

A special value of the spectral zeta function of the non-commutative harmonic oscillators (非可換調和振動子のゼータの特殊値)

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Abstract

The non-commutative harmonic oscillator is a 2×2 -system of harmonic oscillators with a non-trivial correlation. We write down explicitly the special value at $s = 2$ of the spectral zeta function of the non-commutative harmonic oscillator in terms of the complete elliptic integral of the first kind, which is a special case of a hypergeometric function.

1 Introduction

The non-commutative harmonic oscillator $Q = Q(x, \partial_x)$ is defined to be the second-order ordinary differential operator

$$Q(x, \partial_x) = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \left(-\frac{\partial_x^2}{2} + \frac{x^2}{2} \right) + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \left(x\partial_x + \frac{1}{2} \right).$$

The first term is two harmonic oscillators, which are mutually independent, with the scaling constant $\alpha > 0$ and $\beta > 0$, while the second term is considered to be the correlation with a self-adjoint manner. The spectral problem is a 2×2 system of the ordinary differential equations

$$Q(x, \partial_x)u(x) = \lambda u(x)$$

with an eigenstate $u(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix} \in L^2(\mathbf{R})^{\oplus 2}$ and a spectrum $\lambda \in \mathbf{R}$. It is known [8] that under the natural assumption $\alpha\beta > 1$ on the positivity, which is also

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assumed in this paper, the operator Q defines a positive, self-adjoint operator with a discrete spectrum

$$(0 <) \lambda_1 \leq \lambda_2 \leq \dots \rightarrow +\infty$$

The corresponding spectral zeta function is defined to be

$$\zeta_Q(s) = \sum_{n=1}^{\infty} \lambda_n^{-s}.$$

An expression of the special value $\zeta_Q(2)$ is obtained in [2] in terms of a certain contour integral using the solution of a singly confluent type Heun differential equation. It would be indicated that these special values are complicated enough and highly transcendental as reflecting the transcendence of the spectra of the non-commutative harmonic oscillator.

However, in this paper, we prove the following simple expression:

$$\zeta_Q(2) = \frac{\pi^2 (\alpha^{-1} + \beta^{-1})^2}{4 (1 - \alpha^{-1}\beta^{-1})} \left(1 + \left(\frac{\alpha^{-1} - \beta^{-1}}{\alpha^{-1} + \beta^{-1}} {}_2F_1 \left(\frac{1}{4}, \frac{3}{4}; 1; \frac{1}{1 - \alpha\beta} \right) \right)^2 \right). \quad (1)$$

where ${}_2F_1$ is the Gauss hypergeometric series. We also have an expression by using the complete elliptic integral of the first kind as

$$\zeta_Q(2) = \frac{\pi^2 (\alpha^{-1} + \beta^{-1})^2}{4 (1 - \alpha^{-1}\beta^{-1})} \left(1 + \left(\frac{\alpha^{-1} - \beta^{-1}}{\alpha^{-1} + \beta^{-1}} \int_0^{2\pi} \frac{d\theta}{2\pi \sqrt{1 + (\cos \theta)/\sqrt{1 - \alpha\beta}}} \right)^2 \right). \quad (2)$$

In this sense, the special value $\zeta_Q(2)$ is written in terms of a hypergeometric series, which is much tractable and known. Note that each spectrum is related with the monodromy problem of Heun's differential equation, which is far from hypergeometric, see [5], [6]. Only the total of spectra has an extra simple form in some sense.

Here is a brief organization of the paper; In Section 2, we recall the expression of $\zeta_Q(2)$ given in [2], and derive more explicit formula of the generating function appearing in that expression. We prove in Section 3 our main results, the equations (1) and (2). The proof depends on several formulae of hypergeometric series not only for ${}_2F_1$ but also for ${}_3F_2$ such as Clausen's identity.

2 An expression of the generating function

We start from the series-expression of the special value $\zeta_Q(2)$ of the non-commutative harmonic oscillator given in [2, (4.5a)]

$$\zeta_Q(2) = Z_1(2) + \sum_{n=0}^{\infty} Z_n'(2).$$

We introduce notations. Recall that $\alpha > 0$, $\beta > 0$ with $\alpha\beta > 1$. Let us introduce the parameters $\gamma = 1/\sqrt{\alpha\beta}$ and $a = \gamma/\sqrt{1-\gamma^2} = 1/\sqrt{\alpha\beta-1}$ as in [2, (4.1)]. Note that they satisfy $0 < \gamma < 1$ and $a > 0$.

The term $Z_1(2)$ is given in [2, (4.5b)] and $Z'_n(2)$ are given in [2, (4.9)] as

$$Z_1(2) = \frac{(\alpha^{-1} + \beta^{-1})^2}{2(1-\gamma^2)} 3\zeta(2), \quad (3)$$

$$Z'_n(2) = (-1)^n \frac{(\alpha^{-1} - \beta^{-1})^2}{(1-\gamma^2)} \binom{2n-1}{n} \left(\frac{a}{2}\right)^{2n} J_n. \quad (4)$$

The values $\{J_n\}_{n=1,2,\dots}$ are specified by the generating function

$$w(z) := \sum_{n=0}^{\infty} J_n z^n.$$

The function $w(z)$ is a solution of the ordinary differential equation

$$z(1-z)^2 \frac{d^2 w}{dz^2} + (1-3z)(1-z) \frac{dw}{dz} + \left(z - \frac{3}{4}\right) w = 0 \quad (5)$$

which is given in [2, Theorem 4.13] and called a singly confluent Heun's differential equation. The constant term is given by $w(0) = J_0 = 3\zeta(2) = \pi^2/2$. It is easy to see that there exists a unique power-series solution of this homogeneous differential equation (5) with the initial condition $w(0) = \pi^2/2$. The final target $\zeta_Q(2)$ involving these J_n 's with an infinite sum seemed to have no closed expression.

In this section, we give a simple expression of the generating function $w(z)$. We denote by $\partial_z = \partial/\partial z$.

Lemma 1 *The differential equation (5) is equivalent to*

$$4(1-z)\partial_z z \partial_z (1-z)w + w = 0. \quad (6)$$

Proof: This directly follows from Leibniz rule. QED

Lemma 2 *Let $t = z/(z-1)$ be a new independent variable, and $\eta(t) = (1-z)w(z)$ a new unknown function. Then the differential equation (6) is equivalent to*

$$t(1-t)\partial_t^2 \eta + (1-2t)\partial_t \eta - \frac{1}{4}\eta = 0. \quad (7)$$

Proof: The differential equation (6) is equivalent to

$$4(z-1)^2 \partial_z z \partial_z (z-1)w + (z-1)w = 0.$$

Note that $(z-1)(t-1) = 1$ and $\partial_t := \partial/\partial t = -(z-1)^2 \partial_z$. Then

$$4\partial_t t(t-1)\partial_t \eta + \eta = 0.$$

By Leibniz rule, this is equivalent to (7). QED

Proposition 3

$$w(z) = \frac{J_0}{1-z} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{z}{z-1}\right).$$

Proof: Since any power-series solution of (7) in t is a constant multiple of ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; t)$, we have the conclusion. QED

3 The special value

We introduce the auxiliary series

$$g(a) := \frac{2}{J_0} \sum_{n=0}^{\infty} (-1)^n \binom{2n-1}{n} \left(\frac{a}{2}\right)^{2n} J_n$$

so that

$$\zeta_Q(2) = \frac{(\alpha^{-1} + \beta^{-1})^2}{2(1-\gamma^2)} 3\zeta(2) + \frac{(\alpha^{-1} - \beta^{-1})^2}{2(1-\gamma^2)} 3\zeta(2)g(a) \quad (8)$$

$$= \frac{\pi^2 (\alpha^{-1} + \beta^{-1})^2}{4(1 - \alpha^{-1}\beta^{-1})} \left(1 + \left(\frac{\alpha^{-1} - \beta^{-1}}{\alpha^{-1} + \beta^{-1}}\right)^2 g(a)\right) \quad (9)$$

Theorem 4

$$g(a) = {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; -a^2\right)^2.$$

Proof: We note that

$$\binom{2n-1}{n} \left(\frac{1}{2}\right)^{2n} = \frac{1}{2} \times \frac{(2n-1)!!}{(2n)!!} = \frac{1}{2\pi} \int_0^1 \frac{u^n du}{\sqrt{u(1-u)}}.$$

Then, the integration by parts implies that

$$g(a) = \frac{2}{2\pi J_0} \sum_{n=0}^{\infty} (-1)^n \int_0^1 \frac{u^n du}{\sqrt{u(1-u)}} a^{2n} J_n = \frac{1}{\pi J_0} \int_0^1 \frac{w(-a^2 u) du}{\sqrt{u(1-u)}}. \quad (10)$$

By Proposition 3, the function w is written in terms of hypergeometric series ${}_2F_1$. We substitute such an expression into the equation (10), then we obtain

$$g(a) = \frac{1}{\pi} \int_0^1 \frac{1}{1+a^2 u} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{a^2 u}{a^2 u + 1}\right) \frac{du}{\sqrt{u(1-u)}}.$$

We introduce a new variable $v = (1+a^2)u/(1+a^2 u)$. Then

$$g(a) = \frac{1}{\pi} \int_0^1 {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{a^2 v}{1+a^2}\right) \frac{dv}{\sqrt{v(1-v)(1+a^2)}}.$$

Now we use the formula (2.2.2) of [1]

$${}_3F_2(a_1, a_2, a_3; b_1, b_2; x) = \frac{\Gamma(b_2)}{\Gamma(a_3)\Gamma(b_2 - a_3)} \int_0^1 t^{a_3-1} (1-t)^{b_2-a_3-1} {}_2F_1(a_1, a_2; b_1; xt) dt.$$

This shows

$$g(a) = \frac{1}{\sqrt{1+a^2}} {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; \frac{a^2}{1+a^2}\right).$$

By Clausen's identity (in e.g., Exercise 13 of Chapter 2 in [1])

$${}_2F_1\left(a, b; a+b+\frac{1}{2}; x\right)^2 = {}_3F_2\left(2a, 2b, a+b; 2a+2b, a+b+\frac{1}{2}; x\right),$$

we obtain

$$g(a) = \frac{1}{\sqrt{1+a^2}} {}_2F_1\left(\frac{1}{4}, \frac{1}{4}; 1; \frac{a^2}{1+a^2}\right)^2.$$

Moreover by Pfaff formula, Theorem 2.2.5 of [1]

$${}_2F_1(a, b; c; x) = (1-x)^{-a} {}_2F_1(a, c-b; c; x/(x-1)),$$

we obtain

$${}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; -a^2\right) = (1+a^2)^{-1/4} {}_2F_1\left(\frac{1}{4}, \frac{1}{4}; 1; \frac{a^2}{a^2+1}\right).$$

This shows

$$g(a) = {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; -a^2\right)^2.$$

QED

Remark 5 In the earlier version of the paper, it was suggested to make use of the hypergeometric series ${}_3F_2$ with this special parameter $(1/2, 1/2, 1/2; 1, 1)$ by the multi-variable hypergeometric function of type $(3, 6)$, especially by its restriction on the stratum called X_{1b} in [4]. However, we can avoid to use a multi-variable hypergeometric function in the present version as is seen above.

Theorem 4 with the help of the equation (9) shows the equation (1). The equation (2) is shown as follows. By Theorem 3.13 of [1]

$${}_2F_1(a, b; 2a; x) = \left(1 - \frac{x}{2}\right)^{-b} {}_2F_1\left(\frac{b}{2}, \frac{b+1}{2}; a + \frac{1}{2}; \left(\frac{x}{2-x}\right)^2\right),$$

we have

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{2ia}{ia+1}\right) = (1+ia)^{1/2} {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; -a^2\right).$$

Let us recall the definition of the elliptic integral of the first kind;

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \frac{\pi}{2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right).$$

Then we have

$${}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; -a^2\right) = \frac{2}{\pi} (1+ia)^{-1/2} K\left(\frac{2ia}{ia+1}\right) = \frac{2}{\pi} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1+ia \cos 2\theta}} = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{\sqrt{1+ia \cos \theta}},$$

and the equation (2).

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