

An infinitesimal analysis in topology

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1° INTRODUCTION

Algebraic topology is the study of functors from the category $\mathcal{T}op$ of topological spaces into some algebraic categories.

Take, for example, functor $H^*(\)$ of the singular cohomology in coefficient K a coomutative ring. Given a diagram in $\mathcal{T}op$

$$\begin{array}{ccc} & Z & \\ f \uparrow & & \\ X & \hookrightarrow & Y. \end{array}$$

If f extends to Y , one will get in the category $\mathcal{Alg}_K$ of K -algebras a commutative diagram

$$\begin{array}{ccc} H^*(Z) & & \\ H^*(f) \downarrow & \searrow & \\ H^*(X) & \longleftarrow & H^*(Y). \end{array}$$

Further, if $\tilde{f}$ denotes an extension of f , then $H^*(\tilde{f})$ must commute with all the natural transformations, called cohomology operations, from $H^*(\)$ into itself, hence if lucky, one can sometimes decide if an extension exists or not by algebraic means.

If K is a field of positive characteristic, we have a well-known set of cohomology operations constituting a skew graded Hopf algebra $\mathcal{A}$, called the Steenrod algebra. One thus wishes to study the algebra $\mathcal{A}$ and the $\mathcal{A}$ -module structures of $H^*(X)$.

This is a survey to introduce an attempt [KSTY] to throw a new light on the Steenrod algebra using infinitesimal unipotent K -groups.

For simplicity we will fix $K = \mathbb{F}_p$, p odd prime, in what follows.

2° THE STEENROD ALGEBRA

(2.1) The Bockstein operator β is a natural map

$$H^n(X) \longrightarrow H^{n+1}(X) \quad \forall n \in \mathbb{N},$$

induced by the short exact sequence $0 \rightarrow \mathbb{F}_p \rightarrow \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{p} \mathbb{F}_p \rightarrow 0$ such that

$$(1) \quad \beta^2 = 0$$

and

$$(2) \quad \beta(xy) = (\beta x)y + (-1)^n x(\beta y) \quad \forall x \in H^n(X), y \in H^m(X),$$

where the multiplication is the cup product $H^*(X) \otimes_K H^*(X) \rightarrow H^*(X)$ induced by the diagonal mapping $X \rightarrow X \times X$.

Further, one has unique natural maps [SE],(VI.1)

$$p^i : H^n(X) \longrightarrow H^{n+2i(p-1)}(X) \quad \forall i, n \in \mathbb{N},$$

called the Steenrod reduced powers, such that

$$(3) \quad p^0 = \text{id},$$

$$(4) \quad p^i = \begin{cases} x^p & \text{if } x \in H^{2i}(X) \\ 0 & \text{if } x \in H^j(X) \text{ with } j < 2i, \end{cases}$$

(5) (Cartan formula)

$$p^i(xy) = \sum_{j=0}^i p^j(x) p^{i-j}(y) \quad \forall x \in H^n(X), y \in H^m(X)$$

and

(6) (Adem relations)

$$\begin{aligned} \forall a < pb, \quad p^a p^b &= \sum_{t=0}^{\left[\frac{a}{p}\right]} (-1)^{a+t} \binom{(p-1)(b-t)-1}{a-pt} p^{a+b-t} p^t; \\ \forall a \leq pb, \quad p^a \beta p^b &= \sum_{t=0}^{\left[\frac{a}{p}\right]} (-1)^{a+t} \binom{(p-1)(b-t)}{a-pt} \beta p^{a+b-t} p^t \\ &\quad + \sum_{t=0}^{\left[\frac{a-1}{p}\right]} (-1)^{a+t-1} \binom{(p-1)(b-t)-1}{a-pt-1} p^{a+b-t} \beta p^t. \end{aligned}$$

With those in mind we define the Steenrod algebra $\mathcal{A}$ to be

$$T_K(M)/(\mathfrak{B}^2, \text{Adem relations}),$$

where $T_K(M)$ is the tensor algebra over K of a K -linear space M with basis $\mathfrak{B}, \mathfrak{P}^i, i \in \mathbb{Z}^+$, corresponding to β, p^i , respectively. We assign $\mathfrak{B}$ (resp. $\mathfrak{P}^i$) degree 1 (resp. $2i(p-1)$), thus making $\mathcal{A}$ into a graded K -algebra. Put $\mathfrak{P}^0 = 1$.

(2.2) Instead of $\mathcal{A}$ itself we will consider below the graded quotient $S := \mathcal{A}/(\mathfrak{B})$. If $I = (i_1, \dots, i_k) \in \mathbb{N}^k, k \geq 1$, we set in $\mathcal{A}$

$$\mathfrak{P}^I = \mathfrak{P}^{i_1} \dots \mathfrak{P}^{i_k}.$$

By abuse of notations we will denote the image of $\mathfrak{P}^I$ in S by the same letter. We say I is admissible iff either $I = 0$ or

$$\forall \nu \in [1, k], \quad i_\nu \geq 1 \text{ and } i_\nu \geq p i_{\nu+1} \text{ with } i_{k+1} = 0$$

in which case we call $\mathfrak{P}^I$ an admissible monomial.

(2.3) THEOREM (Milnor[Mil]). We have

(i) The admissible monomials form a K -linear basis of S . In particular, each homogeneous part S_m is finite dimensional.

(ii) S is generated as K -algebra by $\mathfrak{P}^{p^i}, i \in \mathbb{N}$.

(iii) With comultiplication

$$\Delta_S : \mathfrak{P}^k \rightarrow \sum_{i=0}^k \mathfrak{P}^i \otimes \mathfrak{P}^{k-i}$$

and the counit $\mathfrak{P}^I \mapsto 0 \forall I$ admissible $\neq 0$, S forms a cocommutative graded bialgebra.

(2.4) If $A = \coprod_{i \geq 0} A_i$ is a graded K -bialgebra with $A_0 = K$, then A admits a unique antipode σ_A , making A into a graded Hopf algebra, due to R. Thom [MM], (8.7): if $a \in A_n$, one defines $\sigma_A(a)$ inductively on the degree of the elements by

$$\sigma_A(a) = -a - \sum_i \sigma_A(a_i) a'_i$$

if $\Delta_A(a) = 1 \otimes a + a \otimes 1 + \sum_i a_i \otimes a'_i$ with a_i and a'_i homogeneous of less degrees than a . In particular, S carries a structure of cocommutative graded Hopf algebra.

(2.5) Let $S^{*gr} = \coprod_{i \geq 0} S_i^*$ be the graded dual of S . Using the identification $(S \otimes_K S)^{*gr} \simeq S^{*gr} \otimes_K S^{*gr}$ via

$$(f \otimes g)(a \otimes b) = f(a)g(b),$$

S^{*gr} comes equipped with a structure of commutative graded Hopf algebra.

Let $I_k = (p^{k-1}, p^{k-2}, \dots, p^1, p^0), k \geq 1$, and $\xi_k \in S^{*gr}$ the dual of $\mathfrak{P}^{I_k}$ with respect to the basis of admissible monomials, so $\deg(\xi_k) = 2(p^k - 1)$.

(2.6) THEOREM (Milnor[Mil]). S^{*gr} is the polynomial algebra $K[\xi_1, \xi_2, \dots]$ in indeterminates $\xi_i, i \geq 1$, with the comultiplication

$$\xi_k \mapsto \sum_{i=0}^k \xi_{k-i}^{p^i} \otimes \xi_i$$

and the counit annihilating all ξ_i .

(2.7) Let $\mathcal{I}_k, k \geq 0$, be the ideal of S^{*gr} generated by

$$\xi_1^{p^k}, \xi_2^{p^{k-1}}, \dots, \xi_k^p, \xi_{k+j}, \quad j \geq 1.$$

Then $\mathcal{I}_k$ is a Hopf ideal, hence $S^{*gr}/\mathcal{I}_k \simeq K[\xi_1, \dots, \xi_k]/(\xi_1^{p^k}, \dots, \xi_k^p)$ is a finite dimensional graded Hopf algebra. In turn, its dual $\mathcal{S}(k)$ is a Hopf subalgebra of $\mathcal{S}$.

One can show that $\mathcal{S}(k)$ is generated as K -algebra by

$$\mathfrak{P}^{p^i}, \quad 0 \leq i \leq k-1,$$

hence $\mathcal{S} = \bigcup_{k \geq 1} \mathcal{S}(k)$ by (2.3). In [KSTY] $\mathcal{S}(k)$ is denoted by $P(k-1)$.

3° INFINITESIMAL UNIPOTENT GROUPS

(3.1) An (affine) K -group (scheme) $\mathfrak{G}$ is a representable functor from the category $K\mathcal{A}lg$ of commutative K -algebras into the category $\mathfrak{G}rp$ of groups: there is commutative Hopf K -algebra $K[\mathfrak{G}]$ such that

$$(1) \quad \mathfrak{G}(_) = K\mathcal{A}lg(K[\mathfrak{G}], _).$$

If $m_{\mathfrak{G}}$ (resp. $\Delta_{\mathfrak{G}}, \epsilon_{\mathfrak{G}}, \sigma_{\mathfrak{G}}$) is the multiplication (resp. comultiplication, counit, antipode) of $K[\mathfrak{G}]$, then for each $R \in K\mathcal{A}lg$, $\mathfrak{G}(R)$ is a group under the multiplication

$$(2) \quad \begin{array}{ccc} \mathfrak{G}(R) \times \mathfrak{G}(R) & \xrightarrow{\quad \quad} & \mathfrak{G}(R) \\ \downarrow \wr & \searrow \wr & \parallel \\ K\mathcal{A}lg(K[\mathfrak{G}] \otimes_K K[\mathfrak{G}], R) & \xrightarrow{K\mathcal{A}lg(\Delta_{\mathfrak{G}}, R)} & K\mathcal{A}lg(K[\mathfrak{G}], R) \end{array}$$

and inversion

$$(3) \quad \begin{array}{ccc} \mathfrak{G}(R) & \xrightarrow{\quad \quad} & \mathfrak{G}(R) \\ \parallel & \searrow \wr & \parallel \\ K\mathcal{A}lg(K[\mathfrak{G}], R) & \xrightarrow{K\mathcal{A}lg(\sigma_{\mathfrak{G}}, R)} & K\mathcal{A}lg(K[\mathfrak{G}], R) \end{array}$$

with the identity element defined by

$$(4) \quad \begin{array}{ccc} \epsilon_K(R) & \dashrightarrow & \mathfrak{G}(R) \\ \parallel & \wr & \parallel \\ K\mathfrak{Alg}(K, R) & \xrightarrow{K\mathfrak{Alg}(\epsilon_{\mathfrak{G}}, R)} & K\mathfrak{Alg}(K[\mathfrak{G}], R). \end{array}$$

A $\mathfrak{G}$ -module is a K -linear space M together with a map $\Delta_M : M \rightarrow M \otimes_K K[\mathfrak{G}]$, called a $K[\mathfrak{G}]$ -comodule map, such that for each $R \in K\mathfrak{Alg}$, the map $\mathfrak{G}(R) \times (M \otimes_K R) \rightarrow M \otimes_K R$ via

$$(5) \quad (x, m \otimes r) \mapsto ((M \otimes_K x) \circ \Delta_M(m))r = \sum_i m_i \otimes r x(a_i)$$

if $\Delta_M(m) = \sum_i m_i \otimes a_i$, makes $M \otimes_K R$ into a $\mathfrak{G}(R)$ -module over R .

(3.2) We say a K -group $\mathfrak{G}$ is algebraic iff the algebra $K[\mathfrak{G}]$ is of finite type over K . In this note we will consider only algebraic K -groups.

Let $\mathcal{I}_{\mathfrak{G}} = \ker(\epsilon_{\mathfrak{G}})$, called the augmentation ideal of the Hopf algebra $K[\mathfrak{G}]$, and set

$$(1) \quad \text{Dist}_m(\mathfrak{G}) = \{\mu \in K[\mathfrak{G}]^* \mid \mu(\mathcal{I}_{\mathfrak{G}}^{m+1}) = 0\}, \quad m \in \mathbb{N}.$$

Then $\text{Dist}(\mathfrak{G}) := \bigcup_{m \in \mathbb{N}} \text{Dist}_m(\mathfrak{G})$ carries a structure of cocommutative Hopf algebra, called the algebra of distributions of $\mathfrak{G}$, with the multiplication given by

$$(\mu\nu)(a) = (\mu \bar{\otimes} \nu) \circ \Delta_{\mathfrak{G}}(a) = \sum_i \mu(a_i) \nu(a'_i) \quad \text{if} \quad \Delta_{\mathfrak{G}}(a) = \sum_i a_i \otimes a'_i,$$

comultiplication $\Delta'_{\mathfrak{G}}$ such that $\Delta'_{\mathfrak{G}}(\mu)(a \otimes b) = \mu(ab)$, counit $\epsilon'_{\mathfrak{G}}$ such that $\epsilon'_{\mathfrak{G}}(\mu) = \mu(1)$, and the antipode $\sigma'_{\mathfrak{G}}$ such that $\sigma'_{\mathfrak{G}}(\mu) = \mu \circ \sigma_{\mathfrak{G}}$, using natural isomorphisms [J], (I.7.4)(2)

$$(2) \quad \begin{array}{ccc} \text{Dist}(\mathfrak{G}) \otimes_K \text{Dist}(\mathfrak{G}) & \xrightarrow{\sim} & \text{Dist}(\mathfrak{G} \times \mathfrak{G}) \\ \uparrow & & \uparrow \\ \sum_{i=0}^m \text{Dist}_i(\mathfrak{G}) \otimes_K \text{Dist}_{m-i}(\mathfrak{G}) & \xrightarrow{\sim} & \text{Dist}_m(\mathfrak{G} \times \mathfrak{G}). \end{array}$$

In particular,

$$(3) \quad \text{Dist}_1^+(\mathfrak{G}) := \{\mu \in \text{Dist}_1(\mathfrak{G}) \mid \mu(1) = 0\}$$

forms a Lie algebra over K , called the Lie algebra of $\mathfrak{G}$, with $[\mu, \nu] := \mu\nu - \nu\mu$.

Any $\mathfrak{G}$ -module M carries a structure of $\text{Dist}(\mathfrak{G})$ -module such that

$$(4) \quad \begin{aligned} \mu m &= (M \bar{\otimes}_K \mu) \circ \Delta_M(m) = \sum_i \mu(a_i) m_i \\ \forall \mu \in \text{Dist}(\mathfrak{G}) \text{ and } m \in M \text{ if } \Delta_M(m) &= \sum_i m_i \otimes a_i. \end{aligned}$$

(3.3) We say a K -group $\mathcal{G}$ is infinitesimal iff $K[\mathcal{G}]$ is finite dimensional over K with the nilpotent augmentation ideal $\mathcal{I}_{\mathcal{G}}$.

If $\mathcal{G}$ is infinitesimal, then $\mathcal{G}(R)$ is a singleton for any integral domain R . Also $\text{Dist}(\mathcal{G}) = K[\mathcal{G}]^*$.

Let $\mathcal{G}$ be an arbitrary K -group again. The map

$$(1) \quad \phi : K[\mathcal{G}] \longrightarrow K[\mathcal{G}] \quad \text{via} \quad a \longmapsto a^p$$

is a homomorphism of Hopf algebras, inducing a morphism of K -groups $F_{\mathcal{G}} := K\mathcal{A}lg(\phi, _): \mathcal{G} \longrightarrow \mathcal{G}$, called the Frobenius morphism of $\mathcal{G}$. Then $\mathcal{G}^1 := \ker(F_{\mathcal{G}}) = \mathcal{G} \times_{\mathcal{G}} e_K$ is a normal subgroup of $\mathcal{G}$, with

$$(2) \quad K[\mathcal{G}^1] \simeq K[\mathcal{G}] \otimes_{K[\mathcal{G}]} (K[\mathcal{G}]/\mathcal{I}_{\mathcal{G}}) \simeq K[\mathcal{G}]/(a^p \mid a \in \mathcal{I}_{\mathcal{G}}),$$

hence $\mathcal{G}^1$ is infinitesimal. More generally, $\mathcal{G}^r := \ker(F_{\mathcal{G}}^r)$, $r \in \mathbb{Z}^+$, is an infinitesimal normal subgroup of G with $K[\mathcal{G}^r] \simeq K[\mathcal{G}]/(a^{p^r} \mid a \in \mathcal{I}_{\mathcal{G}})$, called the r -th Frobenius kernel of $\mathcal{G}$.

(3.4) We now focus on the unipotent K -group $\mathcal{U}_n$ such that $K[\mathcal{U}_n] = K[x_{ij}]_{1 \leq j < i \leq n}$ polynomial algebra in indeterminates x_{ij} , $1 \leq j < i \leq n$, with the comultiplication

$$(1) \quad x_{ij} \longmapsto \sum_{k=j}^i x_{ik} \otimes x_{kj}$$

and the counit $x_{ij} \mapsto 0 \ \forall i, j$, where we agree that $x_{ii} = 1 \ \forall i$. If $R \in K\mathcal{A}lg$, $\mathcal{U}_n(R)$ is isomorphic to the group of $n \times n$ lower triangular unipotent matrices with the entries in R . Also if each x_{ij} is assigned degree 1, then

$$(2) \quad \text{Dist}(\mathcal{U}_n) \simeq K[\mathcal{U}_n]^{*gr} \quad \text{as } K\text{-linear spaces.}$$

Let $\mathcal{U}_{ij}$ be a subgroup of $\mathcal{U}_n$ with $K[\mathcal{U}_{ij}] = K[\mathcal{U}_n]/(x_{st})_{(s,t) \neq (i,j)} \simeq K[x_{ij}]$, hence $\mathcal{U}_{ij}(R)$ consists of the (i,j) -th elementary unipotent matrices with the entries in R . With $\deg(x_{ij}) = 1$, $K[\mathcal{U}_{ij}]$ is a graded commutative cocommutative Hopf algebra, and

$$(4) \quad \text{Dist}(\mathcal{U}_{ij}) \simeq K[\mathcal{U}_{ij}]^{*gr} \simeq S_K(x_{ij})^{*gr}$$

the graded dual of the symmetric algebra in x_{ij} [B], (III.11), with the dual monomial basis $X_{ijk} : X_{ijk}(x_{ij}^{\ell}) = \delta_{k\ell} \ \forall k, \ell \in \mathbb{N}$. Hence

$$(5) \quad X_{ijk} X_{ij\ell} = \binom{k+\ell}{k} X_{ijk+\ell},$$

and

$$(6) \quad \Delta'_{\mathcal{U}_{ij}}(X_{ijk}) = \sum_{\ell=0}^k X_{ij\ell} \otimes X_{ij\ell-k}.$$

Under the multiplication one has a natural bijection

$$(7) \quad \prod_{1 \leq j < i \leq n} \mathcal{U}_{ij}(R) \xrightarrow{\sim} \mathcal{U}_n(R) \quad \forall R \in K\mathcal{A}lg,$$

where the product is taken in any order, hence also a K -linear isomorphism

$$(8) \quad \bigotimes_{i,j} \text{Dist}(\mathcal{U}_{ij}) \xrightarrow{\sim} \text{Dist}(\mathcal{U}_n),$$

which is, however, not an isomorphism of Hopf algebras if $n \geq 3$.

We now arrange the $\mathcal{U}_{ij}$ in (7) and (8) in the increasing order such that

$$(9) \quad (i, j) \succ (s, t) \quad \text{iff} \quad i > s \text{ or } i = s \text{ with } j < t,$$

and fix the arrangement in taking the product once and for all. Then under (8)

$$\left(\prod_{i,j} X_{ij,k} \right)_{k \in \mathbb{N}^{\frac{n(n-1)}{2}}} \quad \text{with the product taken in the order of (9)}$$

forms a K -linear basis of $\text{Dist}(\mathcal{U}_n)$ dual to the monomial basis $\prod_{i,j} x_{ij}^h$ of $K[\mathcal{U}_n]$ in the sense of (2).

4° THE STEENROD ALGEBRA REVISITED

(4.1) In 1988 Tezuka M. found a homomorphism of bigebras

$$(1) \quad \psi : K[\mathcal{U}_n] = K[x_{ij}]_{1 \leq j < i \leq n} \longrightarrow K[\xi_1, \xi_2, \dots] = S^{*gr} \quad \text{via} \quad x_{ij} \longmapsto \xi_{i-j}^{p^{j-1}}.$$

Further, by assigning x_{ij} degree $2(p^{i-j} - 1)p^{j-1}$ we can make $K[\mathcal{U}_n]$ into a graded bigebra. Then by the unicity of the antipode on graded bigebras (2.4), ψ is actually a homomorphism of graded Hopf algebras.

Now $\text{im} \psi = K[\xi_1, \dots, \xi_n]$ is a Hopf subalgebra of S^{*gr} , so let us write ψ again for the homomorphism of Hopf algebras $K[\mathcal{U}_n] \rightarrow K[\xi_1, \dots, \xi_n]$ induced by ψ . If

$$\pi_1 : K[\mathcal{U}_n] \longrightarrow K[x_{ij}]_{i,j} / (x_{ij}^{p^n})_{i,j} = K[\mathcal{U}_n^n]$$

and

$$\pi_2 : K[\xi_1, \dots, \xi_{n-1}] \longrightarrow K[\xi_1, \dots, \xi_{n-1}] / (\xi_1^{p^{n-1}}, \xi_2^{p^{n-2}}, \dots, \xi_{n-1}^{p^1}) = S(n-1)^*$$

are the natural maps, we get a commutative diagram of surjective homomorphisms of Hopf algebras

$$(2) \quad \begin{array}{ccc} K[\mathcal{U}_n] & \xrightarrow{\psi} & K[\xi_1, \dots, \xi_n] \\ \pi_1 \downarrow & \curvearrowright & \downarrow \pi_2 \\ K[\mathcal{U}_n^n] & \xrightarrow{\psi_n} & S(n-1)^*. \end{array}$$

Dualizing ψ_n ,

(4.2) THEOREM [KSTY], (3.3). We have an imbedding of Hopf algebras $S(n-1)$ into $\text{Dist}(\mathcal{U}_n^n)$.

(4.3) Hence any $\mathcal{GL}_n$ -module carries a structure of $S(n-1)$ -module upon restriction, enabling one to exploit the representation theory of $\mathcal{GL}_n$ in the study of $S(n-1)$ -modules, where $\mathcal{GL}_n$ is the K -group such that $\mathcal{GL}_n(R)$ is the group of $n \times n$ invertible matrices with the entries in R , $R \in K\mathcal{A}lg$.

(4.4) To illustrate an application, let us first recall some representation theory of $\mathcal{GL}_n$. Let E be the natural n -dimensional $\mathcal{GL}_n$ -module of basis $e_1, \dots, e_n$. If $\lambda = (\lambda_1, \dots, \lambda_{n-1})$ is a partition of $r = \sum_{i=1}^{n-1} \lambda_i$, let $(\lambda'_1, \dots, \lambda'_m)$ be the transposed partition of λ , and put

$$(1) \quad \Phi_\lambda = \left(\sum_{\sigma \in \mathfrak{S}_{\lambda'_1}} \text{sgn}(\sigma) e_{\sigma(1)} \otimes \dots \otimes e_{\sigma(\lambda'_1)} \right) \otimes \dots \otimes \left(\sum_{\sigma \in \mathfrak{S}_{\lambda'_m}} \text{sgn}(\sigma) e_{\sigma(1)} \otimes \dots \otimes e_{\sigma(\lambda'_m)} \right)$$

in $E^{\otimes r}$. After R. Carter and G. Lusztig [CL] we call $\text{Dist}(\mathcal{GL}_n)\Phi_\lambda$ in $E^{\otimes r}$ the Weyl module of highest weight λ , and denote it by $V(\lambda)$.

In case λ is column p -regular, i.e.,

$$(2) \quad 0 \leq \lambda_i - \lambda_{i+1} \leq p-1 \quad \forall i \in [1, n-1] \quad \text{with} \quad \lambda_n = 0,$$

one can show [KSTY], (3.7)

$$(3) \quad V(\lambda) = S(n-1)\Phi_\lambda.$$

(4.5) Let Y be the complex $(p^n - 1)$ -projective space. Then

$$(1) \quad H^*(Y) \simeq K[x]/(x^{p^n}) \quad \text{as graded } K\text{-algebras,}$$

where x is an indeterminate of degree 2. Hence $H^*(Y)$ admits a structure of S -module. Explicitly,

$$(2) \quad \mathfrak{P}^i(x^j) \equiv \binom{j}{i} x^{j+i(p-1)} \pmod{x^{p^n}}.$$

If V is a K -linear span of $x, x^p, \dots, x^{p^{n-1}}$ in $H^*(Y)$, V is stable under the action of $S(n-1)$. Further, there is an isomorphism

$$(3) \quad \theta : E \xrightarrow{\sim} V \quad \text{via} \quad e_i \longmapsto x^{p^{i-1}}, \quad i \in [1, n]$$

of $S(n-1)$ -modules [KSTY], (4.3), which induces by the Künneth formula or by the Cartan formula an imbedding of $S(n-1)$ -modules

$$(4) \quad \theta^{\otimes r} : E^{\otimes r} \hookrightarrow H^*(Y^r) \quad \text{via} \quad e_{i_1} \otimes \dots \otimes e_{i_r} \longmapsto x^{p^{i_1-1}} \otimes \dots \otimes x^{p^{i_r-1}} \quad \forall r \in \mathbb{Z}^+.$$

In particular, the Weyl module $V(\lambda)$ with $r = \sum_{i=1}^{n-1} \lambda_i$ imbeds in $H(Y^r)$ as an $S(n-1)$ -submodule.

(4.6) Fix a column p -regular partition $\lambda = (\lambda_1, \dots, \lambda_{n-1})$ of $r = \sum_{i=1}^{n-1} \lambda_i$ with its transpose $(\lambda'_1, \dots, \lambda'_m)$.

Composed several times with the cup product, $\theta^{\otimes r}$ of (4.4)(4) yields an $S(n-1)$ -homomorphism

$$(1) \quad \theta' : E^{\otimes r} \longrightarrow H(Y^{\lambda'_1}) \quad \text{such that} \quad e_{i_1} \otimes \dots \otimes e_{i_r} \longmapsto \bigotimes_{j=1}^{\lambda'_1} x_j^{e(j)},$$

where $e(j) = p^{i_j-1} + p^{i_j+\lambda'_1-1} + \dots + p^{i_j+\lambda'_1+\dots+\lambda'_{k(j)}-1}$ with $k(j) = \max\{i \mid 1 \leq i \leq m, \lambda'_i \geq j\} - 1$. Set $\theta_\lambda = \theta'|_{V(\lambda)}$. One finds

$$(2) \quad \theta_\lambda(\Phi_\lambda) \neq 0.$$

(4.7) THEOREM (SMITH, MITCHELL[MIT], [KSTY], (4.10)). If $\lambda = ((n-1)(p-1), (n-2)(p-1), \dots, p-1)$, then θ_λ imbeds $V(\lambda)$, called the Steinberg module that is free over $S(n-1)$, into $H(Y^{n-1})$ as $S(n-1)$ -modules.

(4.8) Further, we have a curious

PROPOSITION [KSTY], (4.10). If $\lambda_1 \leq p-1$ and if $V(\lambda)$ is $\mathcal{OL}_n$ -simple, then θ_λ imbeds $V(\lambda)$ into $H(Y^{\lambda'_1})$ as a $S(n-1)$ -submodule.

(4.9) To further our speculation, Mabuchi [Ma] has verified that in case $n \geq 3$ and $\lambda = (p, 1, \dots, 1)$ with 1 appearing $(n-2)$ -times,

$$(1) \quad \theta_\lambda \text{ is injective iff } V(\lambda) \text{ is } \mathcal{OL}_n\text{-simple.}$$

Computer work by his fellow student Takeno S. has also checked (1) for all column p -regular λ in case $n = 3$ and $p \leq 7$.

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