

FINITELY GENERATED SEMIGROUPS WITH SUCH A PRESENTATION THAT ALL THE CONGRUENCE CLASSES ARE CONTEXT-FREE LANGUAGES*

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Abstract In this paper, we investigate finitely generated semigroups with such a presentation that all the congruence classes are context-free languages.

A monoid M is called *finitely generated* if there exists a finite set of X and there exists a surjective homomorphism of X^* to M which maps an empty word onto the identity element of M .

1. Presentations of monoids

Definition 1 . (1) Let X be finite alphabets and R a subset of $X^* \times X^*$. Then R is string-rewriting system.

(2) For $u, v \in X^*$, $(w_1, w_2) \in R$, $uw_1v \Rightarrow_R uw_2v$.

The congruence μ_R on X^* generated by $\Rightarrow_R$ is the Thue congruence defined by R .

(3) A monoid M is (finitely) presented if there exists a (finite) set of X , there exists a surjective homomorphism ϕ of X^* to S and there exists a (finie) string-rewriting system R consisting of pairs of words over X such that the Thue congruence μ_R is the congruence $\{(w_1, w_2) \in X^* \times X^* \mid \phi(w_1) = \phi(w_2)\}$.

Definition 2 . A monoid M has a presentation with [finite, regular, contex-free] congruence classes if there exists a finite set X and there exists a surjective homomorphism ϕ of X^+ to M such that for each words $w \in X^+$, $\phi^{-1}(\phi(w))$ is a [finite, regular, contex-free] language.

*This is an absrtact and the paper will appear elsewhere.

2. Syntactic monoids of languages and finitely generated presented monoids

Definition 3 . Let A be finite alphabets and A^* the set of words over A . A subset L of A^* is called a language. The syntactic congruence σ_L on A^* is defined by $w\sigma_L w'$ ($w, w' \in A^*$) if and only if $\{(x, y) \in A^* \times A^* \mid xwy \in L\} = \{(x, y) \in A^* \times A^* \mid xw'y \in L\}$. Then a factor monoid A^*/σ_L is called the syntactic monoid of L .

Example 1 . $A = \{a_1, \dots, a_n\}$. For any $w = b_1b_2 \dots b_r$, let $w^R = b_r \dots b_2b_1$. Let $L = \{ww^R \mid w \in A^*\}$. Then

- (1) L is a context-free language which is not accepted by any deterministic pushdown automata.
 - (2) $\text{Syn}(L)$ is the free monoid A^* on A .
- That is, $\phi : A^* \rightarrow \text{Syn}(L) (w \mapsto \sigma_L w)$ is an isomorphism.

Definition 4 . Let M be a monoid and m an element of M . The syntactic congruence σ_m on M is defined by $s\sigma_m t$ ($s, t \in M$) if and only if $\{(x, y) \in M \times M \mid xsy = m\} = \{(x, y) \in M \times M \mid xty = m\}$.

The factor monoid M/σ_m is called the syntactic monoid of M at m .

Lemma 1. Let L be a language of X^* . Then L is a union of σ_L -classes in X^* .

Proposition 1. Let L be a language of A^* and L^c the complement of the set L in A^* . Then $\text{Syn}(L) = \text{Syn}(L^c)$.

Theorem 1 . Let L be a language of X^* . Then the following are equivalent :

- (1) L is a σ_L -class in X^* .
- (2) $xLy \cap L \neq \emptyset ((x, y \in X^*) \Rightarrow xLy \subseteq L$.
- (3) L is an inverse image $\phi^{-1}(m)$ of a homomorphism ϕ of X^* to a monoid M .

Theorem 2 . (Shoji [S]) Let M be a finitely generated monoid and ϕ a surjective homomorphism of A^* to M . For m an element of M , let $L = \phi^{-1}(m)$.

Then the syntactic monoid $\text{Syn}(L) = A^*/\sigma_L$ of L is isomorphic to the syntactic monoid M/σ_m of M at m .

3. Finitely generated semigroups with such a presentation that all the congruence classes are context-free languages

Theorem 3 . (Shoji [S]) *A finitely generated semigroup S has a presentation with regular congruence classes if and only if for any $s \in S$, S/σ_s is a finite semigroup.*

Theorem 4 . (Shoji [S]) *Let S be a finitely generated semigroup.*

Then S has a presentation with finite congruence classes if and only if the following are satisfied :

- (1) *S has no idempotent.*
- (2) *For any $s \in S$, S/σ_s is a finite nilpotent semigroup with a zero element 0.*

Example 2 . *Let $A = \{a, b\}$ and a context-free language $L = \{a^n b^n, b^n a^n | n \in \mathbb{N}\}$. Then all of σ_L -classes are $\{1\}$, $\{ab\}$, $\{a^n\}$, $\{b^n\}$, $c_n = \{a^{p+n} b^p | p \in \mathbb{N}\}$, $d_n = \{a^q b^{q+n} | q \in \mathbb{N}\}$, $\{ba\}$, $e_n = \{b^p a^{p+n} | p \in \mathbb{N}\}$, $f_n = \{b^{q+n} a^q | q \in \mathbb{N}\}$. Hence $\text{Syn}(L)$ has a regular cross-section. Also, $\text{Syn}(L) - \{1\}$ is a $\mathcal{D}$ -class. $\text{Syn}(L)$ has a representation with context-free congruence classes.*

Example 3 . *Let $A = \{a, b\}$ and $G : S \rightarrow SSS|aSb|\epsilon$. Then G is a context-free grammar and its accepted language $L(G)$ equals to $\{a^n b^n | n \geq 0\}$.*

The syntactic monoid $\text{Syn}(L(G))$ has the presentation $A^/\{ab = 1\}$. It is easily seen that $\text{Syn}(L(G))$ has a representation with context-free congruence classes.*

Example 4 . *Let $A = \{a_1, \dots, a_r\} \cup \{b_1, \dots, b_r\}$ and $F(A)$ the free inverse semigroup over A . Then there exists the canonical map $\phi : A^* \rightarrow F(A)$ ($b_i \mapsto a_i^{-1}$) such that for each $w \in F(A)$, $\phi^{-1}(w)$ is not a context-free language. Thus, Free inverse semigroups do not have a representation with context-free congruence classes.*

Remark. Even a monogenic free inverse smigroup do not have any representation with context-free congruence classes.

Result 1 . *For every finitely generated group G , there exists a language L of A^* such that G is isomorphic to $\text{Syn}(L)$.*

Result 2 . (Muller and Schupp [MS]) (1) *Every finitely generated virtually free group G has a (monoid)-representation with context-free congruence classes.*

(2) *Conversely, if a finitely generated group G has a (monoid)- representation with context-free congruence classes then G is a virtually free group.*

Theorem 5 . *Let S be a semigroup having a representation with context-free congruence classes. If S is a completely (0-) simple semigroup, then both the $\mathcal{L}$ -classes and the $\mathcal{R}$ -classes of S is finite and the maximal subgroup is virtually free.*

Theorem 6 . *Let S be a finitely generated submonoids of a virtually free group G . Then S is a cancellative monoid having a representation with context-free congruence classes.*

Example 5 . *Let A be finite alphabets containing $\{a, b, c\}$. Let $R = \{(acb, c)\}$ be a string-rewriting system on A^* . The monoid $M = A^*/\mu_R$ has a representation with context-free congruence classes. Moreover, M is a cancellative monoid which is embedded in a group $G = \langle a, b, c \mid c^{-1}ac = b^{-1} \rangle$ which is not virtually context-free.*

Theorem 7 . *Let M_1, M_2 be a finitely generated monoids having a presentation with context-free congruence classes. Then the free product $M_1 * M_2$ of M_1, M_2 has a presentation with context-free congruence classes.*

References

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