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A Bound for the Pressure Integral in a Plasma Equilibrium

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Abstract. An interpolation inequality for the total variation of the gradient of a composite function has been derived by applying the coarea formula. The interpolation inequality has been applied to the study of a bound for the pressure integral concerning a solution of the Grad-Shafranov equation of plasma equilibrium. A weak formulation of the Grad-Shafranov equation has been given to include singular current profiles.

1. Introduction

A simple but essential question in the fusion plasma research is how large plasma energy can be confined by a given magnitude of plasma current.[1-7] In a magnetohydrodynamic equilibrium of a plasma, the thermal pressure force ∇p is balanced by the magnetic stress $\boldsymbol{j}\times\boldsymbol{B}$, where $\boldsymbol{B}$ is the magnetic flux density, $\boldsymbol{j} = \nabla\times\boldsymbol{B} / \mu_0$ is the current density in the plasma and μ_0 is the vacuum permeability. The plasma equilibrium equation $\nabla p = \boldsymbol{j}\times\boldsymbol{B}$ thus relates the pressure and the current. We want to estimate the maximum of the total pressure with respect to a fixed total current. Mathematically this problem reduces to an a priori estimate for the pressure integral with respect to a solution of the equilibrium equation with a given magnitude of current.

Here we assume a simple two dimensional plasma equilibrium. Let $\Omega \subset \mathbf{R}^2$ be a bounded domain. We consider an infinitely long plasma column; Ω corresponds to

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the cross section of a column containing the plasma. If there is no longitudinal magnetic field, the equilibrium equations are

$$-\Delta\psi = P'(\psi) \qquad \text{in } \Omega, \tag{1.1}$$

$$\psi = c \qquad \text{on } \partial\Omega, \tag{1.2}$$

$$\int_\Omega (-\Delta\psi)\, dx = \mu_0 I, \tag{1.3}$$

where ψ is the flux function, $P = \mu_0 p$, $P(t)$ is a nonnegative function from $\mathbf{R}$ to $\mathbf{R}$, $P' = dP(t)/dt$, I is a given positive constant and c is an unknown constant. We assume $P' \geq 0$. Since $-\Delta\psi/\mu_0$ parallels the current density, I represents the total plasma current.

In this paper we study a bound for the total variation of the gradient of $P(\psi)$ in Ω. A crucial step is to establish an interpolation inequality to estimate the total variation of the gradient of $P(\psi)$ in Ω. Our estimate reads

$$\int_\Omega |\nabla P(\psi(x))|\, dx \leq 2\left(P_{max}\int_\Omega -\Delta\psi\, dx\right)^{1/2}\left(\int_\Omega P'(\psi(x))\, dx\right)^{1/2} \tag{1.4}$$

provided that $-\Delta\psi \geq 0$ in Ω and $\psi = c$ on $\partial\Omega$, and that $P' \geq 0$ with $P(c) = 0$, where c is a constant and P_{max} is the maximum of $P(\psi)$ over Ω. We prove this estimate by using the coarea formula.[8,9] In section 2 we prove (1.4) and extend it for discontinuous P. In this case the meaning of the equation $-\Delta\psi = P'(\psi)$ is not clear. We shall give a meaning for discontinuous P in section 3.

2. An interpolation inequality

Our goal in this section is to estimate the total variation of $\nabla(P(\psi))$ (as a vector-valued measure), where P is monotone and $-\Delta\psi \geq 0$. We first derive the estimate for smooth ψ.

Theorem 2.1. *Let Ω be a bounded domain in $\mathbf{R}^n$ and c be a constant. Suppose that $P \in C^1(\mathbf{R})$ with $P' \geq 0$ and $P(c) = 0$, and that $\psi \in C^m(\Omega) \cap C^0(\overline{\Omega})$ with*

$$-\Delta\psi \geq 0 \quad \textit{in}\ \Omega, \tag{2.1}$$

$$\psi = c \qquad \textit{on}\ \partial\Omega,$$

where $m \geq 2$ *and* $m \geq n$. *Let* P_{max} *denote*

$$P_{max} = \sup_{x\in\Omega} P(\psi(x)) \ . \tag{2.2}$$

Then

$$\int_\Omega |\ \nabla P(\psi(x))\ |\ dx \ \leq 2\left(P_{max}\int_\Omega (-\Delta\psi)\ dx\right)^{1/2}\left(\int_\Omega P'(\psi(x))\,dx\right)^{1/2} . \tag{2.3}$$

Proof. If $-\Delta\psi \equiv 0$, then $\psi \equiv c$ on Ω, so (2.3) holds with zero for both sides. If $P'(\psi) \equiv 0$ on Ω or $P_{max} = 0$, then either $\psi \equiv c$ or $P \equiv 0$. Again (2.3) holds in this case, so we may assume that both integrals in the right hand side of (2.3) is nonzero. We may also assume that the L^1 norm of $-\Delta\psi$ is finite.

For $K > 0$ denote the set of $x \in \Omega$ for which $|\ \nabla\psi(x)\ | > K$ by D. Let E denote the complement of D in Ω. From the definition it follows that

$$\int_E |\ \nabla P(\psi(x))\ |\ dx = \int_E P'(\psi)|\nabla\psi|\ dx$$

$$\leq K\int_E P'(\psi)\,dx \leq K\int_\Omega P'(\psi)\,dx\,, \tag{2.4}$$

since $P' \geq 0$.

By the maximum principle to (2.1), we observe that $\psi \geq c$ on Ω so $0 = P(c) \leq P(\psi) \leq P_{max}$ on Ω. Applying the coarea formula (see e.g. Ref. 8 and 9) yields

$$\int_D |\ \nabla P(\psi)\ |\ dx = \int_{-\infty}^{+\infty} \mathcal{H}^{n-1}(S_t)\,P'(t)\,dt = \int_c^{\psi_{max}} \mathcal{H}^{n-1}(S_t)\,P'(t)\,dt \tag{2.5}$$

with

$$S_t = D \cap L_t, \quad L_t = \{\, x \in \Omega;\ \psi(x) = t \,\}, \qquad \psi_{max} = \sup_{x \in \Omega} \psi(x) \ ,$$

where $\mathcal{H}^{n-1}$ denotes the $n-1$ dimensional Hausdorff measure. Since $|\nabla\psi| > K$ on D it follows that

$$\begin{aligned}\mathcal{H}^{n-1}(S_t) &= \int_{S_t} |\nabla\psi|\, |\nabla\psi|^{-1}\, d\mathcal{H}^{n-1} \\ &\le K^{-1} \int_{L_t} |\nabla\psi|\, d\mathcal{H}^{n-1} \ .\end{aligned}$$

Since $\psi \in C^n(\Omega)$, Sard's theorem[10] implies that L_t is C^n submanifold in Ω for almost every t (a.e. t). Note that $\psi > c$ in Ω and $\psi = c$ on $\partial\Omega$. Thus for $U_t = \{x \in \Omega\ ; \psi(x) > t\,\}$ we observe $\overline{U_t} \subset \Omega$ for $t > c$. For a.e. $t > c$, L_t is C^n boundary of U_t. Since L_t is t-level set of ψ, $\boldsymbol{n} = \nabla\psi / |\nabla\psi|$ is a unit normal vector field. Applying Green's formula yields

$$\int_{L_t} |\nabla\psi|\, d\mathcal{H}^{n-1} = \int_{L_t} \nabla\psi \cdot \boldsymbol{n}\, d\mathcal{H}^{n-1} = \int_{U_t} (-\Delta\psi)\, dx, \quad t > c \,.$$

From $-\Delta\psi \ge 0$ it now follows that

$$\int_{L_t} |\nabla\psi|\, d\mathcal{H}^{n-1} \le \int_{\Omega} (-\Delta\psi)\, dx \,.$$

Wrapping up these two estimates we obtain

$$\mathcal{H}^{n-1}(S_t) \le K^{-1} \int_{\Omega} (-\Delta\psi)\, dx \,.$$

Applying this estimate to (2.5) yields

$$\int_D \left|\nabla P(\psi)\right| dx \ \leq K^{-1} P_{max} \int_\Omega (-\Delta\psi)\, dx\,, \tag{2.6}$$

where P_{max} is defined in (2.2). Summing (2.4) and (2.6) we obtain

$$\int_\Omega \left|\nabla P(\psi)\right| dx \ \leq K \int_\Omega P'(\psi)\, dx \ + K^{-1} P_{max} \int_\Omega (-\Delta\psi)\, dx \tag{2.7}$$

for arbitrary $K > 0$. Taking

$$K = \left[P_{max} \int_\Omega (-\Delta\psi)\, dx \ / \int_\Omega P'(\psi)\, dx \right]^{1/2}$$

in (2.7) yields (2.3).

Q.E.D.

If ψ is not C^2, one should interpret $-\Delta\psi \geq 0$ in the distribution sense. As well known[11] a nonnegative distribution is a nonnegative Radon measure. Let μ be a finite Radon measure on a bounded domain Ω in $\mathbf{R}^n$. The unique solvability of the Dirichlet problem

$$-\Delta\psi = \mu \quad \text{in } \Omega, \tag{2.8a}$$

$$\psi = c \quad \text{on } \partial\Omega \ \ (c\text{: constant}) \tag{2.8b}$$

is now well known for smooth boundary $\partial\Omega$. We solve this problem by using a result of Simader[12] when the boundary is C^1. Let $W^{1,q}(\Omega)$ denote the L^q Sobolev space of order one ($1 < q < \infty$). Let $W_0^{1,q}(\Omega)$ be the subspace $\{u \in W^{1,q}(\Omega); u = 0 \text{ on } \partial\Omega\}$. We denote by $W^{-1,q}(\Omega)$ the dual space of $W_0^{1,q'}(\Omega)$ where $1/q = 1 - 1/q'$.

Lemma 2.2 (Theorem 4.6 of Simader[12]). *Let Ω be a bounded domain with C^1 boundary in $\mathbf{R}^n$. Assume that $1 < q < \infty$. For each $f \in W^{-1,q}(\Omega)$ there is a unique solution $\Phi \in W_0^{1,q}(\Omega)$ for $-\Delta\Phi = f$ in Ω. Moreover the mapping from f to Φ is bounded linear from $W_0^{1,q}(\Omega)$ to $W^{-1,q}(\Omega)$, i.e.,*

$$\|\Phi\|_{1,q} \le C\|f\|_{-1,q} \tag{2.9}$$

with a constant $C = C(\Omega, q, n)$.

Corollary 2.3. *Let* Ω *be a bounded domain with* C^1 *boundary in* $\mathbf{R}^n$. *For a finite Radon measure* μ *on* Ω *there is a unique solution* ψ *of* (2.8a, b) *such that* $\psi \in W^{1,r}(\Omega)$ *for* $1 < r < n/(n-1)$.

Proof. Observe that $r' > n$ implies $W_0^{1,r'}(\Omega) \subset C(\overline{\Omega})$ by the Sobolev inequality. This yields $\mu \in W^{-1,r}(\Omega)$ by a duality, where $1/r = 1 - 1/r'$. Applying Lemma 2.2 with $f = \mu$ obtains a unique solution ψ by $\psi = \Phi + c$.

Q.E.D.

Theorem 2.4. *Let* Ω *be a bounded domain with* C^1 *boundary in* $\mathbf{R}^n$. *Let* c *be a constant. Suppose that* $P \in C^1(\mathbf{R})$ *with* $P' \ge 0$ *and* $P(c) = 0$. *Suppose that* $\psi \in W^{1,r}(\Omega)$ *for some* r *such that* $1 < r < n/(n-1)$, *and that* ψ *satisfies*

$$-\Delta\psi \ge 0 \quad \text{in } \Omega \text{ (in the distribution sense)},$$
$$\psi = c \quad \text{on } \partial\Omega.$$

Let ψ_{max} *be the essential supremum of* ψ *over* Ω. *Assume that* P *and* P' *are bounded on* $[c, \psi_{max})$. *Then*

$$\int_\Omega |\nabla P(\psi(x))|\,dx \le 2\left(P_{max}\|-\Delta\psi\|_1\right)^{1/2}\left(\int_\Omega P'(\psi(x))\,dx\right)^{1/2}, \tag{2.10}$$

where $P_{max} = \sup\{P(\sigma); c \le \sigma \le \psi_{max}\}$ *and* $\|\cdot\|_1$ *denotes the total variation of a measure on* Ω.

For the proof of this Theorem, the reader is referred to Ref. 13.

We next extend the inequality (2.9) when a nondecreasing function P is not necessarily continuous. Let us give an interpretation of each integral appeared in (2.9). Instead of the integral $\int_\Omega P'(\psi)\,dx$, we consider

$$\left[P'(\psi)\right] = \inf \varliminf_{l\to\infty} \int_\Omega P_l'(\psi)\, dx .$$

Here the infimum is taken over all sequence $P_l \in C^1(\mathbf{R})$ with $P_l' \geq 0$ such that $P_l(\psi) \to P(\psi)$ in $L^s(\Omega)$ for some $1 \leq s < \infty$ as $l \to \infty$ and that $(P_l)_{max} \to \text{ess sup}_\Omega P(\psi)$. We say $\{P_l\}$ is an <u>admissible approximation</u> of P if these properties hold. If P is itself C^1 and satisfies the assumptions in Theorem 2.4, P itself is an admissible approximation so for such a P we have

$$\left[P'(\psi)\right] \leq \int_\Omega P'(\psi)\, dx .$$

Since $\int_\Omega |\nabla P(\psi)|\, dx$ is the total variation of $\nabla P(\psi)$ on Ω, i.e.

$$\|\nabla P(\psi)\|_1 = \int_\Omega |\nabla P(\psi(x))|\, dx$$

$$:= \sup \left\{ \int_\Omega P(\psi(x))\, \nabla\cdot\varphi(x)\, dx\, ;\ \varphi \in C_0^1(\Omega), |\varphi(x)| \leq 1 \text{ on } \Omega \right\},$$

it is easy to see

$$\|\nabla P(\psi)\|_1 \leq \varliminf_{l\to\infty} \int_\Omega |\nabla P_l(\psi)|\, dx$$

for any admissible approximation $\{P_l\}$ of P since sup $\varliminf \leq \varliminf$ sup. We have thus proved the following assertion.

Theorem 2.5. *Assume the hypotheses of Theorem 2.4 concerning c, Ω and ψ. Let P be a nondecreasing function on* $\mathbf{R}$ *with $P(c) = 0$. Then*

$$\|\nabla P(\psi)\|_1 \leq 2\left(P_{max}\|-\Delta\psi\|_1\right)^{1/2}\left[P'(\psi)\right]^{1/2} \tag{2.11}$$

provided that $P_{max} =$ ess $\sup_\Omega P(\psi)$ *is finite.*

Remark 2.6. If $P(\sigma)=\sigma$, the inequality (2.10) is an interpolation inequality

$$\| \nabla\psi \|_1 \le 2\left(P_{max}\|-\Delta\psi\|_1\right)^{1/2}|\Omega|^{1/2},$$

where $|\Omega|$ denotes the Lebesgue measure of Ω.

3. Weak solution of the Grad-Shafranov equation

We shall give a meaning of $-\Delta\psi = P'(\psi)$ when a nondecreasing function P is not continuous and ψ is not smooth.

Definition 3.1. Suppose that $\psi \in W^{1,r}(\Omega)$ for some r, $1<r<\infty$ and that P is nondecreasing. We say ψ and P satisfy

$$-\Delta\psi = P'(\psi) \quad \text{in } \Omega$$

if the following properties hold.

(i) $-\Delta\psi \ge 0$ on Ω in the distribution sense.

(ii) There is an admissible sequence $\{P_l\}$ such that

$$\lim_{l\to\infty} \int_\Omega \left(-\Delta\psi - P_l'(\psi)\right)\varphi \, dx = 0$$

for all $\varphi \in C(\overline{\Omega})$.

Theorem 3.2. *Let Ω be a bounded domain with C^1 boundary in $\mathbf{R}^n$. Let c be a constant. Assume that P is a nondecreasing function on $\mathbf{R}$. Assume that $\psi \in W^{1,r}(\Omega)$ for some r, $1<r<n/(n-1)$ and that ψ satisfies*

$$-\Delta\psi = P'(\psi) \quad \text{in } \Omega \ (\textit{in the sense of Definition 3.1})$$

$$\psi = c \qquad \text{on } \partial\Omega.$$

Then

$$\left\| \nabla P(\psi) \right\|_1 \leq 2\, P_{max}^{1/2} \mu_0 I, \tag{3.1}$$

where

$$I = \mu_0^{-1} \int_\Omega (-\Delta\psi)\, dx = \mu_0^{-1} \left\| -\Delta\psi \right\|_1 .$$

Proof. We may assume $P_{max} < \infty$. By Definition 3.1 (ii) with $\varphi \equiv 1$ we observe that

$$[P'(\psi)] \leq \lim_{l \to \infty} \int_\Omega P_l'(\psi)\, dx = \int_\Omega (-\Delta\psi)\, dx = \left\| -\Delta\psi \right\|_1$$

since $-\Delta\psi \geq 0$. The inequality (2.11) yields (3.1).

Q.E.D.

4. Discussions

In plasma physics, the poloidal beta ratio, which is define by

$$\beta = \int_\Omega p\, dx \,/\, (I^2\mu_0/8\pi) = 8\pi \int_\Omega P(\psi)\, dx \,/ \left(\int_\Omega (-\Delta\psi)\, dx \right)^2 ,$$

is an important quantity to characterize a plasma equilibrium. In the case of the space dimension $n = 2$, the Payne-Rayner inequality[14] applies to the estimate of β, and one finds $\beta \leq 1$. A general toroidal equilibrium problem includes two different effects; In the equilibrium equation (1.1), $-\Delta\psi$ should be replaced by a more complicated term including the toroidal curvature effect, and a new term should be added on the right-hand side, which represents the diamagnetic effect of the longitudinal magnetic field. Limitation of β in such a situation has been discussed by many authors, while no rigorous estimate of the bound have been given. Extension of the Payne-Rayner inequality will be discussed elsewhere to estimate the bound for β.

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