

Harmonic Relations between Green's Functions and Green's Matrices for Boundary Value Problems II

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1 Introduction

In previous papers [3], [4], we remarked that there is a harmonic relation between the Green functions $G(x, \xi)$ for

$$\begin{cases} -\frac{d}{dx}(p(x)\frac{du}{dx}) = f(x), & a < x < b \\ u(a) = u(b) = 0, & p(x) > 0 \text{ in } [a, b] \end{cases} \quad (1.1)$$

and the Green matrix $A_0^{-1} = (g_{ij})$ for the discretized system

$$\begin{cases} a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b, & x_{i+\frac{1}{2}} = \frac{1}{2}(x_i + x_{i+1}) \\ -\frac{p_{i+\frac{1}{2}} \frac{U_{i+1}-U_i}{h_{i+1}} - p_{i-\frac{1}{2}} \frac{U_i-U_{i-1}}{h_i}}{\frac{h_{i+1}+h_i}{2}} = f_i, & h_i = x_i - x_{i-1}, \quad i = 1, 2, \dots, n \\ U_0 = U_{n+1} = 0 \end{cases} \quad (1.2)$$

or

$$HA_0U = f$$

with

$$H = \begin{pmatrix} \frac{2}{h_1+h_2} & & \\ & \ddots & \\ & & \frac{2}{h_n+h_{n+1}} \end{pmatrix},$$

$$A_0 = \begin{pmatrix} a_1 + a_2 & -a_2 & & \\ -a_2 & a_2 + a_3 & -a_3 & \\ & \ddots & \ddots & \ddots \\ & & -a_n & a_n + a_{n+1} \end{pmatrix}, \quad a_i = \frac{1}{h_i} p_{i-\frac{1}{2}} \quad (1.4)$$

$$U = (U_1, \dots, U_n)^t, f = (f_1, \dots, f_n)^t$$

In fact, we have

$$G(x_i, x_j) = \begin{cases} \left(\int_a^b \frac{ds}{p(s)} \right)^{-1} \int_a^x \frac{ds}{p(s)} \int_\xi^b \frac{ds}{p(s)} & (x \leq \xi) \\ \left(\int_a^b \frac{ds}{p(s)} \right)^{-1} \int_a^\xi \frac{ds}{p(s)} \int_x^b \frac{ds}{p(s)} & (x \geq \xi) \end{cases}$$

and

$$g_{ij} = \begin{cases} \left(\sum_{k=1}^{n+1} \frac{h_k}{p_{k-\frac{1}{2}}} \right)^{-1} \left(\sum_{k=1}^i \frac{h_k}{p_{k-\frac{1}{2}}} \right) \left(\sum_{k=j+1}^{n+1} \frac{h_k}{p_{k-\frac{1}{2}}} \right) & i \leq j \\ \left(\sum_{k=1}^{n+1} \frac{h_k}{p_{k-\frac{1}{2}}} \right)^{-1} \left(\sum_{k=1}^j \frac{h_k}{p_{k-\frac{1}{2}}} \right) \left(\sum_{k=i+1}^{n+1} \frac{h_k}{p_{k-\frac{1}{2}}} \right) & i \geq j. \end{cases}$$

Hence,

$$G(x_i, x_j) = g_{ij} + O(h^2) \quad \forall i, j, \quad h = \max_i h_i,$$

if $p \in C^{1,1}[a, b]$.

On the other hand, the finite element approximation $v_n = \sum_{i=1}^n \hat{U}_i \phi_i$ with piecewise linear polynomials is determined by solving

$$\sum_{j=1}^n \left(\int_a^b p(x) \phi_i' \phi_j' dx \right) \hat{U}_j = \int_a^b f(x) \phi_i(x) dx, \quad i = 1, 2, \dots, n \quad (1.5)$$

with respect to $\hat{U}_j$, where $\phi_i, i = 1, 2, \dots, n$ are piecewise linear polynomials satisfying $\phi_i(x_j) = \delta_{ij}$. The equations(1.5) can be written in the matrix-vector form

$$\hat{\hat{A}} \hat{U} = \hat{f},$$

where $\hat{\hat{A}} = \hat{\hat{A}}_0$ is obtained by replacing a_i in (1.4) by

$$\begin{aligned} \hat{a}_i &= \frac{1}{h_i} \rho_i, & \rho_i &= \frac{1}{h_i} \int_{x_{i-1}}^{x_i} p(x) dx, \\ \hat{f} &= (\hat{f}_1, \dots, \hat{f}_n), & \hat{f}_i &= \int_{x_{i-1}}^{x_{i+1}} f(x) \phi_i(x) dx. \end{aligned}$$

Then it can also be shown that $\hat{A}^{-1} = (\hat{g}_{ij})$ satisfies

$$\hat{g}_{ij} = \begin{cases} \left(\sum_{k=1}^{n+1} \frac{h_k}{\rho_k} \right)^{-1} \left(\sum_{k=1}^i \frac{h_k}{\rho_k} \right) \left(\sum_{k=j+1}^{n+1} \frac{h_k}{\rho_k} \right) & i \leq j \\ \left(\sum_{k=1}^{n+1} \frac{h_k}{\rho_k} \right)^{-1} \left(\sum_{k=1}^j \frac{h_k}{\rho_k} \right) \left(\sum_{k=i+1}^{n+1} \frac{h_k}{\rho_k} \right) & i \geq j, \end{cases}$$

which indicates a similar harmony between the Green function $G(x, \xi)$ and the corresponding discrete Green function:

$$G(x_i, x_j) = \hat{g}_{ij} + O(h^2) \quad \forall i, j,$$

if $p \in C^{1,1}[a, b]$.

The purpose of this paper is to establish a similar relation for the Green function $G(x, \xi)$ for

$$Lu \equiv -\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)\frac{du}{dx} + r(x)u = f(x), \quad a < x < b \quad (1.6)$$

$$u(a) = u(b) = 0$$

and the discrete Green function $G_h(x_i, x_j)$ (Green matrix) for the discretized system

$$\begin{cases} L_h U \equiv -\frac{p_{i+\frac{1}{2}} \frac{U_{i+1}-U_i}{h_{i+1}} - p_{i-\frac{1}{2}} \frac{U_i-U_{i-1}}{h_i}}{\frac{h_{i+1}+h_i}{2}} + q_i \frac{U_{i+1}-U_{i-1}}{h_{i+1}+h_i} + r_i U_i = f_i, & i = 1, 2, \dots, n \\ U_0 = U_{n+1} = 0, \end{cases} \quad (1.7)$$

provided that $p(x) \in C^{3,1}$, $q(x), r(x) \in C^{1,1}[a, b]$, $p(x) > 0$, $r(x) \geq 0$ in $[a, b]$

2 Results

The discrete Green function $G_h(x_i, x_j)$ for the operator L_h is defined as the solution of the linear system

$$\begin{cases} L_h G_h(x_i, x_j) = \frac{2}{h_{j+1}+h_j} \delta_{ij}, & i, j = 1, 2, \dots, n \\ G_h(x_i, x_j) = 0, & i = 0, n+1, \quad 1 \leq j \leq n, \end{cases}$$

where δ_{ij} stands for the Kronecker symbol. This means that the $n \times n$ matrix $(G_h(x_i, x_j))$ is the inverse of the matrix $A = A_0 + Q + D$, where A_0 is defined by (1,3),

$$Q = \begin{pmatrix} 0 & & 0 \\ -\frac{q_2}{2} & \ddots & \frac{q_2}{2} \\ & \ddots & \\ & & \ddots & \frac{q_{n-1}}{2} \\ & & & -\frac{q_n}{2} & 0 \end{pmatrix}, \quad D = \begin{pmatrix} \frac{r_1(h_1+h_2)}{2} & & & & \\ & \frac{r_2(h_2+h_3)}{2} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \frac{r_n(h_n+h_{n+1})}{2} \end{pmatrix}$$

We first prove the following lemma:

Lemma 2.1. *Given positive integers N_a and N_b , we have*

$$\sum_{j=1}^n G_h(x_i, x_j) \frac{h_j + h_{j+1}}{2} = \begin{cases} O(h) & \text{if } i \leq N_a \text{ or } i \geq n+1 - N_b, \\ O(1) & \text{otherwise.} \end{cases}$$

Proof. Let $\phi(x) \in C^2[a, b]$ be the solution of the problem $Lu = 1$ in (a, b) and $u(a) = u(b) = 0$. Then we have

$$\sum_{j=1}^n G_h(x_i, x_j) \frac{h_j + h_{j+1}}{2} \leq 2\phi(x_i) \quad \forall i$$

(cf. Matsunaga-Yamamoto[2]), which proves Lemma 2.1. □

Then we have the following result.

Theorem 2.2. *If $p \in C^{3,1}[a, b]$, $q(x), r(x) \in C^{1,1}[a, b]$, $p(x) > 0$, $r(x) \geq 0$ in $[a, b]$, then*

$$G_h(x_i, x_j) - G(x_i, x_j) = \begin{cases} O(h^3) & (i \leq N_a \text{ or } i \geq n+1 - N_b), \\ O(h^2) & (\text{otherwise}). \end{cases}$$

Proof. Let $\{V_i\}$ be any mesh function defined on $I = \{x_0, x_1, \dots, x_n, x_{n+1}\}$ such that $V_0 = V_{n+1} = 0$. Then it is easy to see

$$V_i = \sum_{j=1}^n G_h(x_i, x_j) \frac{h_j + h_{j+1}}{2} L_h V_j, \quad i = 1, 2, \dots, n$$

Hence

$$G(x_i, x_j) = \sum_{k=1}^n G_h(x_i, x_k) \frac{h_k + h_{k+1}}{2} L_h G(x_k, x_j), \quad i, j = 1, 2, \dots, n \quad (2.1)$$

Furthermore, a careful computation leads to

$$L_h G(x_k, x_j) = \begin{cases} \frac{2}{h_k + h_{k+1}} [(h_k^2 - h_{k+1}^2) \phi_{kj} + O(h_k^3 + h_{k+1}^3)] & (k \neq j) \\ \frac{2}{h_j + h_{j+1}} [1 + (h_{j+1}^2 \phi_j^+ - h_j^2 \phi_j^-) + O(h_j^3 + h_{j+1}^3)] & (k = j), \end{cases}$$

where

$$\begin{aligned} \phi_{kj} &= \frac{1}{6} p_k \frac{\partial^3 G(x_k, x_j)}{\partial x^3} + \frac{1}{4} (p'_k - q_k) \frac{\partial^2 G(x_k, x_j)}{\partial x^2} + \frac{1}{8} p''_k \frac{\partial G(x_k, x_j)}{\partial x}, \\ \phi_j^+ &= \frac{1}{4} q_j \frac{\partial^2 G(x_j + 0, x_j)}{\partial x^2} - \frac{1}{8} p''_j \frac{\partial G(x_j + 0, x_j)}{\partial x} - \frac{1}{4} p'_j \frac{\partial^2 G(x_j + 0, x_j)}{\partial x^2} \\ &\quad - \frac{1}{6} p_j \frac{\partial^3 G(x_j + 0, x_j)}{\partial x^3}, \\ \phi_j^- &= \frac{1}{4} q_j \frac{\partial^2 G(x_j - 0, x_j)}{\partial x^2} - \frac{1}{8} p''_j \frac{\partial G(x_j - 0, x_j)}{\partial x} - \frac{1}{4} p'_j \frac{\partial^2 G(x_j - 0, x_j)}{\partial x^2} \\ &\quad - \frac{1}{6} p_j \frac{\partial^3 G(x_j - 0, x_j)}{\partial x^3}. \end{aligned}$$

Substituting this relation into (2.1) yields

$$\begin{aligned} G(x_i, x_j) &= \sum_{\substack{k=1 \\ k \neq j}}^n G_h(x_i, x_k) \{ (h_k^2 - h_{k+1}^2) \phi_{kj} + O(h_{k+1}^3 + h_k^3) \} \\ &\quad + G_h(x_i, x_j) \{ 1 + (\phi_j^+ h_{j+1}^2 - \phi_j^- h_j^2) + O(h_{j+1}^3 + h_j^3) \} \end{aligned}$$

or

$$\begin{aligned} G(x_i, x_j) - G_h(x_i, x_j) &= \sum_{k \neq j} G_h(x_i, x_k) \{ (h_k^2 - h_{k+1}^2) \phi_{kj} + O(h_{k+1}^3 + h_k^3) \} \\ &\quad + G_h(x_i, x_j) \{ (\phi_j^+ h_{j+1}^2 - \phi_j^- h_j^2) + O(h_{j+1}^3 + h_j^3) \} \end{aligned} \quad (2.2)$$

Hence there exists a constant $C_1 > 0$ such that

$$|G(x_i, x_j) - G_h(x_i, x_j)| \leq C_1 h \sum_{\substack{k \neq j \\ k=1}}^n G_h(x_i, x_k) (h_k + h_{k+1}) + O(h^2),$$

and, by Lemma 2.1, we have

$$G_h(x_i, x_j) = G(x_i, x_j) + O(h). \quad (2.3)$$

Substituting this into (2.2) we have

$$\begin{aligned}
G(x_i, x_j) - G_h(x_i, x_j) &= \sum_{k \neq j} G(x_i, x_k)(h_k^2 - h_{k+1}^2)\phi_{kj} + G(x_i, x_j)(\phi_j^+ h_{j+1}^2 - \phi_j^- h_j^2) + O(h^2) \\
&= h_1^2 G(x_i, x_1)\phi_{1j} + \sum_{k=1}^{j-2} h_{k+1}^2 [G(x_i, x_{k+1})\phi_{k+1j} - G(x_i, x_k)\phi_{kj}] \\
&\quad - h_j^2 [G(x_i, x_{j-1})\phi_{j-1j} + G(x_i, x_j)\phi_j^-] \\
&\quad + h_{j+1}^2 [G(x_i, x_j)\phi_j^+ + G(x_i, x_{j+1})\phi_{j+1j}] \\
&\quad + \sum_{k=j+1}^{n-1} h_{k+1}^2 [G(x_i, x_{k+1})\phi_{k+1j} - G(x_i, x_k)\phi_{kj}] \\
&\quad - h_{n+1}^2 [G(x_i, x_n)\phi_{nj}] + O(h^2) \\
&= O(h^2) \quad (\text{an improvement of (2.3)})
\end{aligned}$$

since

$$\begin{aligned}
&G(x_i, x_{k+1})\phi_{k+1j} - G(x_i, x_k)\phi_{kj} \\
&= [G(x_i, x_{k+1}) - G(x_i, x_k)]\phi_{k+1j} + G(x_i, x_k)[\phi_{k+1j} - \phi_{kj}] \\
&= O(h_{k+1})\phi_{k+1j} + G(x_i, x_k)h_{k+1} = O(h_{k+1}), \quad \text{etc.}
\end{aligned}$$

Replacing $O(h)$ in (2.3) by $O(h^2)$ and repeating similar argument as above, we obtain for $i \leq N_a$ or $i \geq n+1 - N_b$

$$G(x_i, x_j) - G_h(x_i, x_j) = O(h^3).$$

This proves Theorem 2.2. □

We can apply Theorem 2.2 to derive the superconvergence of the Shortley-Weller approximation applied to the semilinear problem

$$\begin{cases} -\frac{d}{dx}(p(x)\frac{du}{dx}) + q(x)\frac{du}{dx} + f(x, u) = 0, & a < x < b \end{cases} \quad (2.4)$$

$$\begin{cases} u(a) = \alpha, & u(b) = \beta \end{cases} \quad (2.5)$$

with any nodes (1.2):

Theorem 2.3. *In addition to the assumptions of Theorem 2.2, assume that f is continuous on $\mathcal{R} : a \leq x \leq b$, $-\infty < u < +\infty$. Furthermore, assume that f is continuously differentiable with respect to u on $\mathcal{R}$ and $f_u \geq 0$. Then the finite difference method*

$$\begin{cases} -\frac{p_{i+\frac{1}{2}} \frac{U_{i+1}-U_i}{h_{i+1}} - p_{i-\frac{1}{2}} \frac{U_i-U_{i-1}}{h_i}}{\frac{h_{i+1}+h_i}{2}} + q_i \frac{U_{i+1}-U_{i-1}}{h_{i+1}+h_i} + f(x_i, U_i) = 0, & i = 1, 2, \dots, n, \\ U_0 = \alpha, & U_{n+1} = \beta \end{cases} \quad (2.6)$$

for solving (2.4)–(2.5) is superconvergent with any nodes (1.2):

$$u_i - U_i = \begin{cases} O(h^3), & i \in \Gamma = \{1, 2, \dots, N_a, n - N_b + 1, n - N_b + 2, \dots, n\} \\ O(h^2), & i \notin \Gamma \end{cases}$$

as $h \rightarrow 0$, where N_a and N_b are arbitrary given positive integers.

Remark. If the boundary conditions (2.5) are replaced by

$$\alpha_1 u(a) + \alpha_2 u'(a) = \alpha \quad \text{and} \quad \beta_1 u(b) + \beta_2 u'(b) = \beta,$$

where $\alpha_2 \beta_2 \neq 0$, $\alpha_1 \alpha_2 \geq 0$ and $\beta_1 \beta_2 \geq 0$, then it can be shown that the corresponding Shortley-Weller approximation (2.6) is quadratic convergent with any nodes (1.2):

$$u_i - U_i = O(h^2) \quad \forall i$$

as $h \rightarrow 0$. However, superconvergence can not be expected in general.

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