),$$

where d is an integer greater than 1, and a subgroup H of $C(z)^\times$ by

$$H = \{g^\tau g^{-1} \mid g \in C(z)^\times\}.$$

Lemma 3 (A special case of Theorem 2 in Kubota [1], see also Nishioka [7, Theorem 3]). *Let $f_i \in C((z))^\times$ $(i = 1, \dots, h)$ satisfy*

$$f_i = a_i f_i^\tau,$$

where $a_i \in C(z)^\times$ $(1 \le i \le h)$, and let $f_i \in C((z))$ $(i = h+1, \dots, m)$ satisfy

$$f_i = f_i^\tau + b_i,$$

where $b_i \in C(z)$ $(h+1 \le i \le m)$. Suppose that a_i and b_i have the following properties:

(i) *$a_1, \dots, a_h$ are multiplicatively independent modulo H.*

(ii) *If $c_i \in C$ $(h+1 \le i \le m)$ are not all zero, there is no element g of $C(z)$ such that*

$$g^\tau - g = \sum_{i=h+1}^{m} c_i b_i.$$

Then the functions f_i $(1 \le i \le m)$ are algebraically independent over $C(z)$.

Proof of Lemma 2. Letting $M = m!$ and iterating (5), we see that $f_{ij\lambda}(z)$ with $1 \le \lambda \le r$ satisfies

$$f_{ij\lambda}(z) = a^*_{ij\lambda}(z) f_{ij\lambda}(z^{d_i^M}), \tag{7}$$

where

$$a^*_{ij\lambda}(z) = \prod_{l=0}^{M/j-1} a_{ij\lambda}(z^{d_i^{jl}}) \qquad (1 \le i \le t,\ 1 \le j \le m,\ 1 \le \lambda \le r),$$

and iterating (6), we see that $f_{ij\lambda}(z)$ with $r+1 \le \lambda \le s$ satisfies

$$f_{ij\lambda}(z) = f_{ij\lambda}(z^{d_i^M}) + b^*_{ij\lambda}(z), \tag{8}$$

where

$$b^*_{ij\lambda}(z) = \sum_{l=0}^{M/j-1} b_{ij\lambda}(z^{d_i^{jl}}) \qquad (1 \le i \le t,\ 1 \le j \le m,\ r+1 \le \lambda \le s).$$

Hence for each fixed i $(1 \le i \le t)$, we can show that $f_{ij\lambda}(z)$ $(1 \le j \le m,\ 1 \le \lambda \le s)$ are algebraically independent over $\mathbb{C}(z)$ by applying Lemma 3 with $d = d_i^M$, whose notation will be used in the following: If the assumption (i) of Lemma 3 is not satisfied, namely if $a^*_{ij\lambda}$ $(1 \le j \le m,\ 1 \le \lambda \le r)$ are multiplicatively dependent modulo H, then there is an element g of $\mathbb{C}(z)^\times$ such that

$$\prod_{j=1}^{m} \prod_{\lambda=1}^{r} (a^*_{ij\lambda})^{e_{ij\lambda}} = g^\tau / g$$

with $e_{ij\lambda} \in \mathbb{Z}$ $(1 \le j \le m,\ 1 \le \lambda \le r)$ not all zero, and hence $F = g \prod_{j=1}^{m} \prod_{\lambda=1}^{r} f_{ij\lambda}^{e_{ij\lambda}}$ satisfies $F = F^\tau$, which holds only if $F \in \mathbb{C}$ (cf. Loxton and van der Poorten [5, Lemma 1]), and so

$$\prod_{j=1}^{m} \prod_{\lambda=1}^{r} f_{ij\lambda}^{e_{ij\lambda}} \in \mathbb{C}(z)^\times,$$

namely $f_{ij\lambda}$ $(1 \le j \le m,\ 1 \le \lambda \le r)$ are multiplicatively dependent modulo $\mathbb{C}(z)^\times$. If the assumption (ii) of Lemma 3 is not satisfied, namely if there is an element g of $\mathbb{C}(z)$ such that

$$g^\tau = g + \sum_{j=1}^{m} \sum_{\lambda=r+1}^{s} c_{ij\lambda} b^*_{ij\lambda}$$

with $c_{ij\lambda} \in \mathbb{C}$ $(1 \le j \le m,\ r+1 \le \lambda \le s)$ not all zero, then $G = g + \sum_{j=1}^{m} \sum_{\lambda=r+1}^{s} c_{ij\lambda} f_{ij\lambda}$ satisfies $G = G^\tau$, which holds only if $G \in \mathbb{C}$ (cf. Loxton and van der Poorten [5, Lemma 1]), and so

$$\sum_{j=1}^{m} \sum_{\lambda=r+1}^{s} c_{ij\lambda} f_{ij\lambda} \in \mathbb{C}(z),$$

namely $f_{ij\lambda}$ $(1 \le j \le m,\ r+1 \le \lambda \le s)$ are linearly dependent over $\mathbb{C}$ modulo $\mathbb{C}(z)$. Therefore the assumptions (i) and (ii) of Lemma 3 are fulfilled, thereby $f_{ij\lambda}(z)$ $(1 \le j \le m,\ 1 \le \lambda \le s)$ are algebraically independent over $\mathbb{C}(z)$ for each fixed i $(1 \le i \le t)$.

Since $f_{ij\lambda}(z)$ $(1 \le j \le m,\ 1 \le \lambda \le r)$ satisfy (7) and $f_{ij\lambda}(z)$ $(1 \le j \le m,\ r+1 \le \lambda \le s)$ satisfy (8) for each fixed i $(1 \le i \le t)$, the lemma is proved by applying Lemma 1. □

Lemma 4. *Let $d \ge 3$ be an integer and let $\xi \in \mathbb{C}^\times$. Define*

$$f_{\lambda l}(z) = \sum_{k=0}^{\infty} \xi^k \left(\frac{z^{d^k}}{1 + a_\lambda z^{d^k} + b z^{2d^k}} \right)^l \quad (\lambda = 1, \dots, r;\ l = 1, \dots, L),$$

where $a_1, \dots, a_r \in \mathbb{R}$ are distinct and $b \in \mathbb{R} \setminus \{0, 1\}$. Then $f_{\lambda l}(z)$ $(1 \le \lambda \le r,\ 1 \le l \le L)$ are linearly independent over $\mathbb{C}$ modulo $\mathbb{C}(z)$.

Proof. Suppose on the contrary there exist complex numbers $c_{\lambda l}$ $(1 \le \lambda \le r,\ 1 \le l \le L)$, not all zero, such that

$$g(z) = \sum_{\lambda=1}^{r} \sum_{l=1}^{L} c_{\lambda l} f_{\lambda l}(z) \in \mathbb{C}(z).$$

Then $g(z)$ satisfies the functional equation

$$g(z) = \xi g(z^d) + Q(z), \qquad Q(z) = \sum_{\lambda=1}^{r} \sum_{l=1}^{L} \frac{c_{\lambda l} z^l}{(1 + a_\lambda z + b z^2)^l}. \tag{9}$$

Let p_λ and q_λ $(|p_\lambda| \le |q_\lambda|)$ be the roots of $1 + a_\lambda z + b z^2$ and let $S = \{\lambda \in \{1, \dots, r\} \mid c_{\lambda l}\ (1 \le l \le L)$ are not all zero$\}$. Since $a_1, \dots, a_r$ are distinct, p_λ, q_λ $(\lambda \in S)$ are the poles of $Q(z)$. Changing the indices λ if necessary, we may assume that $1 \in S$ and that p_1 and q_μ for some $\mu \in S$ are the poles of $Q(z)$ with the smallest and the largest absolute values among its poles, respectively.

First we assume that $|p_1| < 1 < |q_\mu|$. Noting that, for any $0 < r \le R$, the poles of $g(z)$ lie on the domain $\{z \in \mathbb{C} \mid r \le |z| \le R\}$ if and only if those of $g(z^d)$ lie on the domain $\{z \in \mathbb{C} \mid r^{1/d} \le |z| \le R^{1/d}\}$, we see by (9) that p_1 and q_μ are poles of $g(z)$. Let $p_1^{1/d}$ and $q_\mu^{1/d}$ denote the d-th roots of p_1 and q_μ, respectively, with the smallest positive argument among the d-th roots of p_1 and q_μ. Then $p_1^{1/d}$ and $q_\mu^{1/d}$ are poles of $g(z^d)$.

Here we assert that there exist positive integers s and t such that $\zeta := p_1^{1/d^s}$ and $\eta := q_\mu^{1/d^t}$ are poles of $Q(z)$. Assume on the contrary that p_1^{1/d^k} is not a pole of $Q(z)$ for any $k \geq 1$ or that q_μ^{1/d^k} is not a pole of $Q(z)$ for any $k \geq 1$. In the first case $p_1^{1/d}$ must be a pole of $g(z)$ by (9). Then p_1^{1/d^2} is a pole of $g(z^d)$ but not of $Q(z)$. Hence p_1^{1/d^2} is a pole of $g(z)$ by (9). Repeating this process, we see that p_1^{1/d^k} $(k \geq 1)$ are poles of $g(z)$, which is a contradiction since $g(z) \in \mathbb{C}(z)$. In the second case the same argument for $q_\mu^{1/d}$ leads to a contradiction. Hence the assertion is proved.

Since $d \geq 3$ and so $\zeta, \eta \notin \mathbb{R}$, noting that ζ and η are roots of quadratic polynomials with real coefficients, we have $b\zeta\overline{\zeta} = b\eta\overline{\eta}\,(= 1)$. This implies $|p_1|^{2/d^s} = |q_\mu|^{2/d^t}$, which contradicts $|p_1| < 1 < |q_\mu|$. Hence $|p_1| \geq 1$ or $|q_\mu| \leq 1$.

Next assume that $1 < |p_1| \leq |q_\mu|$. By the same argument as above, there exists a positive integer t such that $\eta = q_\mu^{1/d^t} \notin \mathbb{R}$ is a pole of $Q(z)$ and $b\eta\overline{\eta} = 1$. If $z = q_\mu$ is a simple root of $1 + a_\lambda z + bz^2$ for some λ, then by the definition of p_1 and q_μ we see that $z = p_1$ is also a root of $1+a_\lambda z+bz^2$. Thus we get $b\eta\overline{\eta} = bp_1q_\mu\,(= 1)$ and so $|p_1| = |q_\mu|^{2/d^t-1}$, which contradicts $1 < |p_1| \leq |q_\mu|$. Hence $z = q_\mu$ must be a double root of $1+a_\lambda z+bz^2$ for some λ. Then we see $q_\mu = \pm b^{-1/2}$ and thus $|b\eta\overline{\eta}| = |b|^{1-1/d^t} = 1$. This implies $|b| = 1$ and so $|q_\mu| = 1$, which again contradicts $1 < |p_1| \leq |q_\mu|$. Also in the case of $|p_1| \leq |q_\mu| < 1$ we get a contradiction by the same argument.

Therefore $|p_1| = |q_\mu| = |q_1| = 1$ and so $b = -1$ since $bp_1q_1 = 1$ and $b \in \mathbb{R} \setminus \{1\}$. Hence the poles of $Q(z)$ are real and so only ± 1 can be the poles of $Q(z)$. Then ± 1 can be poles of $g(z)$ or $g(z^d)$ by (9) and hence at least one of ± 1 is a pole of $g(z)$. If 1 is a pole of $g(z)$, then $\zeta_d = e^{2\pi\sqrt{-1}/d} \notin \mathbb{R}$ is a pole of $g(z^d)$ but not of $Q(z)$. Hence ζ_d is a pole of $g(z)$ by (9). Then $\zeta_{d^2} \notin \mathbb{R}$ is a pole of $g(z^d)$ but not of $Q(z)$. Hence ζ_{d^2} is a pole $g(z)$ by (9). Repeating this process, we see that ζ_{d^k} $(k \geq 1)$ are poles of $g(z)$, which is a contradiction since $g(z) \in \mathbb{C}(z)$. If -1 is a pole of $g(z)$, then a contradiction is deduced by a similar argument. □

Lemma 5. *Let $d \geq 3$ be an integer and let $\xi \in \mathbb{C}^\times$. Define*

$$g_\lambda(z) = \sum_{k=0}^{\infty} \xi^k \frac{a_\lambda z^{d^k} + 2bz^{2d^k}}{1 + a_\lambda z^{d^k} + bz^{2d^k}} \quad (\lambda = 1, \ldots, r),$$

where $a_1, \ldots, a_r \in \mathbb{R}$ are distinct and $b \in \mathbb{R} \setminus \{0, 1\}$. Then $g_1(z), \ldots, g_r(z)$ are linearly independent over $\mathbb{C}$ modulo $\mathbb{C}(z)$.

Proof. Suppose there exist complex numbers c_λ $(1 \leq \lambda \leq r)$, not all zero, such that

$$G(z) = \sum_{\lambda=1}^{r} c_\lambda g_\lambda(z) \in \mathbb{C}(z).$$

Then $G(z)$ satisfies the functional equation

$$G(z) = \xi G(z^d) + Q(z), \qquad Q(z) = \sum_{\lambda=1}^{r} \frac{c_\lambda(a_\lambda z + 2bz^2)}{1 + a_\lambda z + bz^2}.$$

The lemma is proved in the same way as in the proof of Lemma 4. □

Lemma 6. *Let $\xi \in \mathbb{C}^\times$ and $q_1(z), \ldots, q_s(z) \in \mathbb{C}(z)$ with $q_\lambda(0) = 0$. Let $d \geq 2$ and $m \geq 1$ be integers. Define*

$$\varphi_{j\lambda}(z) = \sum_{k=0}^{\infty} \xi^{jk} q_\lambda(z^{d^{jk}}) \quad (j = 1, \ldots, m;\ \lambda = 1, \ldots, s)$$

and

$$f_{\lambda\mu}(z) = \sum_{k=0}^{\infty} (\zeta_\mu \xi)^k q_\lambda(z^{d^k}) \quad (\lambda = 1, \ldots, s;\ \mu = 1, \ldots, m!),$$

where $\zeta_1, \ldots, \zeta_{m!}$ are distinct $m!$-th roots of unity. If, for each fixed μ $(1 \leq \mu \leq m!)$, $f_{\lambda\mu}(z)$ $(1 \leq \lambda \leq s)$ are linearly independent over $\mathbb{C}$ modulo $\mathbb{C}(z)$, then so are $\varphi_{j\lambda}(z)$ $(1 \leq j \leq m,\ 1 \leq \lambda \leq s)$.

Proof. Suppose on the contrary there exist $c_{j\lambda} \in \mathbb{C}$ $(1 \leq j \leq m,\ 1 \leq \lambda \leq s)$, not all zero, such that

$$g(z) = \sum_{j=1}^{m} \sum_{\lambda=1}^{s} c_{j\lambda} \varphi_{j\lambda}(z) \in \mathbb{C}(z).$$

We define sequences $\{b_k^{(j)}\}_{k \geq 0}$ $(1 \leq j \leq m)$ by

$$\begin{aligned}
\{b_k^{(1)}\}_{k\geq 0} &= \{1, 1, 1, \ldots\}, \\
\{b_k^{(2)}\}_{k\geq 0} &= \{1, 0, 1, 0, 1, 0, \ldots\}, \\
\{b_k^{(3)}\}_{k\geq 0} &= \{1, 0, 0, 1, 0, 0, 1, 0, 0, \ldots\}, \\
\{b_k^{(4)}\}_{k\geq 0} &= \{1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, \ldots\}, \\
&\cdots .
\end{aligned}$$

Since

$$\varphi_{j\lambda}(z) = \sum_{k=0}^{\infty} b_k^{(j)} \xi^k q_\lambda(z^{d^k}) \qquad (1 \leq j \leq m,\ 1 \leq \lambda \leq s),$$

we have

$$g(z) = \sum_{j=1}^{m} \sum_{\lambda=1}^{s} c_{j\lambda} \sum_{k=0}^{\infty} b_k^{(j)} \xi^k q_\lambda(z^{d^k}) = \sum_{\lambda=1}^{s} \sum_{k=0}^{\infty} \left(\sum_{j=1}^{m} c_{j\lambda} b_k^{(j)} \right) \xi^k q_\lambda(z^{d^k}).$$

Since $\{\sum_{j=1}^{m} c_{j\lambda} b_k^{(j)}\}_{k \geq 0}$ $(1 \leq \lambda \leq s)$ are periodic sequences whose periods divide $m!$, there exist distinct $m!$-th roots of unity $\zeta_1, \ldots, \zeta_{m!}$ such that

$$\sum_{j=1}^{m} c_{j\lambda} b_k^{(j)} = \sum_{\mu=1}^{m!} a_{\lambda\mu} \zeta_\mu^k \quad (1 \leq \lambda \leq s),$$

where $a_{\lambda\mu} \in \mathbb{C}$ $(1 \leq \lambda \leq s,\ 1 \leq \mu \leq m!)$ are not all zero, since $c_{j\lambda}$ $(1 \leq j \leq m,\ 1 \leq \lambda \leq s)$ are not all zero and the sequences $\{b_k^{(j)}\}_{k \geq 0}$ $(1 \leq j \leq m)$ are linearly independent over $\mathbb{C}$.

Hence

$$\begin{aligned} g(z) &= \sum_{\lambda=1}^{s}\sum_{k=0}^{\infty}\left(\sum_{\mu=1}^{m!} a_{\lambda\mu}\zeta_\mu^k\right)\xi^k q_\lambda(z^{d^k}) \\ &= \sum_{\lambda=1}^{s}\sum_{\mu=1}^{m!} a_{\lambda\mu}\sum_{k=0}^{\infty}(\zeta_\mu\xi)^k q_\lambda(z^{d^k}) \\ &= \sum_{\lambda=1}^{s}\sum_{\mu=1}^{m!} a_{\lambda\mu} f_{\lambda\mu}(z), \end{aligned}$$

which means that $f_{\lambda\mu}(z)$ $(1 \le \lambda \le s,\ 1 \le \mu \le m!)$ are algebraically dependent over $\mathbb{C}(z)$. Since $f_{\lambda\mu}(z)$ satisfies

$$f_{\lambda\mu}(z) = \zeta_\mu \xi f_{\lambda\mu}(z^d) + q_\lambda(z)$$

and $\zeta_1\xi, \dots, \zeta_{m!}\xi$ are distinct, by Loxton and van der Poorten's theorem [5, Theorem 2] or by Kubota's result [1, Corollary 9], the functions $f_{\lambda\mu}(z)$ $(1 \le \lambda \le s)$ are linearly dependent over $\mathbb{C}$ modulo $\mathbb{C}(z)$ for some μ, which contradicts the assumption of the lemma. □

By Lemma 6 with $\xi = 1$ and Lemma 4 with $\xi = \zeta_\mu$, we immediately have the following:

Lemma 7. *Let $d \ge 3$ be an integer and let*

$$\varphi_{j\lambda l}(z) = \sum_{k=0}^{\infty}\left(\frac{z^{d^{jk}}}{1 + a_\lambda z^{d^{jk}} + bz^{2d^{jk}}}\right)^l \quad (j = 1, \dots, m;\ \lambda = 1, \dots, r;\ l = 1, \dots, L),$$

where $a_1, \dots, a_r \in \mathbb{R}$ are distinct and $b \in \mathbb{R} \setminus \{0, 1\}$. Then $\varphi_{j\lambda l}(z)$ $(1 \le j \le m,\ 1 \le \lambda \le r,\ 1 \le l \le L)$ are linearly independent over $\mathbb{C}$ modulo $\mathbb{C}(z)$.

Lemma 8. *Let $d \ge 3$ be an integer and let*

$$\Phi_{j\lambda}(z) = \prod_{k=0}^{\infty}\left(1 + \frac{a_\lambda z^{d^{jk}}}{1 + bz^{2d^{jk}}}\right) \quad (j = 1, \dots, m;\ \lambda = 1, \dots, r),$$

where $a_1, \dots, a_r \in \mathbb{R}^\times$ are distinct and $b \in \mathbb{R} \setminus \{0, 1\}$. Then $\Phi_{j\lambda}(z)$ $(1 \le j \le m,\ 1 \le \lambda \le r)$ are multiplicatively independent modulo $\mathbb{C}(z)^\times$.

Proof. Suppose there exist integers $e_{j\lambda}$ $(1 \le j \le m,\ 1 \le \lambda \le r)$ such that

$$G(z) = \prod_{j=1}^{m}\prod_{\lambda=1}^{r}\Phi_{j\lambda}(z)^{e_{j\lambda}} \in \mathbb{C}(z)^\times.$$

Taking the logarithmic derivative of $G(z)$, we obtain

$$\frac{zG'(z)}{G(z)} = z\frac{d}{dz}\log G(z) = \sum_{j=1}^{m}\sum_{\lambda=1}^{r} e_{j\lambda}\sum_{k=0}^{\infty} d^{jk}\left(\frac{a_\lambda z^{d^{jk}} + 2bz^{2d^{jk}}}{1 + a_\lambda z^{d^{jk}} + bz^{2d^{jk}}} - \frac{2bz^{2d^{jk}}}{1 + bz^{2d^{jk}}}\right) \in \mathbb{C}(z).$$

On the other hand, by Lemma 6 with $\xi = d$ and Lemma 5 with $\xi = \zeta_\mu d$, we see that

$$\sum_{k=0}^{\infty} d^{jk} \frac{a_\lambda z^{d^{jk}} + 2bz^{2d^{jk}}}{1 + a_\lambda z^{d^{jk}} + bz^{2d^{jk}}} \quad (1 \le j \le m,\ 1 \le \lambda \le r)$$

and

$$\sum_{k=0}^{\infty} d^{jk} \frac{2bz^{2d^{jk}}}{1 + bz^{2d^{jk}}} \quad (1 \le j \le m)$$

are linearly independent over $\mathbb{C}$ modulo $\mathbb{C}(z)$, and thus so are

$$\sum_{k=0}^{\infty} d^{jk} \left(\frac{a_\lambda z^{d^{jk}} + 2bz^{2d^{jk}}}{1 + a_\lambda z^{d^{jk}} + bz^{2d^{jk}}} - \frac{2bz^{2d^{jk}}}{1 + bz^{2d^{jk}}} \right) \qquad (1 \le j \le m,\ 1 \le \lambda \le r).$$

Hence $e_{j\lambda} = 0$ $(1 \le j \le m,\ 1 \le \lambda \le r)$. □

3 Proof of Theorem 1

Proof of Theorem 1. Let

$$D = \{d \in \mathbb{N} \mid d \neq a^n \text{ for any } a, n \in \mathbb{N} \text{ with } n > 1\}.$$

Then

$$\mathbb{N} \setminus \{1\} = \bigcup_{d \in D} \{d, d^2, \ldots\} \quad \text{(disjoint union)}.$$

We note that if $d, d' \in D$ are distinct, then $\log d / \log d' \notin \mathbb{Q}$. It is enough to prove the algebraic independency of the numbers

$$\prod_{k=0}^{\infty} \left(1 - \frac{a}{R_{d^{jk}}}\right) \qquad (d \in D \cap S,\ j \in \mathbb{N},\ a \in \mathbb{L} \setminus \{R_{d^{jk}}\}_{k \ge 0})$$

and

$$\sum_{k=0}^{\infty} \frac{1}{(R_{d^{jk}} - b)^l} \qquad (d \in D \cap S,\ j \in \mathbb{N},\ b \in \mathbb{L} \setminus \{R_{d^{jk}}\}_{k \ge 0},\ l \in \mathbb{N}).$$

Hence it suffices to prove that, for any distinct $d_1, \ldots, d_t \in D \cap S$, for any $m \in \mathbb{N}$, for any distinct $a_1, \ldots, a_r \in \mathbb{L} \setminus \{R_{d_i^{jk}}\}_{k \ge 0}$, and for any $L \in \mathbb{N}$, the numbers

$$\prod_{k=0}^{\infty} \left(1 - \frac{a_\lambda}{R_{d_i^{jk}}}\right) \qquad (1 \le i \le t,\ 1 \le j \le m,\ 1 \le \lambda \le r)$$

and

$$\sum_{k=0}^{\infty} \frac{1}{(R_{d_i^{jk}} - a_\lambda)^l} \qquad (1 \le i \le t,\ 1 \le j \le m,\ 1 \le \lambda \le r,\ 1 \le l \le L)$$

are algebraically independent.

Let α, β ($|\alpha| \geq |\beta|$) be the roots of $X^2 - A_1X - A_2 = 0$. We can express $\{R_n\}_{n\geq 0}$ as follows:

$$R_n = g_1\alpha^n + g_2\beta^n,$$

where $g_1 = (R_1 - \beta R_0)/(\alpha - \beta), g_2 = (\alpha R_0 - R_1)/(\alpha - \beta) \in \mathbb{Q}(\sqrt{\Delta})$ with $\Delta = A_1^2 + 4A_2$. Since $\Delta > 0$, $A_1A_2 \neq 0$, and $R_1^2 \neq R_0R_2$, we see that $|\alpha| > |\beta|$, $|\alpha| > 1$, and $g_1g_2 \neq 0$. Since $A_2 = -\alpha\beta = \pm 1$ and since d_i is odd, we see that

$$R_{d_i^k} = g_1\alpha^{d_i^k} - g_2A_2\alpha^{-d_i^k} \quad (k \geq 0).$$

Let K be an algebraic number field including $a_1, \ldots, a_r$ and $\sqrt{\Delta}$. Define $\Phi_{ij\lambda}(z) \in K[[z]]$ $(1 \leq i \leq t,\ 1 \leq j \leq m,\ 1 \leq \lambda \leq r)$ by

$$\Phi_{ij\lambda}(z) = \prod_{k=0}^{\infty}\left(1 - \frac{a_\lambda g_1^{-1} z^{d_i^{jk}}}{1 - g_1^{-1}g_2A_2z^{2d_i^{jk}}}\right)$$

and $\Psi_{ij\lambda l}(z) \in K[[z]]$ $(1 \leq i \leq t,\ 1 \leq j \leq m,\ 1 \leq \lambda \leq r,\ 1 \leq l \leq L)$ by

$$\Psi_{ij\lambda l}(z) = \sum_{k=0}^{\infty}\left(\frac{z^{d_i^{jk}}}{1 - a_\lambda g_1^{-1} z^{d_i^{jk}} - g_1^{-1}g_2A_2z^{2d_i^{jk}}}\right)^l.$$

Then each of $\Phi_{ij\lambda}(z)$, $\Psi_{ij\lambda l}(z)$ converges in $|z| < 1$ and

$$\Phi_{ij\lambda}(\alpha^{-1}) = \prod_{k=0}^{\infty}\left(1 - \frac{a_\lambda}{R_{d_i^{jk}}}\right) \qquad (1 \leq i \leq t,\ 1 \leq j \leq m,\ 1 \leq \lambda \leq r),$$

$$\Psi_{ij\lambda l}(\alpha^{-1}) = g_1^l \sum_{k=0}^{\infty} \frac{1}{(R_{d_i^{jk}} - a_\lambda)^l} \qquad (1 \leq i \leq t,\ 1 \leq j \leq m,\ 1 \leq \lambda \leq r,\ 1 \leq l \leq L).$$

Since $R_0 = 0$ if $g_1^{-1}g_2 = -1$ and since $A_1R_0 = 2R_1$ if $g_1^{-1}g_2 = 1$, the assumption of the theorem implies $-g_1^{-1}g_2A_2 \neq 1$ and thus we can apply Lemmas 7 and 8. Since each $\Phi_{ij\lambda}(z)$ satisfies the functional equation

$$\Phi_{ij\lambda}(z) = \left(1 - \frac{a_\lambda g_1^{-1} z}{1 - g_1^{-1}g_2A_2z^2}\right)\Phi_{ij\lambda}(z^{d_i^j})$$

and since each $\Psi_{ij\lambda l}(z)$ satisfies the functional equation

$$\Psi_{ij\lambda l}(z) = \Psi_{ij\lambda l}(z^{d_i^j}) + \left(\frac{z}{1 - a_\lambda g_1^{-1} z - g_1^{-1}g_2A_2z^2}\right)^l,$$

by Lemmas 2, 7, and 8, the values $\Phi_{ij\lambda}(\alpha^{-1})$ $(1 \leq i \leq t,\ 1 \leq j \leq m,\ 1 \leq \lambda \leq r)$ and $\Psi_{ij\lambda l}(\alpha^{-1})$ $(1 \leq i \leq t,\ 1 \leq j \leq m,\ 1 \leq \lambda \leq r,\ 1 \leq l \leq L)$ are algebraically independent. □

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