

Thesis

**Study of AdS/CFT dictionary
and an existence condition of holographic duals**

Kakeru Sugiura

Department of Physics, Kyoto University

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Abstract

The holographic principle, or the AdS/CFT correspondence, allows us to describe gravity theory in terms of a lower-dimensional quantum field theory (QFT) without gravity. This framework provides a promising candidate for the formulation of quantum gravity theory. Because the whole mechanism of this duality has not yet been uncovered, it is important to establish the AdS/CFT dictionary, a set of explicit maps between two theories.

In the literature, the AdS/CFT dictionary has been studied under the assumption that a holographic QFT is given. However, to decide whether a QFT has a dual description of gravity theory is a non-trivial problem, and it is necessary to develop some criteria. In particular, something practical and easy to analyze is desirable.

Photon spheres are characteristic of general black holes, and thus provide a suitable touchstone for the emergence of holographic spacetimes. The fluctuation of fields on a black hole spacetime is quasinormal modes (QNMs). Because the dynamics of QNMs is dictated by the effective potential that has a peak at the photon sphere, their spectra should also be affected by the photon sphere. Meanwhile, the well-verified part of the AdS/CFT dictionary suggests the coincidence of the spectra on the gravity side and the QFT side.

In this thesis, based on our work [1], we provide a spectral analysis of the Schwarzschild AdS black hole near its photon sphere. We find that the QNMs show a peculiar spectral pattern that consists of long-lived modes and their sudden decay, in the presence of the photon sphere and the AdS boundary. Our large angular momentum analysis owes to an analogue to solvable Schrödinger equations, such as an inverted harmonic oscillator and the Pöschl-Teller model, with a Dirichlet boundary condition. Through the AdS/CFT dictionary, it predicts a corresponding peculiar subsector in the large angular momentum spectrum of thermal holographic QFTs on a sphere. The appearance of such a subsector, “photon-sphere subsector”, would work as an existence condition of holographic duals.

Contents

1	Introduction	4
2	Review of AdS/CFT dictionary I: bulk to boundary	9
2.1	Motivation for developing dictionary	9
2.2	Dictionary for CFT correlators	12
2.2.1	Pure AdS in Poincaré patch	12
2.2.2	Pure AdS in global patch	15
2.2.3	BTZ black hole	17
2.2.4	Analytic structure of correlators	19
3	Review of Quasinormal modes	22
3.1	Quasinormal modes in AdS/CFT	22
3.2	WKB approximation	24
3.2.1	Stokes phenomenon	25
3.2.2	Inverted harmonic oscillator	27
3.2.3	Long-lived quasinormal modes in AdS black hole	30
4	Review of AdS/CFT dictionary II: boundary to bulk	34
4.1	Lightcone cuts method	34
4.1.1	General formalism	34
4.1.2	Pure AdS in global patch	36
4.1.3	Summary and comments	38
4.2	Holographic inverse scattering	38
4.2.1	Inverse scattering theory	39
4.2.2	CFT correlator as scattering data	42
4.2.3	BTZ black hole	44
4.2.4	Summary and comments	46
5	Photon-sphere subsector: an existence condition of holographic duals	47
5.1	Photon sphere in AdS black hole	47
5.2	From quasinormal modes to quantum mechanics	50
5.2.1	Potential hill relates to the quantum-mechanical model	50
5.2.2	Large l expansion	51
5.2.3	Analytic potential at high temperature	53
5.3	Solving quasinormal modes respecting AdS boundary	55
5.3.1	Inverted harmonic oscillator	55

Contents

5.3.2	Pöschl-Teller model	58
6	Conclusion and discussion	61
	Acknowledgements	69
A	Appendix for Chapter 4	70
A.1	Calculation details in the lightcone cuts method	70
B	Appendix for Chapter 5	73
B.1	Representation of $SL(2; \mathbb{R})$	73
B.1.1	Relation between $SO(2, 1)$, $SU(1, 1)$, and $SL(2; \mathbb{R})$	73
B.1.2	Representation of $\mathfrak{su}(1, 1)$	75
B.2	Solvable models and symmetry	78
B.2.1	Inverted harmonic oscillator	78
B.2.2	Pöschl-Teller model	83
	References	88

Chapter 1

Introduction

To understand the world is to understand who we are. To find where gravity comes from is to find where we come from. This world is composed of elementary particles and is governed by the particles. Matters are made from electrons and quarks, and photons and gluons connect them. Diverse physical phenomena over an extensive range of length scales are, in contrast to their huge variety of appearance, believed to emerge from the dynamics of remarkably few kinds of elementary particles. The standard model, consisting of seventeen particles, is a milestone in this kind of reductionism, and quantum theory provides its fundamental framework.

Down to the present day, only our familiar gravity has not been unified with quantum theory, among the four fundamental interactions weaving this world. If we simply quantize general relativity, quantum fluctuations of gravity cannot be ignored at a short distance scale, and non-renormalizable divergences break the predictability of the theory. Despite a long history of attempts, we have not yet arrived at any satisfactory formulation of the quantum theory of gravity. However, quantum gravity is not a mere dream of theorists. Without precise knowledge of the quantum nature of gravity, we cannot recognize the world correctly. It should describe, to say nothing of gravitational interactions between the elementary particles, the microscopic structure of spacetimes such as the resolution of singularities in black holes [2] and in the early universe [3], which is seemingly a macroscopic phenomenon. It would be an indispensable part of our intellect and would teach us the ends of the wandering journey of our universe from the past to the future.

One of the clues to the nature of quantum gravity is the information paradox of black holes [4] (see [5] for a detailed review). Classically, a black hole is distinguished by only a few parameters (its mass, angular momentum, and charges), and any other information cannot be seen by an external observer. In addition, considering the quantum effects of matter fields, the black hole evaporates through Hawking radiation, and finally, thermal radiation is left behind [6]. No longer can the observer extract information about the erstwhile black hole from the featureless radiation. Hence, it seems as if the information stored inside the black hole disappeared. More precisely, this process seems to violate a principle of quantum theory, unitarity. Once a particle in a pure state is thrown into the black hole, it becomes a thermal mixed state after time evolution. On the other hand, the time evolution of a closed quantum system must be unitary, and the pure state should remain pure. General relativity or quantum theory, which is wrong?

The holographic principle [7, 8] provides a convincing approach to this paradox. This asserts that the quantum gravity theory (bulk theory) describing our universe is equivalent to a lower-dimensional quantum theory with no gravity (boundary theory). An explicit and quantitative realization of this principle is given by the AdS/CFT correspondence [9], which is a duality be-

tween a $(d + 1)$ -dimensional gravity theory on an asymptotically anti-de Sitter (AdS) spacetime and a d -dimensional conformal field theory (CFT). Under this correspondence, gravity is considered as an *emergent* phenomenon from the non-gravitating system. If the gravity theory has a field theoretic description, its unitarity guarantees that the information problem does not exist in principle. In fact, the time evolution of the black hole entropy can be estimated through the AdS/CFT correspondence, and it shows the Page curve [10], which indicates the conservation of the information (see [11]). This is one of the reasons why we can expect that the consistent quantum gravity theory can be formulated based on the holographic principle.

Although the holographic principle has played a key role in recent developments towards quantum gravity, the vast majority of this correspondence is waiting to be uncovered. Our mission is to establish a complete set of boundary-to-bulk maps, *the AdS/CFT dictionary*. We hope to study unknown quantum gravity through well-known quantum field theories (QFTs). In other words, we desire to define quantum gravity theory by extending the holographic dictionary from the range of classical gravity. For this purpose, we must tackle the development of three classes of dictionary:

1. What type of QFTs can be dual to gravity theory? Are there any typical characteristics that distinguish them from non-holographic QFTs? What are the existence conditions of holographic duals?
2. From a holographic QFT, how do the constituents of bulk gravity theory (metric of background geometry, field components, and their dynamics) emerge from boundary data?
3. Suppose that all ingredients of the theories are given, what are the relations of physical quantities in the holographically dual two theories?

The problems in class 3 have been studied most widely¹ and practically since the first page of the AdS/CFT dictionary was written by [18, 19]. They presented the equivalence of the partition functions or the coincidence of the spectra on the gravity side and the CFT side. Afterwards, the computation of the spectra is extended to the Minkowski signature [20]. These works intend to understand strongly coupled CFTs through the calculation of classical gravity, in contrast to our aim. However, these parts of the dictionary provide the bottom-up evidence for the validity of the AdS/CFT correspondence, so we will review these calculations in Chap. 2.

On the gravity side, the spectrum is nothing but (quasi-)normal modes, the fluctuation spectrum of fields in the background spacetime. Especially in a black hole spacetime, a frequency of the field can have an imaginary part, and the field dumps or grows while oscillating. Such modes are called quasinormal modes (QNMs) and have broad significance from astrophysics to the AdS/CFT correspondence. According to the AdS/CFT dictionary, gravity theory on a black hole geometry would be dual to a CFT in a thermal state, and QNM frequencies in bulk gravity theory are precisely the same as the poles of the retarded Green function in the boundary CFT. Hence, to understand the relaxation process in thermalization of CFTs, QNMs in AdS black holes have attracted much attention. We will review the notion of QNMs in Chap. 3.

In this way, the mainstream of the AdS/CFT study focuses on applications to CFTs. Instead, for our purpose, it is necessary to understand bulk gravity theory in terms of boundary QFTs. This kind of inverse problem, including the problems in class 2 above, is called bulk reconstruction. While the term “bulk reconstruction” often means the program to construct bulk fields from boundary CFT operators in the literature [21–32] (see also [33, 34] for reviews),

¹We also note that the dictionary for the entanglement entropy in CFTs [12, 13] provided crucial progress in the information paradox [14–17].

the most essential ingredient of gravity theory would be the metric of the bulk spacetime. This is the very gravitational degree of freedom, as well as where the dynamics of fields occurs.

Among many attempts to reconstruct the metric, the lightcone cuts method [35] is notable. This method can be applied to any asymptotically AdS spacetime because it is covariantly formulated and independent of the spacetime dimension. Meanwhile, there are several restrictions: first, this method can only reconstruct the conformal metric, that is, the metric up to a non-rigid conformal factor. While there are some attempts [36–39] to identify the factor, no general prescription has been known yet. Second, the method requires $(d + 3)$ -point correlators in CFT_d , which are not computable in general. The other approaches of metric reconstruction [40–50] contain restricted (typically, only applicable to static and spherically/planar symmetric cases) but comparatively practical ones. However, even these reconstructions are still complicated and costly. We will review some reconstruction methods in Chap. 4.

Even if we can reconstruct a bulk metric from a given QFT, we will confront another problem. Because a typical reconstruction method is just a set of calculation procedures, it could be applied to a QFT regardless of whether the QFT is actually holographic or not. It means that we might “reconstruct” a non-physical spacetime. Then, how can we distinguish true geometry from artifacts? Naively, it is expected that the problems in class 3 would fail for a fake metric. That is, a quantity calculated from the non-physical metric would contradict the corresponding result of the QFT. However, such confirmations would also take much effort. While several meaningful criteria for holographic theories have been proposed [51–54], they are conceptual rather than practical.

At this point, the problems in class 1 arises: it is necessary to find an existence condition for holographic duals and a simpler and more practical way to detect the emergence of a holographic spacetime. One possible candidate for such a condition is a fundamental principle like causality. If a causality condition in a boundary CFT is translated through the AdS/CFT dictionary, it gives a non-trivial restriction on the bulk gravity side [55], and vice versa [56]. So, these holographic causality conditions would exclude the non-physical geometries.

Another approach, which is the main subject of this thesis, is extracting a typical behavior of holographic QFTs caused by gravitational dynamics. For instance, [57, 58] suggested that the existence of a gravity dual is confirmed by measurements of a response function in a given thermal QFT. If the thermal QFT is holographic, there is a black hole in the bulk, and we can observe its Einstein ring [59–61] on the boundary side. Furthermore, [62] proposed a way to discover, in tabletop experiments, condensed matter systems that are holographically dual to quantum gravity, spacetime-emergent materials, by observing the correspondent of the Hawking-Page transition [63, 64]. These are not only practical progress in detecting holographic duals, but also provide essential clues to elucidate the mechanism of the holographic principle and the nature of quantum gravity.

In this thesis, under these circumstances, we study an existence condition of holographic duals and found that *photon spheres*, which are the characteristic of general black holes, can be an indicator of the spacetime emergence.

In general, for spherical black holes, the photon sphere is a spherical region in the bulk spacetime on which photons (or other particles) can circulate the black hole forever. The orbit is unstable: the circulating particle can be swallowed by the black hole or escape from it, once perturbed infinitesimally. The effective motion of the particle is dictated by the (null) geodesics under an effective potential that has a peak at the photon sphere. This potential hill is formed by the competition of the gravitational attraction and the centrifugal force caused by the angular momentum of the particle. Since a bulk field, once quantized, should correspond to a particle

orbiting the black hole, the spectra of the QNMs should also be affected by the photon sphere in general.

The QNMs in AdS black holes have a long history: they were first studied in [65–67], and the various QNM analyses can be found in [68–90] (see [91–94] for detailed reviews). Especially, the study in the framework of the WKB approximation by Festuccia and Liu [95] is related to our work.

Another interesting aspect of the photon sphere is relevant to an extension of the AdS/CFT correspondence to asymptotically flat black holes. Recently, it was shown that a symmetry group $SL(2; \mathbb{R})$ dictates the QNM spectra near the photon sphere/ring of the asymptotically flat Schwarzschild and Kerr black holes [96, 97]. The analysis was further generalized to warped geometries [98–100], and related properties were studied [101–103]. The photon sphere in asymptotically flat spacetimes is expected to work as a holographic screen, as an analog of the AdS boundary in the AdS/CFT correspondence.

In Chap. 5, motivated by these works, we study QNMs near the photon sphere in the Schwarzschild AdS black hole. We find, in the large angular momentum limit, a peculiar form of the QNM spectra associated with the photon sphere. Through the AdS/CFT dictionary, this shows the existence of a peculiar subsector, “photon-sphere subsector”, in the large angular momentum spectra of thermal holographic CFTs on a sphere. Therefore, owing to the universality of the existence of the photon sphere, it is suggested that the existence of the photon-sphere subsector plays the role of an existence condition of holographic duals for general QFTs.

Organization of this thesis

This thesis is based on our paper [1]. The organization is as follows.

In Chap. 2, we review the notion of the AdS/CFT dictionary and some bulk-to-boundary maps. After the validity of the top-down AdS_5/CFT_4 correspondence is briefly described, we emphasize the necessity of developing the dictionary to accumulate bottom-up evidence of the AdS_{d+1}/CFT_d correspondence. As a starting point, the equivalence of the partition functions between the bulk and boundary theory, the GKPW relation [18, 19], is introduced. Then, Son-Starinets prescription [20] for calculating correlation functions is demonstrated by some examples. It provides the basis for the CFT interpretation of our QNM analysis.

In Chap. 3, we review the notion of QNMs. The definition of QNMs in gravity theory is introduced, and their significance in the AdS/CFT context is mentioned. After that, as one of the semi-analytic methods to calculate QNM spectra, the WKB approximation is elaborated. We apply the WKB analysis to the system of an inverted harmonic oscillator and exemplify that two different boundary conditions corresponding to QNMs in an asymptotically flat and AdS black hole induce different spectra. The analysis contains not only a review, but also a new result of this thesis. Then, we review the WKB analysis performed in the Schwarzschild AdS black hole background by Festuccia and Liu [95], which provides a different approach from our analysis in Chap. 5. Through these analysis, a practical limit of the WKB method becomes evident.

In Chap. 4, we review some reconstruction methods of bulk metrics from boundary correlators. The general formalism of the lightcone cuts method by Engelhardt and Horowitz [35] is described, and its application to the pure AdS_{d+1} geometry is analytically demonstrated based on our work [39]. We also describe a more restricted but more practical method by Fan and Yang [46, 47], which is based on inverse scattering theory in quantum mechanics. This method is numerically applied to the BTZ black hole, and we find its metric to be well reproduced.

In Chap. 5, we describe our QNM analysis based on [1]. First, we look at the existence condition of the photon sphere in the Schwarzschild AdS geometry. In fact, there exists a minimum angular momentum so that the effective potential has a potential hill, and we provide an estimation of it.

Then, we map the wave equation for the QNMs to a Schrödinger equation with a potential hill. The photon sphere is located at the top of the potential hill, around which the potential can be approximated by an inverted harmonic oscillator. At a large angular momentum, the approximation works better, as expected from the results in the asymptotically flat case [97]. We find that the large angular momentum limit reduces the quantum mechanics to a simple form: a potential hill at the photon sphere and a Dirichlet boundary at the AdS boundary. We obtain an explicit relation between the energy spectrum of the Schrödinger equation and the QNM spectrum, which is translated to the dual CFT spectrum through the AdS/CFT dictionary. We also find an analytic expression of the potential in the Schrödinger equation at high temperature.

The Schrödinger equation is not solvable. Thus, to grasp a possible universal feature of the energy spectra, we resort to solvable quantum mechanics that shares the feature of our Schrödinger equation. We use an inverted harmonic oscillator and the Pöschl-Teller model, which both have a potential hill but a different potential tail. In fact, the solvable models have already been utilized in the QNM analysis [104–106] since they can provide an analytic expression for the energy spectrum.

In solving those, we find that the existence of the AdS boundary is essential. Even when we bring the effective location of the AdS boundary to the spatial infinity (which is possible by taking the large angular momentum limit), the effect of the Dirichlet boundary remains crucial, and the spectrum keeps differing from that of the asymptotically flat case.

The two solvable models present a spectral pattern similar to each other, despite the distinct behavior of the potential tail. Therefore, we claim that our system of the QNMs near the photon sphere in the Schwarzschild AdS black hole should share this peculiar spectrum.

In Chap. 6, we summarize this thesis and discuss some directions for future research.

Chapter 2

Review of AdS/CFT dictionary I: bulk to boundary

In this chapter, we review the notion of the AdS/CFT dictionary and some attempts to develop it. In Sec. 2.1, we give a brief review of the AdS/CFT correspondence and emphasize the importance of studying the AdS/CFT dictionary to unveil the nature of quantum gravity. In Sec. 2.2, as one of the most fundamental and established parts of the dictionary, we describe the GKPW relation [18, 19] and the Son-Starinets prescription [20], through which the correlation functions in a CFT can be calculated from the bulk equation of motion in a given geometry.

2.1 Motivation for developing dictionary

Let us start with a historical overview. The semi-classical analysis on a black hole spacetime has revealed that the unification of general relativity and quantum theory brings the thermal properties to gravitational systems. Especially, the black hole has the Bekenstein-Hawking entropy [6, 107]

$$S_{\text{BH}} = \frac{A}{4G}, \tag{2.1}$$

which is proportional to the surface area A of its horizon. Since this should express the number of microscopic states of the black hole, this formula suggests that the quantum degrees of freedom of the black hole should be localized, not inside, at the surface of it, the horizon. Pushing forward this idea, the holographic principle [7, 8] has arisen; a quantum gravity theory should be equivalent to a lower-dimensional quantum theory without gravity. It implies that our universe has emerged from a world with no gravity.

A superstring theory has led to the first realization of the holographic principle [9] (see also [108] for a detailed review). The string theory consists of open and closed strings as dynamical degrees of freedom and D-branes as non-perturbative objects. The open and closed string modes contain the gauge field and the spin-two tensor field, the gravitational field, respectively. A D-brane is where the endpoints of the open strings are moving and, at the same time, a source of the closed strings. Such a duality between the open and closed strings gives a foundation of holography.

In concrete, we can consider the setup where N D3-branes are piled up in the type IIB superstring theory on the ten-dimensional flat spacetime. In this setup, an open string has the

2.1. Motivation for developing dictionary

degrees of freedom that correspond to the choice of the branes it ends on. This is the same as the degree of the $N \times N$ matrix and the open string expresses a field in the adjoint representation of the $U(N)$ gauge symmetry. Then, writing down the low-energy effective field theory of the open strings ending on the D-branes, we obtain the four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. This theory has the conformal symmetry and is an example of CFT_4 .

On the other hand, in terms of the closed strings, the stack of the D-branes are regarded as a gravitational source. The gravitational effect is described by the black 3-brane geometry [109]

$$ds^2 = \frac{1}{\sqrt{H(r)}}(-dt^2 + d\mathbf{x}_3^2) + \sqrt{H(r)}(dr^2 + r^2 d\Omega_5^2), \quad (2.2)$$

$$H(r) := 1 + \frac{r_0^4}{r^4}, \quad r_0^4 := 4\pi g_s N l_s^4,$$

which is a classical solution of the type IIB supergravity. Here, $\mathbf{x}$ represents the three space coordinates along the D3-branes, $d\Omega_5$ is the metric of $\mathbb{S}^5$, and g_s , l_s are the coupling constant and the length scale of the strings, respectively. Now, by taking the near-horizon limit $r \ll r_0$ to focus on the region near the D3-branes and transforming the coordinate as $z := r_0^2/r$, we obtain

$$ds^2 = \frac{r_0^2}{z^2}(-dt^2 + dz^2 + d\mathbf{x}_3^2) + r_0^2 d\Omega_5^2. \quad (2.3)$$

This is $\text{AdS}_5 \times \mathbb{S}^5$, the product of the five-dimensional Poincaré AdS spacetime and the five-dimensional sphere each with radius r_0 .

Eventually, from the duality of the open and closed strings in the superstring theory, the duality between the $\mathcal{N} = 4$ $SU(N)$ supersymmetric Yang-Mills theory on the four-dimensional flat spacetime and the type IIB classical supergravity theory on $\text{AdS}_5 \times \mathbb{S}^5$ has been suggested.¹ This is called the $\text{AdS}_5/\text{CFT}_4$ correspondence.

Strictly speaking, the above argument is justified in a certain parameter region. First, to treat the superstring theory, which would involve quantum theory of gravity (closed strings), as a classical theory, the loop corrections of quantum gravity must be negligible. While the coupling constant of gravity is the (ten-dimensional) Newton constant $G^{(10)}$, whose dimension is $[\text{length}]^8$, the expansion coefficients of loop corrections should be dimensionless. So, the coefficients would be powers of $R^4 G^{(10)}$, where R is the scalar curvature of the background spacetime. Since both AdS_5 and $\mathbb{S}^5$ in (2.3) has constant curvatures proportional to r_0^{-2} , the classical limit requires $G^{(10)} \ll r_0^8$. Because the string theory brings us $G^{(10)} = 8\pi^6 g_s^2 l_s^8$,

$$N \gg 1 \quad (2.4)$$

has to be satisfied so that the quantum corrections are negligible. In terms of the gauge theory, this is the limit where the gauge group $SU(N)$ is large enough.

Next, the length scale of the system must be large enough compared to the string length scale l_s so that the superstring theory is well approximated by the supergravity theory, whose degrees of freedom are particles. This is equivalent to the higher-order corrections in the expansion in powers of l_s being negligible in the string theory, and it implies $l_s \ll r_0$. Meanwhile, the coupling constant g_{YM} of the gauge theory drawn from the open string sector can be written as $g_{\text{YM}}^2 = 2\pi g_s$. Therefore, when we define the 't Hooft coupling $\lambda := g_{\text{YM}}^2 N$, which is the effective coupling constant and should be fixed even in the large N limit,

$$\lambda \gg 1 \quad (2.5)$$

¹It is known that the $U(1)$ part of the original $U(N)$ gauge group are decoupled after taking the near-horizon limit.

2.1. Motivation for developing dictionary

has to be satisfied so that the stringy effects are negligible. This means, in the language of the gauge theory, the strong coupling regime.

In summary, what the superstring theory suggests is the duality between the *classical* gravity theory and the *strong-coupling* gauge theory. Although it is still beneficial that complicated (and often impractical) analyses on the strong-coupling regime can be reduced to some problem in classical gravity, such a duality would be inadequate to study quantum gravity. However, if we assume the AdS/CFT correspondence would be valid for the whole theory, not only in the parameter region (2.4) and (2.5), the gravity theory could be investigated by analysis on the gauge theory. In other words, the quantum theory of gravity could be defined by the gauge theory through the AdS/CFT correspondence.

If the quantum gravity theory defined through the AdS/CFT correspondence can describe only perturbation on $\text{AdS}_5 \times \mathbb{S}^5$, the significance will be very restricted. Nowadays, however, it is believed that (quantum) gravity theory on more general asymptotically AdS spacetimes has a dual CFT description, even with no top-down suggestion like one from string theory. This is mentioned as the $\text{AdS}_{d+1}/\text{CFT}_d$ correspondence.

Such a conjecture is supported by various circumstantial evidence and can be expected from the symmetry of the theories. An asymptotically AdS_{d+1} spacetime has the $\text{SO}(2, d)$ symmetry as the asymptotic symmetry at infinity, and it coincides with the conformal symmetry on the d -dimensional flat spacetime. Especially, the pure AdS spacetime, which has the $\text{SO}(2, d)$ symmetry as the isometry, would correspond to the vacuum state of a CFT. When some excitation arises in the bulk pure AdS spacetime, while the asymptotic symmetry would be kept, the bulk symmetry would be reduced. So, asymptotically AdS spacetimes would be dual to excited states of the CFT.

Despite many attempts, the details of the AdS/CFT correspondence are still shrouded in mystery and waiting to be uncovered. Then, some fundamental questions arise: which QFT has a holographic dual? When a QFT is given, is there some procedure to confirm that there is a dual gravitational description, to determine what kind of spacetime emerges, and to extract insight into (quantum) gravity theory? Or, more practically, if a condensed matter system is realized in a laboratory, can we study (quantum) gravity in tabletop experiments? To answer these questions, more bottom-up evidence and a deeper understanding of the mechanism of the AdS/CFT correspondence are needed. Developing the AdS/CFT dictionary offers indispensable building blocks of such a program. They are essential steps to elucidate, not only the mechanism of the holographic principle, but also the nature of quantum gravity.

On the other hand, we should not forget that gravity theory on an asymptotically AdS spacetime is just a toy model of our universe. The cosmological constant Λ can be locally regarded as $\Lambda = 0$ in the real world. We live in the flat Minkowski spacetime, and the black holes in our universe are well described by an asymptotically flat Kerr geometry [60, 61]. Meanwhile, observational cosmology suggests that the universe is undergoing an accelerating expansion, and its global structure has a positive curvature $\Lambda > 0$ [110], that is, de Sitter (dS) like. An AdS geometry with $\Lambda < 0$ shows completely different structure from them.

Some realizations of the holographic principle for $\Lambda \geq 0$ have been known up to the present, which include the Kerr/CFT correspondence [111–113], the celestial holography [114–118], for asymptotically flat spacetimes; the dS/CFT correspondence [119–121] for the global patch, the static-patch holography [122–127], and an explicit dS_3/CFT_2 model [128, 129], for asymptotically dS spacetimes. Nevertheless, the AdS/CFT correspondence is still the best established and concretized model of holography. Therefore, it is natural and promising to attempt to reach a holographic dual of our universe by developing and generalizing the AdS/CFT dictionary.

2.2 Dictionary for CFT correlators

Let us consider the $\text{AdS}_{d+1}/\text{CFT}_d$ correspondence. The equivalence of the bulk and boundary theory in the AdS/CFT correspondence is nothing but the equivalence of the partition functions of the theories. Treating gravity classically means approximating the bulk partition function by the on-shell action evaluated on a background spacetime, since the classical configuration corresponds to a saddle point of the path integral. Here we consider in the bulk theory a field ϕ_i that is non-normalizable and, if the AdS boundary is located at $z = 0$, satisfies the boundary condition

$$\phi_i(z = 0, x) \propto J_i(x), \quad (2.6)$$

where x is the spacetime coordinate on the boundary. In terms of the dual CFT, J_i should be interpreted as a source and coupled to a boundary operator $\mathcal{O}_i$.² Then, the coincidence of the partition functions can be expressed as

$$e^{iS_{\text{AdS}}[\phi]} \Big|_{\text{on-shell, } \phi_i(0, x) = J_i(x)} = \int \mathcal{D}\Phi e^{iS_{\text{CFT}}[\Phi] + i \int d^d x \mathcal{O}_i(x) J_i(x)}. \quad (2.7)$$

This is one of the most fundamental and well-established parts of the AdS/CFT dictionary, known as the GKPW relation [18, 19].

Because an n -point function in the CFT can be defined as

$$\langle \mathcal{O}_{i_1}(x_1) \cdots \mathcal{O}_{i_n}(x_n) \rangle := \frac{1}{i} \frac{\delta}{\delta J_{i_1}(x_1)} \cdots \frac{1}{i} \frac{\delta}{\delta J_{i_n}(x_n)} \int \mathcal{D}\Phi e^{iS_{\text{CFT}}[\Phi] + i \int d^d x \mathcal{O}_i(x) J_i(x)} \Big|_{J=0}, \quad (2.8)$$

we can calculate the correlation function through (2.7) and (2.8) if the action and a classical solution of the dual gravity theory are known.

2.2.1 Pure AdS in Poincaré patch

Here we show a few examples to demonstrate how to use this dictionary. First, let us consider, on the pure AdS_{d+1} spacetime in the Poincaré patch

$$ds^2 = \frac{R^2}{z^2} (-dt^2 + dz^2 + d\mathbf{x}_{d-1}^2) \quad (2.9)$$

with the AdS radius R , the scalar field Φ with mass μ , whose action is

$$S_{\text{scalar}} = \int \sqrt{-g} d^{d+1}x \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} \mu^2 \Phi^2 \right). \quad (2.10)$$

Considering the Fourier mode in terms of the boundary spacetime coordinates x

$$\Phi(z, x) = \tilde{\Phi}(z, k) e^{-i\Omega t + i\mathbf{k} \cdot \mathbf{x}}, \quad (2.11)$$

²To find what $\mathcal{O}_i$ is, in general, some top-down derivation is needed. In the context of the original $\text{AdS}_5/\text{CFT}_4$ correspondence, for example, if ϕ_i is a fluctuation of the bulk gravitational field, Maxwell field, and dilaton (minimally coupled massless scalar field), then $\mathcal{O}_i$ would be the boundary energy-momentum tensor, conserved current, and glueball $\text{tr } F^2$, respectively [130].

2.2. Dictionary for CFT correlators

the Klein-Gordon equation can be written as

$$\partial_z^2 \tilde{\Phi} - \frac{d-1}{z} \partial_z \tilde{\Phi} - \left(\frac{\mu^2 R^2}{z^2} + \mathbf{k}^2 - \Omega^2 \right) \tilde{\Phi} = 0. \quad (2.12)$$

If we assume $\Phi(z \simeq 0) \sim z^{\bar{\Delta}}$ near the AdS boundary and substitute it for (2.12), we obtain

$$\bar{\Delta} = \frac{d}{2} \pm \sqrt{\mu^2 R^2 + \frac{d^2}{4}} \quad (=:\Delta_{\pm}) \quad (2.13)$$

as the solutions. Therefore, the asymptotic behavior of the scalar field can be expressed as

$$\Phi(z \simeq 0, x) \simeq A(x) z^{\Delta_-} (1 + \dots) + B(x) z^{\Delta_+} (1 + \dots), \quad (2.14)$$

or, in the momentum space,

$$\tilde{\Phi}(z \simeq 0, k) \simeq \Phi^{(-)}(k) z^{\Delta_-} (1 + \dots) + \Phi^{(+)}(k) z^{\Delta_+} (1 + \dots). \quad (2.15)$$

Since $\Delta_- < 0$, Φ diverges at the boundary. This is due to the divergence of the spacetime metric. So, introducing a cutoff at $z = \varepsilon$ ($0 < \varepsilon \ll 1$), we modify the boundary condition (2.6) as

$$\Phi(\varepsilon, x) = \varepsilon^{\Delta_-} J(x). \quad (2.16)$$

Then, the source in the dual CFT is given by

$$J(x) = A(x), \quad (2.17)$$

and Φ would be coupled to a boundary operator $\mathcal{O}$ through the interaction term

$$S_{\text{int}} = \int d^d x \mathcal{O}(x) A(x). \quad (2.18)$$

In addition, the coefficient of the normalizable term $B(x)$ corresponds to the response, that is, the one-point function of $\mathcal{O}$ in the presence of the source $J(x)$:

$$B(x) \propto \langle \mathcal{O}(x) \rangle_J. \quad (2.19)$$

The conformal dimension Δ of $\mathcal{O}$ can be calculated from (2.18). Because the scale transformation $x^\mu \rightarrow x'^\mu = \lambda x^\mu$ in the boundary CFT can be interpreted as the isometry in the bulk spacetime, it also acts the z -coordinate like $z \rightarrow z' = \lambda z$. The invariance of the scalar field Φ under this transformation requires, according to (2.14), that $A(x)$ must be transformed as $A(x) \rightarrow \lambda^{-\Delta_-} A(x)$ under the scale transformation. On the other hand, the operator $\mathcal{O}$ with conformal dimension Δ is transformed as $\mathcal{O}(x) \rightarrow \lambda^{-\Delta} \mathcal{O}(x)$. Therefore, the conformal invariance of the deformation term (2.18), which transforms as $S_{\text{int}} \rightarrow \lambda^{d-\Delta-\Delta_-} S_{\text{int}}$, requires

$$\Delta = d - \Delta_- = \Delta_+. \quad (2.20)$$

Let us move on to the calculation of correlation functions of the CFT operator $\mathcal{O}$. Here we focus on the two-point function in momentum space [20] (see [131, 132] for other kinds of correlators). The basis of the calculation is the asymptotic behavior of the bulk field (2.15).

2.2. Dictionary for CFT correlators

Since $\Phi^{(-)}$ and $\Phi^{(+)}$ correspond to a source and the response, respectively, the Green function can be calculated up to an overall constant factor³ as

$$G_R(k) \propto \frac{\Phi^{(+)}(k)}{\Phi^{(-)}(k)} \quad (2.21)$$

by imposing an appropriate boundary condition on $\tilde{\Phi}$. In the pure Poincaré AdS_{d+1} background, for spacelike momenta $k^2 > 0$, the equation of motion (2.12) has two independent solutions

$$z^{d/2} I_\nu(kz) \quad \text{and} \quad z^{d/2} K_\nu(kz), \quad (2.22)$$

where I_ν and K_ν are the first and second kinds of modified Bessel functions and

$$\nu := \sqrt{\mu^2 R^2 + \frac{d^2}{4}} = \Delta - \frac{d}{2}. \quad (2.23)$$

Here we assume for simplicity that the conformal dimension Δ is a non-integer. We should impose the boundary condition that the solution $\tilde{\Phi}$ should be regular at $z = \infty$. The modified Bessel functions have the following asymptotic forms as $|z| \rightarrow \infty$:

$$I_\nu(z) \simeq \frac{1}{\sqrt{2\pi z}} e^z \quad \left(\text{for } |\arg z| < \frac{\pi}{2} \right), \quad (2.24)$$

$$K_\nu(z) \simeq \sqrt{\frac{\pi}{2z}} e^{-z} \quad \left(\text{for } |\arg z| < \frac{3\pi}{2} \right). \quad (2.25)$$

Hence, the solution proportional to K_ν is chosen, and its asymptotic behavior near the AdS boundary $z = 0$ is

$$\tilde{\Phi}(z \simeq 0, k) \sim \frac{\Gamma(\nu)}{2^{1-\nu}} k^{-\nu} z^{\Delta_-} + \frac{\Gamma(-\nu)}{2^{1+\nu}} k^\nu z^{\Delta_+}. \quad (2.26)$$

Therefore, the Green function can be calculated from (2.21) as

$$G_R(k) \propto \frac{\Gamma(-\nu)}{\Gamma(\nu)} \frac{(k^2)^\nu}{2^{2\nu}} \quad (\text{for } k^2 > 0). \quad (2.27)$$

For timelike momenta $k^2 < 0$, two independent solutions of the equation of motion (2.12) can be chosen as

$$z^{d/2} H_\nu^{(1)}(qz) \quad \text{and} \quad z^{d/2} H_\nu^{(2)}(qz), \quad (2.28)$$

where $q := \sqrt{-k^2}$ and $H_\nu^{(1,2)}$ are the first and second kinds of Hankel functions. Because they have the asymptotic form as $|z| \rightarrow \infty$

$$H_\nu^{(1)}(z) \simeq \sqrt{\frac{2}{\pi z}} \exp \left[i \left(z - \frac{2\nu+1}{4} \pi \right) \right] \quad (\text{for } -\pi < \arg z < 2\pi), \quad (2.29)$$

$$H_\nu^{(2)}(z) \simeq \sqrt{\frac{2}{\pi z}} \exp \left[-i \left(z - \frac{2\nu+1}{4} \pi \right) \right] \quad (\text{for } -2\pi < \arg z < \pi), \quad (2.30)$$

³Here we describe a simplified prescription, which is sufficient to investigate the analytic structure of Green functions. The constant factor can be calculated from the bulk action in the complete analysis [20].

2.2. Dictionary for CFT correlators

the regularity at $z = \infty$ cannot specify the solution. Therefore, instead, we should impose the boundary condition that the solution should be an outgoing wave at $z \rightarrow \infty$. This condition corresponds to defining the *retarded* Green function in the dual CFT.⁴ Then, the solution proportional to $H_\nu^{(1)}$ is chosen for $\Omega > 0$ and the other is chosen for $\Omega < 0$ (remember the $e^{-i\Omega t}$ factor in (2.11)). The asymptotic behavior near the AdS boundary is

$$\tilde{\Phi}(z \simeq 0, k) \sim \mp \frac{i\Gamma(\nu)}{\pi} \left(\frac{q}{2}\right)^{-\nu} z^{\Delta_-} + \frac{1 \pm i \cot(\pi\nu)}{\Gamma(1+\nu)} \left(\frac{q}{2}\right)^\nu z^{\Delta_+} \quad (\text{for } \Omega \gtrless 0), \quad (2.31)$$

and the retarded Green function can be calculated as

$$G_R(k) \propto \frac{\Gamma(-\nu)}{\Gamma(\nu)} e^{-i\pi\nu \operatorname{sgn} \Omega} \frac{(-k^2)^\nu}{2^{2\nu}} \quad (\text{for } k^2 < 0). \quad (2.32)$$

While we have worked with the Lorentzian signature, we can also calculate with the Euclidean signature. Since $k^2 = -\Omega^2 + \mathbf{k}^2 = \Omega_E^2 + \mathbf{k}^2$ is always positive after the Wick rotation, the calculation is almost parallel to that of the spacelike case above⁵. On the other hand, the conformal symmetry on $\mathbb{R}^d$ determines the Euclidean two-point function in the dual CFT as

$$\langle \mathcal{O}(x) \mathcal{O}(0) \rangle \propto \frac{1}{|x|^{2\Delta}}. \quad (2.33)$$

The momentum-space correlator is defined by

$$G_E(k) := \int d^d x \langle \mathcal{O}(x) \mathcal{O}(0) \rangle e^{-ik \cdot x}. \quad (2.34)$$

Substituting

$$\frac{1}{|x|^{2\Delta}} = \frac{1}{\Gamma(\Delta)} \int_0^\infty ds s^{\Delta-1} e^{-s|x|^2} \quad (2.35)$$

for (2.34), the x -integral becomes the Gauss integral

$$\int d^d x e^{-s|x|^2} e^{-ik \cdot x} = \left(\frac{\pi}{s}\right)^{d/2} e^{-k^2/4s}. \quad (2.36)$$

Note that $\Delta - d/2 = \nu$, and we obtain

$$G_E(k) \propto \frac{\pi^{d/2} \Gamma(-\nu)}{\Gamma(\Delta)} \frac{(k^2)^\nu}{2^{2\nu}}. \quad (2.37)$$

This is actually the same as the results of the bulk calculation (2.27): the dictionary works!

2.2.2 Pure AdS in global patch

Next, we consider as the background geometry the pure AdS_{d+1} spacetime in the global patch. Its metric is given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-1}^2, \quad f(r) = 1 + \frac{r^2}{R^2}. \quad (2.38)$$

⁴If we want to calculate the advanced Green function, we should choose the solution that is ingoing at $z \rightarrow \infty$.

⁵One finds, in the complete analysis, that the overall constant factor has the additional minus sign.

2.2. Dictionary for CFT correlators

The topology of the AdS boundary is $\mathbb{R} \times \mathbb{S}^{d-1}$. The scalar field (2.10) on this geometry can be decomposed to the Fourier mode as

$$\Phi(r, t, \text{angles}) = \tilde{\Phi}_{lm}(r, \Omega) e^{-i\Omega t} Y_{lm}(\text{angles}), \quad (2.39)$$

where Y_{lm} is the spherical harmonics, the eigenfunction of the Laplacian on $\mathbb{S}^{d-1}$:

$$\Delta_{\mathbb{S}^{d-1}} Y_{lm} = -l(l + d - 2) Y_{lm}. \quad (2.40)$$

Here $l (= 0, 1, \dots)$ is the (total) angular momentum and m represents the other $d - 2$ quantum numbers collectively. Note that, although the momentum k in (2.11) is continuous, the angular momentum l can take only discrete values. The Klein-Gordon equation becomes

$$\partial_r^2 \tilde{\Phi}_{lm} + \left(\frac{d-1}{r} + \frac{f'}{f} \right) \partial_r \tilde{\Phi}_{lm} + \frac{1}{f} \left[\frac{\Omega^2}{f} - \frac{l(l+d-2)}{r^2} - \mu^2 \right] \tilde{\Phi}_{lm} = 0, \quad (2.41)$$

which allows the following asymptotic behavior

$$\tilde{\Phi}_{lm}(r \rightarrow \infty, \Omega) \simeq \Phi_{lm}^{(-)}(\Omega) r^{-\Delta_-} (1 + \dots) + \Phi_{lm}^{(+)}(\Omega) r^{-\Delta_+} (1 + \dots) \quad (2.42)$$

near the AdS boundary located at $r = \infty$.

This equation of motion has two independent solutions

$$\frac{r^l}{f(r)^{\Omega R/2}} F \left(\begin{matrix} \frac{\Delta_- + l - \Omega R}{2}, & \frac{\Delta_+ + l - \Omega R}{2} \\ \frac{d}{2} + l \end{matrix}; -\frac{r^2}{R^2} \right), \quad (2.43)$$

$$\frac{r^{-\hat{l}}}{f(r)^{\Omega R/2}} F \left(\begin{matrix} \frac{\Delta_- - \hat{l} - \Omega R}{2}, & \frac{\Delta_+ - \hat{l} - \Omega R}{2} \\ \frac{d}{2} - \hat{l} \end{matrix}; -\frac{r^2}{R^2} \right), \quad (2.44)$$

where $\hat{l} := l + d - 2 > 0$ and F is the Gaussian hypergeometric function. Since

$$F \left(\begin{matrix} a, & b \\ c \end{matrix}; -z \right) = 1 + \mathcal{O}(z), \quad (2.45)$$

only the first solution is regular at $r = 0$.⁶ Its asymptotic behavior near the AdS boundary can read from the relation

$$\begin{aligned} F \left(\begin{matrix} a, & b \\ c \end{matrix}; -z \right) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} z^{-a} F \left(\begin{matrix} a, & a-c+1 \\ a-b+1 \end{matrix}; -\frac{1}{z} \right) \\ &+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} z^{-b} F \left(\begin{matrix} b, & b-c+1 \\ b-a+1 \end{matrix}; -\frac{1}{z} \right) \quad (\text{for } |z| > 1, z \notin \mathbb{R}_{<0}), \end{aligned} \quad (2.46)$$

and we obtain

$$\tilde{\Phi}_{lm}(r \rightarrow \infty, \Omega) \sim \frac{\Gamma(\frac{d}{2} + l) \Gamma(\frac{\Delta_+ - \Delta_-}{2})}{\Gamma(\frac{\Delta_+ + l - \Omega R}{2}) \Gamma(\frac{\Delta_+ + l + \Omega R}{2})} \left(\frac{r}{R} \right)^{-\Delta_-} + \frac{\Gamma(\frac{d}{2} + l) \Gamma(-\frac{\Delta_+ - \Delta_-}{2})}{\Gamma(\frac{\Delta_- + l - \Omega R}{2}) \Gamma(\frac{\Delta_- + l + \Omega R}{2})} \left(\frac{r}{R} \right)^{-\Delta_+}. \quad (2.47)$$

⁶We should impose the regularity condition, instead of the outgoing condition as in the pure Poincaré AdS. This is because the pure global AdS spacetime is a finite-size “box”: introducing the tortoise coordinate, the Klein-Gordon equation (2.41) can be rewritten as a Schrödinger-like equation (see Sec. 2.2.3). The entire geometry $r \in [0, \infty)$ is mapped to the finite region $r_* = R(\tan^{-1}(r/R) - \pi/2) \in [-\pi R/2, 0]$ and the potential (5.4) diverges at both ends. Therefore, it is natural to impose that the field should vanish at $r = 0$, and actually our solution (2.43) does.

2.2. Dictionary for CFT correlators

Therefore, the Green function can be calculated as

$$G_R^{lm}(\Omega) \propto \frac{R^{2\nu} \Gamma(-\nu) \Gamma\left(\frac{\Delta+l-\Omega R}{2}\right) \Gamma\left(\frac{\Delta+l+\Omega R}{2}\right)}{\Gamma(\nu) \Gamma\left(\frac{d-\Delta+l-\Omega R}{2}\right) \Gamma\left(\frac{d-\Delta+l+\Omega R}{2}\right)}, \quad (2.48)$$

where the AdS radius R is interpreted as the radius of $\mathbb{S}^{d-1}$ in which the dual CFT lives.

2.2.3 BTZ black hole

So far, we have considered the pure AdS spacetimes, which are dual to vacuum states of CFTs at zero temperature. By contrast, a CFT at finite temperature would be described holographically by a black hole geometry. The dynamics in the AdS_{d+1} Schwarzschild spacetime, the most simplest black hole geometry for $d \geq 3$, is not solvable. In $d = 2$, however, there is the BTZ black hole [133, 134]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\varphi^2, \quad f(r) = \frac{r^2 - r_h^2}{R^2} \quad (2.49)$$

as a vacuum solution of the Einstein equation with a negative cosmological constant. The Hawking temperature, which is identified as the dual CFT temperature, is given by

$$T = \frac{f'(r_h)}{4\pi} = \frac{r_h}{2\pi R^2}. \quad (2.50)$$

For the scalar field $\Phi(r, t, \varphi) = \tilde{\Phi}_l(r, \Omega) e^{-i\Omega t + il\varphi}$, the equation of motion has the same form as (2.41). The exact solution for the massless scalar was obtained in [135], and here we extend their derivation to the massive case. By introducing $\psi(r) := \sqrt{r} \tilde{\Phi}_l$ and the tortoise coordinate

$$r_* := \int_{\infty}^r \frac{ds}{f(s)} = -\frac{R^2}{r_h} \tanh^{-1}\left(\frac{r_h}{r}\right), \quad (2.51)$$

the field equation reduces to a Schrödinger-like equation

$$\left(\frac{d^2}{dr_*^2} + \Omega^2 - V(r_*)\right)\psi = 0, \quad (2.52)$$

whose potential is given by

$$V(r_*) = \frac{1}{4R^4} \left[\frac{r_h^2 + 4l^2 R^2}{\cosh^2(r_h r_*/R^2)} + \frac{r_h^2(3 + 4\mu^2 R^2)}{\sinh^2(r_h r_*/R^2)} \right]. \quad (2.53)$$

Defining the new variable

$$x := \frac{1}{\cosh^2(r_h r_*/R^2)}, \quad (2.54)$$

which maps the horizon $r_* = -\infty$ and the AdS boundary $r_* = 0$ to $x = 0, 1$, respectively, and the new field

$$y(x) := \frac{x^{i\Omega R^2/2r_h}}{(1-x)^{3/4}} \psi, \quad (2.55)$$

2.2. Dictionary for CFT correlators

the equation of motion (2.52) can be rewritten as

$$x(1-x)y'' + [\gamma - (1 + \alpha + \beta)x]y' - \left(\alpha\beta + \frac{\delta}{1-x}\right)y = 0, \quad (2.56)$$

where

$$(\alpha, \beta) = \left(1 + \frac{iR(l - \Omega R)}{2r_h}, 1 - \frac{iR(l + \Omega R)}{2r_h}\right), \quad (2.57)$$

$$\gamma = 1 - \frac{i\Omega R^2}{r_h}, \quad (2.58)$$

$$\delta = \frac{\mu^2 R^2}{4}. \quad (2.59)$$

For the massless ($\mu = 0$) case, the $(1-x)^{-1}y$ term vanishes and this is already a hypergeometric equation.

Also, for the massive case, this can be converted to another hypergeometric equation. When we define $u(x) := (1-x)^{-s}y(x)$, the equation (2.56) becomes

$$x(1-x)u'' + [\gamma - (1 + \alpha + \beta + 2s)x]u' - \frac{\alpha\beta + \delta + \gamma s - (\alpha + s)(\beta + s)x}{1-x}u = 0. \quad (2.60)$$

The independence on x of the coefficient of the u -term requires

$$s = \frac{-(\alpha + \beta - \gamma) \pm \sqrt{(\alpha + \beta - \gamma)^2 + 4\delta}}{2} = \frac{-1 \pm \sqrt{1 + \mu^2 R^2}}{2}. \quad (2.61)$$

To match u with y for the massless case, we choose the plus sign: $s = \Delta/2 - 1$. Then, (2.60) can be written in a hypergeometric form

$$x(1-x)u'' + [\gamma - (1 + \tilde{\alpha} + \tilde{\beta})x]u' - \tilde{\alpha}\tilde{\beta}u = 0, \quad (2.62)$$

where

$$(\tilde{\alpha}, \tilde{\beta}) = \left(\frac{\Delta}{2} + \frac{iR(l - \Omega R)}{2r_h}, \frac{\Delta}{2} - \frac{iR(l + \Omega R)}{2r_h}\right). \quad (2.63)$$

Note that this is the natural generalization of the massless ($\Delta = 2$) case.

There are two independent solutions of (2.62)

$$(1-x)^{\gamma-\tilde{\alpha}-\tilde{\beta}}F\left(\begin{matrix} \gamma-\tilde{\alpha}, & \gamma-\tilde{\beta} \\ \gamma \end{matrix}; x\right), \quad (2.64)$$

$$x^{1-\gamma}(1-x)^{\gamma-\tilde{\alpha}-\tilde{\beta}}F\left(\begin{matrix} 1-\tilde{\alpha}, & 1-\tilde{\beta} \\ 2-\gamma \end{matrix}; x\right) \quad (2.65)$$

that is analytic around the horizon $x = 0$ and converges under the conditions $|x| < 1$ and $\text{Re}(\tilde{\alpha} + \tilde{\beta} - \gamma) > 0$. Since, from the relation

$$\tilde{\Phi}_l(r, \Omega) \equiv \frac{1}{\sqrt{r_h}} \left[\cosh\left(\frac{r_h}{R^2} r_*\right) \right]^{i\Omega R^2/r_h} \left[-\tanh\left(\frac{r_h}{R^2} r_*\right) \right]^\Delta u(x), \quad (2.66)$$

2.2. Dictionary for CFT correlators

the asymptotic behavior

$$\tilde{\Phi}(r \simeq r_h, \Omega) \sim e^{-i\Omega r_*} u(x \simeq 0) \quad (2.67)$$

holds, the outgoing boundary condition at the horizon adopts the first solution (2.64). Its asymptotic behavior near the AdS boundary can read from the relation

$$\begin{aligned} (1-x)^{\gamma-\tilde{\alpha}-\tilde{\beta}} F\left(\begin{matrix} \gamma-\tilde{\alpha}, & \gamma-\tilde{\beta} \\ \gamma \end{matrix}; x\right) &= \frac{\Gamma(\gamma)\Gamma(\tilde{\alpha}+\tilde{\beta}-\gamma)}{\Gamma(\tilde{\alpha})\Gamma(\tilde{\beta})} (1-x)^{\gamma-\tilde{\alpha}-\tilde{\beta}} F\left(\begin{matrix} \gamma-\tilde{\alpha}, & \gamma-\tilde{\beta} \\ \gamma-\tilde{\alpha}-\tilde{\beta}+1 \end{matrix}; 1-x\right) \\ &\quad + \frac{\Gamma(\gamma)\Gamma(\gamma-\tilde{\alpha}-\tilde{\beta})}{\Gamma(\gamma-\tilde{\alpha})\Gamma(\gamma-\tilde{\beta})} F\left(\begin{matrix} \tilde{\alpha}, & \tilde{\beta} \\ \tilde{\alpha}+\tilde{\beta}-\gamma+1 \end{matrix}; 1-x\right) \\ &\quad (\text{for } |\arg(1-z)| < \pi), \end{aligned} \quad (2.68)$$

and we obtain

$$\tilde{\Phi}_l(r \rightarrow \infty, \Omega) \sim \frac{\Gamma(\gamma)\Gamma(\tilde{\alpha}+\tilde{\beta}-\gamma)}{\Gamma(\tilde{\alpha})\Gamma(\tilde{\beta})} \left(\frac{r}{r_h}\right)^{-\Delta_-} + \frac{\Gamma(\gamma)\Gamma(\gamma-\tilde{\alpha}-\tilde{\beta})}{\Gamma(\gamma-\tilde{\alpha})\Gamma(\gamma-\tilde{\beta})} \left(\frac{r}{r_h}\right)^{-\Delta_+}. \quad (2.69)$$

Therefore, using (2.50), (2.58) and (2.63), the retarded Green function can be calculated as

$$G_R^l(\Omega) \propto \frac{r_h^{2\nu} \Gamma(-\nu) \Gamma\left(\frac{\Delta}{2} + \frac{i(l-\Omega R)}{4\pi T R}\right) \Gamma\left(\frac{\Delta}{2} - \frac{i(l+\Omega R)}{4\pi T R}\right)}{\Gamma(\nu) \Gamma\left(1 - \frac{\Delta}{2} + \frac{i(l-\Omega R)}{4\pi T R}\right) \Gamma\left(1 - \frac{\Delta}{2} - \frac{i(l+\Omega R)}{4\pi T R}\right)}, \quad (2.70)$$

where the horizon radius r_h is interpreted as the CFT temperature through (2.50).

2.2.4 Analytic structure of correlators

Let us look into the analytic structure of the three (retarded) Green functions. The retarded Green function obtained from the pure Poincaré AdS geometry

$$G_R(k) \propto \begin{cases} \frac{\Gamma(-\nu)}{\Gamma(\nu)} \frac{(k^2)^\nu}{2^{2\nu}} & (\text{for } k^2 > 0) \\ \frac{\Gamma(-\nu)}{\Gamma(\nu)} e^{-i\pi\nu \operatorname{sgn} \Omega} \frac{(-k^2)^\nu}{2^{2\nu}} & (\text{for } k^2 < 0) \end{cases}$$

is that in a CFT on the Minkowski spacetime $\mathbb{R}^{1,d-1}$ at zero temperature. This has a branch cut along the real Ω -axis for $|\Omega| > |\mathbf{k}|$ (Fig. 2.1a). Strictly speaking, the value of $G_R(\Omega)$ on the real axis should be defined as the boundary value on the upper half plane $G_R(\Omega + i0)$, and $G_R(\Omega + i0)^* = G_R(\Omega - i0)$ holds for real Ω . Hence, the imaginary part of the G_R can be calculated from its discontinuity across the cut as

$$\operatorname{Im} G_R(\Omega) = \frac{1}{2i} [G_R(\Omega + i0) - G_R(\Omega - i0)]. \quad (2.71)$$

In this case, it becomes

$$\operatorname{Im} G_R(\Omega, \mathbf{k}) \propto -\frac{\Gamma(-\nu)}{\Gamma(\nu)} \sin(\pi\nu) \operatorname{sgn} \Omega \theta(-k^2) \frac{|k^2|^\nu}{2^{2\nu}}, \quad (2.72)$$

2.2. Dictionary for CFT correlators

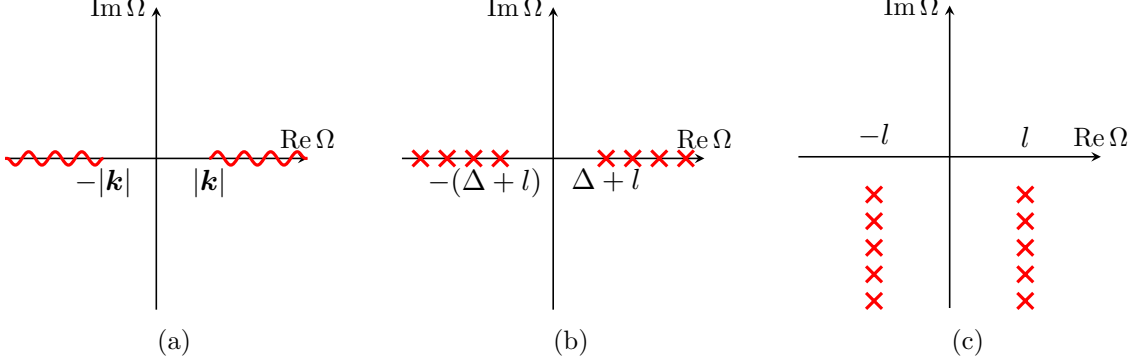


Figure 2.1: The analytic structure of the (retarded) Green functions. (a) The branch cut of $G_R(\Omega, \mathbf{k})$. (b) The poles of $G_R^{lm}(\Omega)$ on the real axis. They correspond to the normal modes in the pure global AdS. (c) The poles of $G_R^l(\Omega)$ in the lower half plane. They correspond to the quasinormal modes around the BTZ black hole.

where θ is the step function. Because the spectral function $\rho(\Omega)$ can be expressed as

$$\rho(\Omega) \propto \text{Im } G_R(\Omega), \quad (2.73)$$

the existence of the branch cut implies that the CFT operator $\mathcal{O}$ has a continuous spectrum. Also, the bulk scalar field Φ is allowed to have continuous modes.

The Green function obtained from the pure global AdS geometry

$$G_R^{lm}(\Omega) \propto \frac{R^{2\nu} \Gamma(-\nu) \Gamma\left(\frac{\Delta+l-\Omega R}{2}\right) \Gamma\left(\frac{\Delta+l+\Omega R}{2}\right)}{\Gamma(\nu) \Gamma\left(\frac{d-\Delta+l-\Omega R}{2}\right) \Gamma\left(\frac{d-\Delta+l+\Omega R}{2}\right)},$$

which is that in a CFT on $\mathbb{R} \times \mathbb{S}^{d-1}$ at zero temperature, does not have such a branch cut. Instead, it has simple poles on the real axis (Fig. 2.1b) at

$$\Omega R = \pm(l + \Delta + 2n) =: \Omega_n R \quad (n = 0, 1, \dots). \quad (2.74)$$

Around a pole, the $\Gamma(-n)$ factor is dominant, and the Green function can be approximated as

$$G_R^{lm}(\Omega \simeq \Omega_n) \sim \frac{\text{Res}(\Gamma, -n)}{(\Omega + i0) - \Omega_n}, \quad (2.75)$$

where an irrelevant factor is omitted. According to the Hilbert formula $1/(x - x_0 \mp i0) = \text{p.v. } 1/(x - x_0) \pm i\pi\delta(x - x_0)$, the spectral function (2.73) becomes

$$\rho^{lm}(\Omega \simeq \Omega_n) \sim -\pi \text{Res}(\Gamma, -n) \delta(\Omega - \Omega_n). \quad (2.76)$$

Hence, the existence of the poles (2.74) on the real axis suggests that the spectra of $\mathcal{O}$ consist of discrete and stable excitations. In terms of the bulk scalar theory, these poles represent the normal modes. They reflect the fact that the pure global AdS spacetime is a finite-size “box” and the field has to vanish at its ends (see footnote 6 on p. 16).

Finally, the retarded Green function obtained from the BTZ black hole geometry

$$G_R^l(\Omega) \propto \frac{r_h^{2\nu} \Gamma(-\nu) \Gamma\left(\frac{\Delta}{2} + \frac{i(l-\Omega R)}{4\pi T R}\right) \Gamma\left(\frac{\Delta}{2} - \frac{i(l+\Omega R)}{4\pi T R}\right)}{\Gamma(\nu) \Gamma\left(1 - \frac{\Delta}{2} + \frac{i(l-\Omega R)}{4\pi T R}\right) \Gamma\left(1 - \frac{\Delta}{2} - \frac{i(l+\Omega R)}{4\pi T R}\right)}$$

2.2. Dictionary for CFT correlators

is that in a CFT on $\mathbb{R} \times \mathbb{S}^1$ at finite temperature T . It has poles, in contrast to the case of zero temperature, in the lower half Ω -plane (Fig. 2.1c) at

$$\Omega R = \pm l - i2\pi T R(\Delta + 2n) =: \Omega_{T,n} R \quad (n = 0, 1, \dots). \quad (2.77)$$

They make the spectral function have peaks at $\Omega = \text{Re } \Omega_{T,n} = \pm l/R$ with finite width $|\text{Im } \Omega_{T,n}| = 2\pi T(\Delta + 2n)$:

$$\rho_n^l \left(\Omega \simeq \pm \frac{l}{R} \right) \sim \frac{\text{Res}(\Gamma, -n) \text{Im } \Omega_{T,n}}{(\Omega \mp l/R)^2 + (\text{Im } \Omega_{T,n})^2}, \quad (2.78)$$

where the subscript n means the contribution from the n -th overtone. Especially in the BTZ case, where the pole frequencies share the same real parts, the spectral function would be the superposition of these peaks: $\rho^l(\Omega \simeq \pm l/R) \sim \sum_n \rho_n^l(\Omega \simeq \pm l/R)$. Therefore, the spectra of $\mathcal{O}$ consist of *unstable* excitations.

From the bulk perspective, the scalar modes corresponding to these complex poles behave as

$$\Phi(t) \sim e^{-i(\text{Re } \Omega_{T,n})t} e^{(\text{Im } \Omega_{T,n})t}. \quad (2.79)$$

Since $\text{Im } \Omega_{T,n} < 0$ in this case, such modes damp as the time t increases. So, these modes have finite decay rate $|\text{Im } \Omega_{T,n}|^{-1}$, as well as energy $\text{Re } \Omega_{T,n}$. These are nothing but quasinormal modes, which represent a perturbation around the black hole. They play an important role in the research of not only black holes themselves but also quantum gravity. The significance of quasinormal modes in the context of the AdS/CFT correspondence will be described in Chap. 3.

Chapter 3

Review of Quasinormal modes

In this chapter, we review the notion of quasinormal modes. In Sec. 3.1, we present the definition of quasinormal modes and describe their significance in the context of the AdS/CFT correspondence. In most cases, it is too challenging to obtain exact solutions for quasinormal modes. So, approximate and numerical approaches have gained considerable attention. Among those, we describe the WKB method in Sec. 3.2. In addition to an overview of this approximation, its application to the system of an inverted harmonic oscillator is demonstrated. Subsequently, based on [95], the WKB analysis of QNMs in an AdS black hole is performed in a large angular momentum limit. It provides a different approach from our large angular momentum analysis, which will be elaborated in Chap. 5.

3.1 Quasinormal modes in AdS/CFT

Quasinormal modes (QNMs) are solutions of a field equation on a black hole geometry that satisfy specific boundary conditions (see [91–94] for detailed reviews). While here we focus on the scalar field for simplicity, QNMs can be defined for higher spin fields [136]. After the separation of variables, the equation of motion for the scalar field $\Phi(r, t, \dots) \sim \psi(r)e^{-i\Omega t}$ reduces to a Schrödinger-like equation

$$\left(\frac{d^2}{dr_*^2} + \Omega^2 - V(r) \right) \psi(r) = 0. \quad (3.1)$$

The form of the potential V depends on the background geometry. For most spacetimes of interest, however, the potential vanishes at the black hole horizon: $V(r_* \rightarrow -\infty) \rightarrow 0$. So, the solutions of (3.1) behave as $\Phi(t, r_* \rightarrow -\infty) \sim e^{-i\Omega(t \pm r_*)}$. Because, classically, nothing should be emitted from the horizon, only outgoing modes should exist. Therefore, we should impose

$$\Phi \sim e^{-i\Omega(t+r_*)} \quad \text{for } r_* \rightarrow -\infty \quad (3.2)$$

at the horizon. An appropriate boundary condition at spatial infinity $r \rightarrow \infty$ ($r_* = 0$), for an asymptotically flat spacetime, is trivial. Since the background metric approaches that of the Minkowski spacetime and $V(r_* \rightarrow 0) \rightarrow 0$, we should impose $\Phi(t, r_* \rightarrow 0) \sim e^{-i\Omega(t-r_*)}$ to discard ingoing modes from spatial infinity. For an asymptotically AdS spacetime of interest in the AdS/CFT context, the so-called AdS boundary exists and the potential diverges at spatial infinity. Therefore, the scalar field should vanish

$$\Phi \rightarrow 0 \quad \text{for } r_* \rightarrow 0 \quad (3.3)$$

3.1. Quasinormal modes in AdS/CFT

at the AdS boundary. These two boundary conditions are what define QNMs in an asymptotically AdS spacetime.

QNM spectra subject to the boundary conditions (3.2) and (3.3) would be quantized in general. Because a black hole absorbs the field and the system is dissipative, in contrast to the normal modes observed in the pure global AdS background (see (2.74)), quasinormal frequencies can have both a real and an imaginary part:

$$\Omega = \Omega_R + i\Omega_I. \quad (3.4)$$

Since, for $\Omega_I < 0$, such modes decay while oscillating, they are called *quasi*-normal.

QNMs play an important role in the context of black hole perturbation theory, that is, the study of the stability of black holes. If we consider back reaction of matter fields, the spacetime metric would be perturbed when a black hole absorbs the fields. Then, the black hole would emit a part of its energy as gravitational waves. If the black hole is stable against such a perturbation, the emission of energy will cease gradually, and the black hole will settle down to another stable state. In this process of relaxation, QNMs dominate the “ringdown” phase, which is usually a long period of damping proper oscillations. If the black hole is unstable, the emission will be more and more severe and take away all the energy of the black hole. These situations occur in the case of $\Omega_I < 0$ and $\Omega_I > 0$, respectively.

In the context of the AdS/CFT correspondence, as is mentioned in Sec. 2.2, the outgoing boundary condition (3.2) at the horizon corresponds to choosing the retarded Green function in a CFT. Furthermore, since the bulk scalar field behaves as

$$\tilde{\Phi}(r \rightarrow \infty, \Omega) \simeq \Phi^{(-)}(\Omega)r^{-\Delta_-}(1 + \dots) + \Phi^{(+)}(\Omega)r^{-\Delta_+}(1 + \dots) \quad (3.5)$$

near the AdS boundary, where $\Delta_{\pm} \geq 0$, the Dirichlet condition (3.3) for QNMs is equivalent to

$$\Phi^{(-)}(\Omega) = 0. \quad (3.6)$$

This implies that the QNM frequencies of Φ in the bulk theory are the same as the poles, or the resonance frequencies, of the retarded Green function of $\mathcal{O}$ in the dual CFT [20, 137]. So, for example, the QNM frequencies in the BTZ background [135] can be identified with the poles of G_R obtained in Sec. 2.2.3 as

$$\Omega R = \pm l - i2\pi T R(\Delta + 2n) =: \Omega_{T,n} R \quad (n = 0, 1, \dots), \quad (3.7)$$

where $\Delta = 1 + \sqrt{1 + \mu^2 R^2}$.

According to the AdS/CFT correspondence, a black hole geometry is dual to a thermal state in the boundary CFT [64], and the black hole works as a thermal bath. A local operator in the CFT produces a bulk particle (field excitation) near the AdS boundary. As the particle falls into the bulk towards the black hole horizon, the dual CFT operator gradually becomes composite and non-local due to the action of the time evolution operator [138]. That is, perturbing the black hole corresponds to perturbing the CFT thermal state, and the decay of the bulk perturbation, described by QNMs, corresponds to the relaxation to thermal equilibrium. Therefore, the timescale t_d of thermalization in the boundary CFT is given by the imaginary part of the lowest QNM frequency [66]

$$t_d \sim |\text{Im } \Omega_{T,n=0}|^{-1}. \quad (3.8)$$

3.2. WKB approximation

3.2 WKB approximation

QNM spectra can hardly be solved analytically in most situations. However, because of their significance from astrophysics to the AdS/CFT context, various attempts have been made to find QNMs by numerical and semi-analytic methods. The QNMs in the Schwarzschild AdS black hole were first studied in [65–67] mainly using numerical computations, and there are many subsequent works [68–90] thus far. Meanwhile, since the field equation on an AdS black hole geometry can be cast to the Schrödinger-like equation, the approximation methods developed in quantum mechanics can be utilized. One of these is the (Jeffreys-)Wentzel-Kramers-Brillouin (WKB) method. In this method, the field is expressed as an exponential function, and its phase is expanded in powers of a small parameter, which corresponds to $\hbar$. This method was first applied to the QNMs in the asymptotically flat Schwarzschild black hole by [139], and afterwards extended to the Schwarzschild AdS case [75, 79, 95]. We also note that, pushing forward this direction, there are some attempts to apply the exact WKB analysis, which is a non-perturbative extension of the ordinary WKB method, to solve QNM spectra beyond the semi-analytic approximation [140–142].

The subject of the WKB analysis is to solve a Schrödinger-like equation

$$\left[\frac{d^2}{dz^2} + \eta^2(u^2 - V(z)) \right] \psi(z) = 0, \quad (3.9)$$

where η is a large parameter. It is assumed that the wave function ψ can be expressed in the form of

$$\psi(z) = e^{\eta S(z)}, \quad (3.10)$$

and the phase function S can be expanded in powers of η^{-1} :

$$S(z) = S_0(z) + \eta^{-1} S_1(z) + \mathcal{O}(\eta^{-2}). \quad (3.11)$$

For our purpose, it is sufficient to consider the potential V independent of η . By substituting (3.10) and (3.11) to (3.9) and comparing the coefficients up to $\mathcal{O}(\eta^{-1})$, we obtain

$$S_0'^2 = V(z) - u^2, \quad (3.12)$$

$$S_1' = -\frac{S_0''}{2S_0'}. \quad (3.13)$$

They can be solved formally, and we find two independent solutions

$$\psi_{\pm}(z_0, z) := \frac{1}{\sqrt{\kappa(z)}} e^{\pm \eta \mathcal{Z}(z_0, z)}, \quad (3.14)$$

where

$$\kappa(z) := \sqrt{V(z) - u^2}, \quad (3.15)$$

$$\mathcal{Z}(z_0, z) := \int_{z_0}^z dz' \kappa(z'), \quad (3.16)$$

and z_0 is an arbitrary point.

3.2. WKB approximation

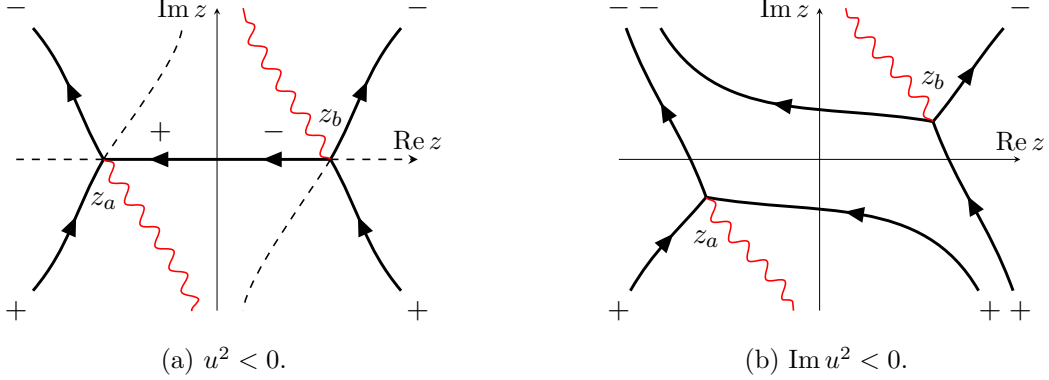


Figure 3.1: A schematic illustration of the Stokes geometry for $V(z) = -z^2/4$. The thick curves, dashed curves, and red wavy curves represent the Stokes curves, anti-Stokes curves, and the branch cuts, respectively. The branch cuts are chosen along an anti-Stokes curve. The arrows along the Stokes curves express the direction of decreasing $\text{Re } \mathcal{Z}$, and the plus and minus symbols are the signs of $\text{Re } \mathcal{Z}$ on each curve. (a) For $u^2 < 0$, two turning points are connected by a Stokes curve and not simple. (b) For $\text{Im } u^2 < 0$, the Stokes curve splits. Here anti-Stokes curves are omitted for simplicity.

3.2.1 Stokes phenomenon

In general, the potential V has turning points z_T , which are defined as the solutions of

$$V(z_T) - u^2 = 0. \quad (3.17)$$

Especially, if $V'(z_T) \neq 0$, the turning point is called a simple turning point. In this chapter, we consider only simple turning points. Note that (3.17) allows complex solutions. What we want is to construct a solution subject to the desirable boundary condition at one end of the system as a linear combination of $\psi_{\pm}$, to extend the solution to the other end, and to determine the energy u in order to satisfy the boundary condition there. However, the approximation (3.14) breaks down at turning points, since the prefactor $\kappa(z_T)^{-1/2}$ diverges. Therefore, we extend our solution along a curve connecting both ends of the system on the complex z -plane, which is routed to avoid turning points. In this procedure, we must pay careful attention to the issue of analytic continuation of the solution, or the structure of the Stokes geometry, which is defined by the behavior of the phase integral $\mathcal{Z}$.

For a turning point z_T , a one-dimensional curve on the complex z -plane defined by

$$\text{Im } \mathcal{Z}(z_T, z) = 0 \quad (3.18)$$

is called the Stokes curve. An endpoint of a Stokes curve is a turning point by definition, and the other is a turning point or a singularity, including a point at infinity. Each region divided by Stokes curves is called a Stokes region. Similarly, a one-dimensional curve defined by

$$\text{Re } \mathcal{Z}(z_T, z) = 0 \quad (3.19)$$

is called the anti-Stokes curve.

Let us consider an inverted harmonic oscillator $V(z) = -z^2/4$ as an example. The Stokes geometry for this potential is depicted in Fig. 3.1. This potential has two turning points z_a and

3.2. WKB approximation

z_b ($\text{Re } z_a \leq \text{Re } z_b$) for a given u . For $u^2 < 0$, $z_{a,b} = \mp 2|u|$ are real and just classical turning points, at which the classically allowed region and the forbidden region switch. In the allowed region, ψ should be purely oscillatory, and this means that there are two anti-Stokes lines on $(-\infty, z_a]$ and $[z_b, \infty)$ along the real z -axis. In contrast, a Stokes line on the forbidden region $[z_a, z_b]$ connects the two turning points.

Moreover, there are two additional (anti-)Stokes lines for each turning point. In the vicinity of a turning point $z = z_T + \varepsilon e^{i\theta}$, $\varepsilon \ll 1$, the potential can be approximated as $V(z') \simeq V(z_T) + V'(z_T)(z' - z_T)$, and we obtain

$$\mathcal{Z}(z_T, z_T + \varepsilon e^{i\theta}) \simeq \frac{2}{3} \sqrt{V'(z_T)} \varepsilon^{3/2} e^{i3\theta/2}. \quad (3.20)$$

For $z_T = z_a$, $V'(z_a) = |u| > 0$ and

$$\text{Im } \mathcal{Z}(z_a, z_a + \varepsilon e^{i\theta}) \propto \varepsilon^{3/2} \sin\left(\frac{3\theta}{2}\right). \quad (3.21)$$

Hence, the Stokes curves depart from z_a at three angles $\theta = 2\pi n_a/3$ ($n_a \in \mathbb{Z}$). For $z_T = z_b$, since $V'(z_b) < 0$, we obtain

$$\text{Im } \mathcal{Z}(z_b, z_b + \varepsilon e^{i\theta}) \propto \varepsilon^{3/2} \cos\left(\frac{3\theta}{2}\right), \quad (3.22)$$

and the output angles of the Stoke curves are $\theta = \pi(2n_b + 1)/3$ ($n_b \in \mathbb{Z}$).

Along the Stokes curves, the phase integral $\mathcal{Z}$ remains real by definition. Because a turning point is also a branch point of the integrand $\kappa(z) = \sqrt{V(z) - u^2}$, the direction of decreasing $\text{Re } \mathcal{Z}$ depends on the location of its branch cut. When we choose the branch cuts as shown in Fig. 3.1, the values of n 's above are determined as $n_a = 0, 1, 2$ and $n_b = 0, -1, -2$. Then, we obtain

$$\text{Re } \mathcal{Z}(z_a, z_a + \varepsilon e^{i\theta}) \propto (-1)^{n_a} \varepsilon^{3/2}, \quad (3.23)$$

$$\text{Re } \mathcal{Z}(z_b, z_b + \varepsilon e^{i\theta}) \propto (-1)^{n_b-1} \varepsilon^{3/2}, \quad (3.24)$$

and the direction of the flow becomes as shown by the wavy curves in Fig. 3.1. Since $\mathcal{Z}(z_T, z_T) = 0$ at a turning point, in the upstream and downstream of its Stokes curves, $\text{Re } \mathcal{Z}(z_T, z) > 0$ and $\text{Re } \mathcal{Z}(z_T, z) < 0$ hold, respectively. Note that the sign of $\mathcal{Z}(z_T, z)$ depends on the choice of the reference turning point z_T .

When u is complex like a QNM frequency, the output angles are shifted as $\theta = 2(\pi n - \arg \sqrt{V'(z_T)})/3$, while the angles between neighboring Stokes curves remain $2\pi/3$. Then, the Stokes curve connecting two turning points splits into two distinct curves (Fig. 3.1b). However, the flow direction of the Stokes curves is unchanged.

If $\text{Re } \mathcal{Z}(z_T, z) > 0$ along a Stokes curve, ψ_+ and ψ_- are called dominant and subdominant on the curve, respectively. This is because the dominant mode is exponentially greater than the subdominant mode. If $\text{Re } \mathcal{Z}(z_T, z) < 0$, ψ_- becomes dominant and ψ_+ becomes subdominant. The dominance along the Stokes curves is shown in Fig. 3.1 by the plus and minus signs.

Because the sign of $\text{Re } \mathcal{Z}$ switches across an anti-Stokes curve, the dominance of $\psi_{\pm}$ is exchanged. By contrast, across a Stokes curve, the dominant mode remains dominant. However, when the solution is analytically continued appropriately from a Stokes region to another, we find that the subdominant contribution mixes with the dominant mode. More specifically, the following connection formula holds (see [143] for a mathematical proof, for example):

3.2. WKB approximation

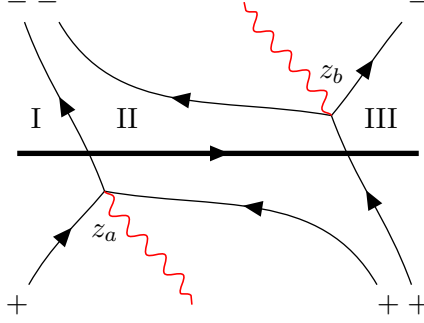


Figure 3.2: A path of the analytic continuation along the real z -axis.

Let U_1 and U_2 two Stokes regions neighboring across a Stokes curve starting from a turning point z_T . We express the dominant and subdominant solutions in U_i along the Stokes curve as $\psi_d^i(z_T, z)$ and $\psi_s^i(z_T, z)$, respectively. We assume that all the turning points for $V(z)$ are simple, and any two turning points cannot be connected by a Stokes curve.¹ Then, the analytic continuation of the solutions obeys the relation

$$\begin{cases} \psi_d^1(z_T, z) = \psi_d^2(z_T, z) \pm i\psi_s^2(z_T, z), \\ \psi_s^1(z_T, z) = \psi_s^2(z_T, z), \end{cases} \quad (3.25)$$

where the $-$ ($+$) sign corresponds to the case where the path of the analytic continuation from U_1 to U_2 crosses the Stokes curve (counter)clockwise around z_T . This discontinuous behavior of WKB solutions is called the Stokes phenomenon.

After all, the central issue of the WKB analysis is to identify the structure of the Stokes geometry and to draw appropriately the path of the analytic continuation. Since the Stokes geometry depends on not only $V(z)$ but also the argument of the energy u , as exemplified in Fig 3.1, this program requires previous knowledge (or speculation) of u . Otherwise, we have to try all possible patterns of the geometry.

3.2.2 Inverted harmonic oscillator

outgoing condition + outgoing condition

Let us consider the spectra of the inverted harmonic oscillator $V(z) = -z^2/4$. First, we impose the boundary condition that there exists only an outgoing mode at $z \rightarrow \pm\infty$, which corresponds to QNMs in an asymptotically flat black hole. This system can be solved exactly, and we know that its energy eigenvalues are actually pure imaginary and that their imaginary parts are negative (see Sec. 5.3 and App. B.2.1). So, we should work in the Stokes geometry shown in Fig. 3.1b. A path of the analytic continuation can be chosen along the real z -axis to avoid the branch cuts, as shown in Fig. 3.2. It passes through three Stokes regions I, II, and III.

The turning points are obtained as

$$z_{a,b} = \mp 2|u| e^{i(\pi + \arg u^2)/2}. \quad (3.26)$$

Because the phase integral $\mathcal{Z}$ behaves as

$$\mathcal{Z}(z_T, z) \simeq \pm \frac{i}{4} z^2 \quad (z \rightarrow \pm\infty), \quad (3.27)$$

¹Note that this assumption excludes the Stokes geometry shown in Fig. 3.1a.

3.2. WKB approximation

ψ_- (ψ_+) is outgoing at $z \rightarrow -\infty$ ($z \rightarrow +\infty$). Hence, we choose the wave function in region I as $\psi^{\text{I}}(z) = \psi_-^{\text{I}}(z_a, z)$, which is normalized at z_a . When our path passes through region I to II, the solution is analytically continued as

$$\psi_-^{\text{I}}(z_a, z) = \psi_-^{\text{II}}(z_a, z) - i\psi_+^{\text{II}}(z_a, z) \quad (3.28)$$

according to the connection formula (3.25). To move into region III, we should decompose $\psi_{\pm}^{\text{II}}$ as

$$\psi_{\pm}^{\text{II}}(z_a, z) = e^{\pm\eta\mathcal{Z}(z_a, z_b)} \psi_{\pm}^{\text{II}}(z_b, z) \quad (3.29)$$

and apply (3.25) to $\psi_{\pm}^{\text{II}}(z_b, z)$. Then, we obtain

$$\psi_-^{\text{I}}(z_a, z) = -i\psi_+^{\text{III}}(z_a, z) + \left(1 + e^{2\eta\mathcal{Z}(z_a, z_b)}\right) \psi_-^{\text{III}}(z_a, z). \quad (3.30)$$

Hence the outgoing boundary condition at $z \rightarrow \infty$ requires that the coefficient of ψ_-^{III} vanishes, that is,

$$\frac{2\eta}{i}\mathcal{Z}(z_a, z_b) - \pi \in 2\pi\mathbb{Z}. \quad (3.31)$$

Here, by changing the variable as $z' =: 2|u|e^{i(\pi+\arg u^2)/2}x$, the phase integral $\mathcal{Z}(z_a, z_b)$ can be calculated as

$$\mathcal{Z}(z_a, z_b) = -\pi u^2. \quad (3.32)$$

Therefore, the energy eigenvalues can be obtained as

$$u^2 = -i\eta^{-1}\left(n + \frac{1}{2}\right) \quad (n = 0, 1, \dots), \quad (3.33)$$

where the restriction $n \geq 0$ comes from the assumption $\text{Im } u^2 < 0$. Although we have calculated in the WKB approximation up to $\mathcal{O}(\eta^{-1})$ so far, this spectrum coincides the exact result (see (5.53)).

Note that, for $\text{Im } u^2 > 0$, the Stokes geometry can be obtained by flipping the dominance sign along the Stokes curves starting from z_b and then reflecting the entire geometry about the imaginary z -axis. So, the analytic continuation along the real z -axis becomes $\psi_+^{\text{I}} = \psi_+^{\text{II}} = \psi_+^{\text{III}}$, which means that no solutions satisfy our boundary condition $\psi_-^{\text{III}} = \psi_-^{\text{III}}$ at $z \rightarrow -\infty$.

outgoing condition + Dirichlet condition²

Next, we consider a different boundary condition that there is only an outgoing mode at $z \rightarrow -\infty$ and the wave function should vanish at $z = L$:

$$\psi(L) = 0, \quad L \gg 1. \quad (3.34)$$

This corresponds to QNMs in an asymptotically AdS black hole and L to the location of the AdS boundary (see Sec. 5.2). So, we assume $\text{Im } u^2 < 0$ again. We begin with a solution in region III

$$\psi^{\text{III}}(z) = \psi_+^{\text{III}}(L, z) - \psi_-^{\text{III}}(L, z) \quad (3.35)$$

²This subsection presents not a review of literature but a new result of this thesis.

3.2. WKB approximation

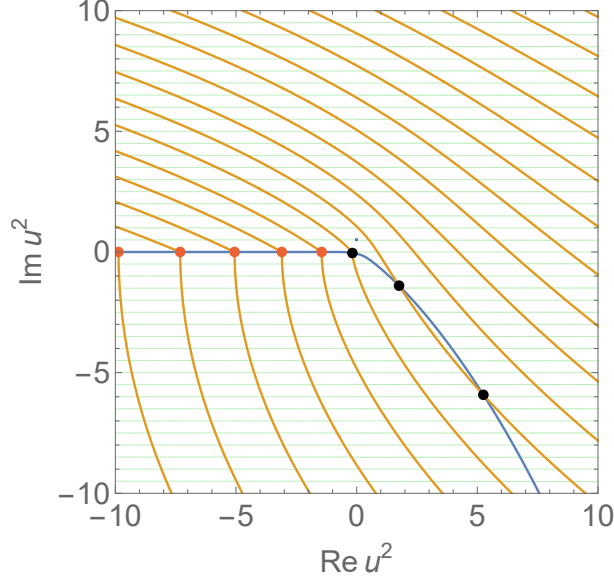


Figure 3.3: The spectrum of the inverted harmonic oscillator $V(z) = -z^2/4$. We have imposed the Dirichlet boundary condition at $z = L = 10$, and have assumed that there is only an outgoing mode at $z \rightarrow -\infty$. The blue curve represents $2\text{Re } \mathcal{Z}(z_b, L) = \text{Re } \tilde{\mathcal{Z}}_{\eta=1}$ and the orange curve represents $2\text{Im } \mathcal{Z}(z_b, L) + \pi/2 \in 2\pi\mathbb{Z}$. While the blobs represent the intersection points of these two curves, they do not precisely lie on the green lines, which represent $\text{Im } \tilde{\mathcal{Z}}_{\eta=1} = 0$ (that is, $\text{Im } u^2 \in \frac{1}{2}\mathbb{Z}$). Numerically, the red blobs satisfies $\text{Im } \tilde{\mathcal{Z}}_{\eta=1} / \text{Re } \tilde{\mathcal{Z}}_{\eta=1} < 10^{-2}$, but the black blobs are not. Note that the blue curve passes slightly below the real u^2 -axis, or the branch cut.

that obviously vanishes at $z = L$, and then analytically continue it along the path in Fig. 3.2 backwards. As a result, we obtain

$$\psi^{\text{III}}(z) = \left[1 + e^{2\eta\mathcal{Z}(z_a, z_b)} \left(1 - i e^{2\eta\mathcal{Z}(z_b, L)} \right) \right] \psi_+^{\text{I}}(L, z) - \left(1 + i e^{-2\eta\mathcal{Z}(z_b, L)} \right) \psi_-^{\text{I}}(L, z). \quad (3.36)$$

The outgoing boundary condition at $z \rightarrow -\infty$ requires that the coefficient of ψ_+^{I} vanishes, that is,

$$1 + e^{2\eta\mathcal{Z}(z_a, z_b)} \left(1 - i e^{2\eta\mathcal{Z}(z_b, L)} \right) = 0. \quad (3.37)$$

Because it is difficult to solve this condition accurately for u^2 , here we look into the qualitative behavior of the spectrum. A sufficient condition of the quantization condition (3.37) is given by

$$2\eta \text{Im } \mathcal{Z}(z_b, L) + \frac{\pi}{2} \in 2\pi\mathbb{Z}, \quad (3.38)$$

$$2\eta \text{Re } \mathcal{Z}(z_b, L) = \tilde{\mathcal{Z}}_{\eta}, \quad (3.39)$$

where

$$\tilde{\mathcal{Z}}_{\eta} := \ln \left(1 + e^{-2\eta\mathcal{Z}(z_a, z_b)} \right), \quad (3.40)$$

and $\mathcal{Z}(z_a, z_b)$ is equal to (3.32). Since L is large enough, $\mathcal{Z}(z_b, L)$ can be approximated as

$$\mathcal{Z}(z_b, L) \simeq \frac{i}{2} \left[\frac{L^2}{2} + 2u^2 \ln L + u^2 (1 - i\pi - \ln u^2) \right]. \quad (3.41)$$

3.2. WKB approximation

The approximate solutions of (3.38) and (3.39) with (3.41) and $\eta = 1$ are plotted in Fig. 3.3. The orange curve is for (3.38), and the blue/green curves are for the real and imaginary parts of (3.39), respectively. Although the intersection points of these three curves should represent the quantized spectrum, there are no precise solutions. This is because the sufficient quantization conditions (3.38) and (3.39) impose three constraints, and the system becomes overdetermined. Nevertheless, referring to the exact result in Fig. 5.5, the WKB spectrum captures the correct qualitative behavior. Their quantitative mismatch would be due to the truncation of the η -expansion in the WKB solution, since we must set $\eta = 1$ to match the exact calculation.

3.2.3 Long-lived quasinormal modes in AdS black hole

Finally, let us analyze QNMs in an AdS black hole in the framework of the WKB method, based on [95]. A scalar field $\Phi = e^{-i\Omega t} Y_{lm}(\text{angles}) \psi(r) r^{(1-d)/2}$ with mass μ in $(d+1)$ -dimensional Schwarzschild AdS spacetime

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-1}^2, \quad f(r) = 1 + \frac{r^2}{R^2} - \left(\frac{r_0}{r}\right)^{d-2}, \quad (3.42)$$

where r_0 corresponds to the black hole temperature, obeys a Schrödinger-like equation

$$\left(\frac{d^2}{dz^2} + \Omega^2 - V(r)\right)\psi(r) = 0. \quad (3.43)$$

Here $z = -r_*(r)$ is the tortoise coordinate, and the potential V is given by

$$V(r) = f \left[\frac{(2l + d - 2)^2 - 1}{4r^2} + \frac{\nu^2 - 1/4}{R^2} + \frac{r_0^{d-2}(d-1)^2}{4r^d} \right]. \quad (3.44)$$

We define

$$p := l + \frac{d-2}{2}, \quad (3.45)$$

and consider the large p limit with fixed energy

$$u := \frac{\Omega}{p}. \quad (3.46)$$

Then, the equation of motion (3.43) can be cast into the form of a WKB formulation (3.9) as

$$\left[\frac{d^2}{dz^2} + p^2(u^2 - V_0(r))\right]\psi(r) = 0, \quad (3.47)$$

where the WKB potential V_0 is

$$V_0(r) = \frac{f(r)}{r^2}. \quad (3.48)$$

The potential in the tortoise coordinate z is illustrated in Fig. 3.4. Approaching the black hole horizon $z = \infty$, the potential goes to zero:

$$V(z \rightarrow +\infty) \rightarrow 0. \quad (3.49)$$

3.2. WKB approximation

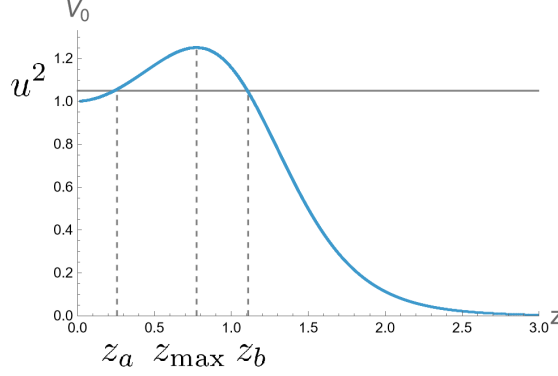


Figure 3.4: The WKB potential V_0 in the tortoise coordinate $z = -r_*$. For this figure we took $d = 3$, $R = 1$, $r_0 = 4\sqrt{3}/9$, and $\mu = 0$.

At the AdS boundary $z = 0$, the potential behaves as

$$V(z = 0) = \frac{1}{R^2}, \quad (3.50)$$

$$V'(z = 0) = 0. \quad (3.51)$$

A WKB solution vanishing at $z = 0$ can be constructed as $\psi(z) = \psi_+(0, z) - \psi_-(0, z)$. Because the phase integral $\mathcal{Z}$ can be approximated as

$$\mathcal{Z}(0, z \simeq 0) \simeq i \sqrt{u^2 - \frac{1}{R^2}} z, \quad (3.52)$$

this solution behaves as $\psi(z \simeq 0) \simeq \sin\left(\sqrt{u^2 - 1/R^2} z\right) \simeq \sqrt{u^2 - 1/R^2} z$. However, it does not reproduce the general asymptotic behavior (3.5) of QNMs, that is,

$$\Phi(r \rightarrow \infty) \sim r^{-\Delta_+} \quad \text{or} \quad \psi(z \simeq 0) \sim z^{\frac{1}{2} + \nu}. \quad (3.53)$$

Note that $z \simeq R^2/r$ near the AdS boundary. So, we fix this problem by referring to the solution of the original equation (3.43) near the boundary.

Around $z \simeq 0$, the original potential (3.44) is approximated as

$$V(z \simeq 0) \simeq \frac{p^2}{R^2} + \frac{\nu^2 - \frac{1}{4}}{z^2}. \quad (3.54)$$

When $u > 1/R^2$, the equation (3.43) with this potential can be solved exactly and has two independent solutions

$$\sqrt{z} J_\nu\left(p\sqrt{u^2 - 1/R^2} z\right) \quad \text{and} \quad \sqrt{z} Y_\nu\left(p\sqrt{u^2 - 1/R^2} z\right), \quad (3.55)$$

where J_ν and Y_ν are the first and second kinds of Bessel functions. Because at $z = 0$ the former vanishes and the latter diverges for $\nu > 0$, we should choose $\sqrt{z} J_\nu$. In the large p limit, for $z \gtrsim 0$, we can expand this solution as

$$\psi(z) \simeq \cos\left(p\sqrt{u^2 - 1/R^2} z - \frac{2\nu + 1}{4}\pi\right), \quad (3.56)$$

3.2. WKB approximation

which can be realized in the modified form of the WKB solution as

$$\psi(z) \simeq \psi_+(0, z) e^{-i\pi(2\nu+1)/4} + \psi_-(0, z) e^{+i\pi(2\nu+1)/4}. \quad (3.57)$$

This is our starting point for the analytic continuation.

Here, to identify the Stokes geometry for V_0 is the most essential and complicated problem. Since the equation for turning points (3.17) becomes a polynomial of degree d in r , it allows d turning points. In addition, [95] pointed out that there are periodically infinite number of turning points in the complex z -plane due to the multiplicity of the tortoise coordinate $z(r)$, corresponding to a choice of the integration contour from r to ∞ . They showed that these infinite turning points contribute to the analytic continuation of the WKB solution for small p and large ν , where the WKB potential is monotonic in z . Although the structure of the Stokes geometry for the monotonic potential can be evaluated, our potential V_0 has a photon sphere, or a local maximum, at $z = z_{\max}$, which is defined as $V'(z_{\max}) = 0$. This makes the pattern of the Stokes geometry for $1/R^2 < \text{Re } u^2 < V(z_{\max})$ quite complicated, even if $\text{Im } u^2$ is almost zero.

So, in [95], it is roughly assumed that we can ignore the periodic structure of the Stokes geometry.³ Moreover, they suggested that the turning points which contribute to the analytic continuation from $z = 0$ to $z = \infty$ are only the two corresponding to the classical turning points z_a and z_b among the d points.

Under these simplifications, the Stokes geometry for V_0 can be regarded as the same structure as that of the inverted harmonic oscillator.⁴ Then, we can take a path of the analytic continuation as shown in Fig. 3.2 again, but now it starts from $z = 0$ in region I and goes to $z = \infty$ in region III. The calculation is almost parallel to that in Sec. 3.2.2. We start from (3.57) as $\psi^{\text{I}}(z)$ and obtain

$$\begin{aligned} \psi^{\text{I}}(z) = & \left(1 - ie^{-2p\mathcal{Z}(0, z_a) + i\pi(2\nu+1)/2}\right) \psi_+^{\text{III}}(0, z) e^{-i\pi(2\nu+1)/4} \\ & + \left[1 + e^{2p\mathcal{Z}(z_a, z_b)} \left(1 + ie^{2p\mathcal{Z}(0, z_a) - i\pi(2\nu+1)/2}\right)\right] \psi_-^{\text{III}}(0, z) e^{+i\pi(2\nu+1)/4}. \end{aligned} \quad (3.58)$$

Because the phase integral $\mathcal{Z}$ can be approximated as

$$\mathcal{Z}(0, z \rightarrow \infty) \simeq i\sqrt{u^2} z, \quad (3.59)$$

the outgoing boundary condition at $z \rightarrow \infty$ requires that the coefficient of ψ_-^{III} vanishes, that is,

$$2p \text{Im } \mathcal{Z}(0, z_a) - \pi(\nu + 1) \in 2\pi\mathbb{Z}. \quad (3.60)$$

$$2p \text{Re } \mathcal{Z}(0, z_a) = \ln \left(1 + e^{-2p\mathcal{Z}(z_a, z_b)}\right). \quad (3.61)$$

The phase integral $\mathcal{Z}(0, z_a)$ is too complicated to calculate in general. So, we focus on $u \gtrsim 1/R$, where a turning point z_a goes to the AdS boundary $z \simeq 0$. Then, we obtain

$$\mathcal{Z}(0, z_a) \simeq \frac{i\pi}{2}(uR - 1), \quad (3.62)$$

and the quantization condition (3.60) gives

$$\text{Re}(uR) \simeq 1 + \frac{1}{p}(2n + \nu + 1) =: u_n R \quad (n = 0, 1, \dots). \quad (3.63)$$

³The problem of the periodic structure can be avoided by working in the complex r -coordinate, not in the tortoise coordinate z . However, the identification of the entire Stokes geometry is still a nontrivial issue.

⁴We will comment on the validity of this assumption in Chap. 6.

3.2. WKB approximation

Recalling the definition of p (3.45) and u (3.46), the energy of the QNMs are given by

$$\text{Re}(\Omega R) \simeq l + \Delta + 2n, \quad (3.64)$$

which coincides with the normal modes in the pure global AdS spacetime (2.74). On the other hand, since $\text{Im } u$ is assumed to be small, the condition (3.61) leads to

$$\text{Im}(\Omega R) \simeq -\frac{1}{\pi} \exp\left(-2p \int_{z_a}^{z_b} dz' \sqrt{V(z') - u_n^2}\right), \quad (3.65)$$

whose magnitude is exponentially small in p , but it is slightly negative.

The WKB analysis so far revealed that there are long-lived quasinormal modes in the QNM spectrum of the Schwarzschild AdS black hole. The existence of these modes was numerically confirmed in [81, 84] with some quantitative corrections. Through the AdS/CFT dictionary, these QNMs suggested that there are long-lived quasiparticles in the spectrum of a thermal CFT on $\mathbb{R} \times \mathbb{S}^{d-1}$. These quasiparticles will be refound in Chap. 5 through another approach. There, in addition, the decay of these modes will be discovered beyond the $\Omega \simeq p/R$ approximation.

Chapter 4

Review of AdS/CFT dictionary II: boundary to bulk

In the previous two chapters, we described the calculation of CFT correlators and spectra from a given bulk geometry through the AdS/CFT correspondence. To investigate (quantum) gravity theory, conversely, we must draw constituents of bulk gravity theory from boundary CFT data. Actually, CFT correlators have enough information to recover the metric of the dual bulk spacetime. This kind of “inverse” problem, called bulk reconstruction, is the subject of this chapter. In Sec. 4.1, we describe the lightcone cuts method [35], by which we can reconstruct the causal structure of an asymptotically AdS spacetime with an arbitrary dimension. However, it requires $(d+3)$ -point correlators in CFT_d and is hardly feasible. So, in Sec. 4.2, we describe a more restricted but more practical approach by [46, 47]. This method, here we call holographic inverse scattering, can reconstruct the entire geometry of a static AdS black brane from the retarded Green function in the dual CFT.

4.1 Lightcone cuts method¹

4.1.1 General formalism

In the lightcone cuts method by Engelhardt and Horowitz [35], (i) first, we should reconstruct a set of bulk subregions (called lightcone cuts) on the AdS boundary from the CFT correlators, and (ii) then, we recover the causal structure of the bulk spacetime from the lightcone cuts. The CFT data are used only in the former part, and the latter is closed in the bulk calculation. So, we begin with the description of (ii).

Let (M, g) be a $(d+1)$ -dimensional asymptotically AdS spacetime², and $J^\pm(p)$ be the causal future/past of $p \in M$. The future/past lightcone cut (cut for short) of p , $C_p^\pm$, is defined as

$$C_p^\pm := \partial J^\pm(p) \cap \partial M, \quad (4.1)$$

where ∂ means the boundary (especially, ∂M denotes the conformal boundary of the spacetime). Roughly speaking, this is the set of the endpoints of the null geodesics departing from p (see Fig. 4.1). In addition, we define $C^\pm[S] := \{C_p^\pm \mid p \in S\}$ as the set of the future/past cuts for $S \subset M$.

¹The description in this section is based on our paper [39].

²In [35], more technical assumptions are imposed on M to rigorously show the facts here.

4.1. Lightcone cuts method

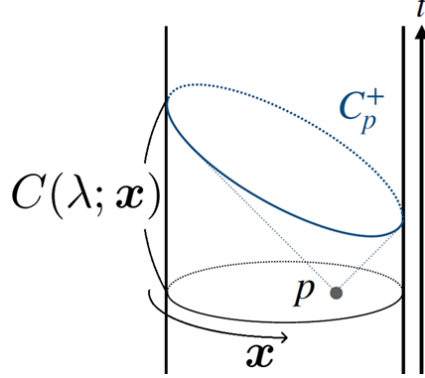


Figure 4.1: The future lightcone cut of a bulk point p . When we write the cut as $t = C(\lambda; \mathbf{x})$ in terms of the boundary coordinate $(t, \mathbf{x})$, the cut is expressed as the height at the boundary.

Under this definition, the following holds ($I^\pm$ denotes chronological future/past):

1. For $p \in I^-(\partial M)$ ($I^+(\partial M)$), there exists a unique future (past) cut of p .
2. Let $p, q \in I^-(\partial M)$ ($I^+(\partial M)$). If and only if $p = q$, $C_p^+ \cap C_q^+$ ($C_p^- \cap C_q^-$) is open and non-empty.
3. If C_p^+ is tangent to C_q^+ at a unique point where both cuts are differentiable, then p and q are null separated. The same thing holds for C_p^- and C_q^- .

Properties 1 and 2 together roughly mean that there is a one-to-one map between $I^-(\partial M)$ and the set $C^+[I^-(\partial M)]$ of the future cuts, as well as between $I^+(\partial M)$ and $C^-[I^+(\partial M)]$. Hence, a cut can be identified with the corresponding bulk point, and the set of cuts can be regarded as the bulk region chronologically connected to ∂M . On the other hand, property 3 says that the set $C^\pm[S]$ of the cuts knows the bulk null vectors on S . Owing to this property, once either $C^+[S]$ or $C^-[S]$ is obtained, the causal structure of S can be constructed.

Let us organize the above facts in gravity theory from the view of the bulk reconstruction. Suppose that we can access all the data of the boundary CFT, whereas we know nothing about the dual bulk gravity theory. If the lightcone cuts $C^\pm[I^\mp(\partial M)]$ is somehow obtained by the CFT data, it will provide the holographic *definition* of the bulk points. That is, the bulk spacetime manifold M can be defined by the set of the lightcone cuts. In this sense, they constitute the most fundamental part of the AdS/CFT dictionary.

Next, we move on to (i). To identify cuts from CFT data, a method using the bulk-point singularity [144] was proposed in [35]. For instance, a CFT two-point function $\langle \mathcal{O}(x)\mathcal{O}(y) \rangle$ can be calculated as the propagator between x and y . When the CFT has a gravity dual, there is a contribution propagating via a point z inside the bulk spacetime. If the bulk point z is null separated from the boundary points x and y , the two-point function will diverge [45]. This can be generalized to higher-point functions. More precisely, the bulk-point singularity is a divergence of correlators in holographic QFTs, and it happens if there exists a bulk point such that it is null separated from all points in the correlator and we can assign momenta to the propagators while conserving the total momentum.³

³Although the existence of the bulk-point singularity is confirmed only for the vacuum correlators dual to the pure AdS, its existence and possibility of application to general geometries are also mentioned in [35].

4.1. Lightcone cuts method

Using this divergence, the set of cuts can be obtained from the CFT data as follows: first, we fix $d + 1$ points on the boundary so that any two points are spacelike separated. The intersection point of their latent $d + 1$ cuts fixes an interaction point z in the bulk. Next, putting two additional boundary points in the chronological future of them, we consider the correlator among those $d + 3$ points. Then, we move the last two points and find their configuration where the correlator diverges. The orbit drawn by the two points gives the holographic definition of the future cut C_z^+ of z . Repeating this procedure, the set of the future cuts $C^+[I^-(\partial M)]$ is obtained. $C^-[I^+(\partial M)]$ is also obtained similarly.

Now, let us assume that we have constructed the set of future cuts $C^+[I^-(\partial M)]$ from the CFT data. Due to properties 1 and 2, we will be able to label these cuts by $d + 1$ continuous parameters $\lambda = (\lambda^0, \dots, \lambda^d)$ (the same number as the bulk dimension) as $C^+(\lambda)$. Here we choose λ so that $C^+(\lambda)$ is smooth with respect to λ . At this stage, the set $\{C^+(\lambda)|\lambda\}$ is regarded as a holographic definition of $I^-(\partial M)$ in the bulk spacetime, and λ is interpreted as a coordinate. The metric to be reconstructed will be written in this coordinate as $g_{\mu\nu}(\lambda)$.

To obtain the metric, we use property 3. We should find infinitesimal variations $\delta\lambda = \delta\lambda(\lambda)$ such that $C^+(\lambda)$ and $C^+(\lambda + \delta\lambda)$ are tangent at exactly one point. If we write the cut as $t = C(\lambda; \mathbf{x})$ with $(t, \mathbf{x})$ being the boundary coordinate (see Fig. 4.1), this condition is written as

$$C(\lambda; \mathbf{x}) = C(\lambda + \delta\lambda; \mathbf{x}), \quad \nabla_{\mathbf{x}} C(\lambda; \mathbf{x}) = \nabla_{\mathbf{x}} C(\lambda + \delta\lambda; \mathbf{x}). \quad (4.2)$$

The former means that both cuts meet at $(t, \mathbf{x})$, and the latter means that they share the same tangent plane there. Property 3 claims that all of such infinitesimal $\delta\lambda$ should be a null vector at λ . Hence, the metric must satisfy

$$g_{\mu\nu}(\lambda)\delta\lambda^\mu\delta\lambda^\nu = 0, \quad (4.3)$$

where μ and ν run from 0 to d . The metric can be determined through the above equation by finding an adequate number of such $\delta\lambda$'s for each λ . However, the fact that (4.3) is homogeneous makes it impossible to fix the normalization of $g_{\mu\nu}(\lambda)$, and hence all we can do is to find $d(d+3)/2$ null vectors and solve for $g_{\mu\nu}(\lambda)$ up to a conformal factor which may depend on λ . If we also want to cover $I^+(\partial M)$, we should do the same procedure for the past cuts. Eventually, we can determine the conformal structure of the emergent spacetime, but the degree of freedom of the conformal transformation cannot be fixed.

4.1.2 Pure AdS in global patch

As an example, we demonstrate the reconstruction of the pure global AdS_{d+1} geometry with the lightcone cuts method, based on our work [39]. Here, instead of the holographic calculation (i), we calculate the cuts in terms of the bulk gravity theory. The metric in question is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-1}^2, \quad f(r) = 1 - \frac{r^2}{L^2}, \quad (4.4)$$

where the metric of $\mathbb{S}^{d-1}$ is explicitly written as

$$d\Omega_{d-1}^2 = \sum_{k=1}^{d-1} \sin^2 \theta_1 \cdots \sin^2 \theta_{k-1} (d\theta_k)^2. \quad (4.5)$$

4.1. Lightcone cuts method

Let $(t, \boldsymbol{\theta}, r) = (T, \boldsymbol{\Theta}, R)$ be a bulk point (we use $\boldsymbol{\theta}$ to collectively express θ_k 's). A future/past null geodesic starting from this point along the θ_1 -direction is determined by solving the equations

$$f(r) \dot{t} = \pm 1, \quad r^2 \dot{\theta}_1 = \text{const.}, \quad 0 = -f(r) \dot{t}^2 + \frac{\dot{r}^2}{f(r)} + r^2 \dot{\theta}_1^2, \quad (4.6)$$

$$(t, \boldsymbol{\theta}, r)|_{\lambda=0} = (T, \boldsymbol{\Theta}, R), \quad (4.7)$$

where λ is an affine parameter, and the dot means the λ -derivative. Note that the first and second equations in (4.6) fix the energy and the angular momentum of the particle. By setting $r = \infty$ and removing the angular momentum, we obtain the one-parameter family of endpoints

$$t = T \pm L \cos^{-1} \left[\frac{R}{\sqrt{R^2 + L^2}} \cos(\theta_1 - \Theta_1) \right] \quad (4.8)$$

parametrized by θ_1 .

To obtain the whole lightcone cuts, we must take all the null geodesics into account, not only those directed along the θ_1 -direction. Instead of solving the geodesic equation for general cases, here we make use of the symmetry of the bulk spacetime. The global pure AdS_{d+1} geometry has the $\text{SO}(d)$ rotation on $\mathbb{S}^{d-1}$ as an isometry. Under this transformation, $|\theta_1 - \Theta_1|$ appearing in (4.8) is replaced with the geodesic distance between $\boldsymbol{\theta}$ and $\boldsymbol{\Theta}$ on $\mathbb{S}^{d-1}$, which we write as $\Omega_{d-1}(\boldsymbol{\theta}, \boldsymbol{\Theta})$ hereafter. Therefore, the whole future/past lightcone cut of the bulk point $(T, \boldsymbol{\Theta}, R)$ is given by

$$t = T \pm L \cos^{-1} \left[\frac{R}{\sqrt{R^2 + L^2}} \cos \Omega_{d-1}(\boldsymbol{\theta}, \boldsymbol{\Theta}) \right]. \quad (4.9)$$

Here, we forget the knowledge about the bulk spacetime and assume that a set of future cuts $C^+(\lambda)$ has been reconstructed, through some holographic calculation, as

$$t = T + L \cos^{-1} [Y \cos \Omega_{d-1}(\boldsymbol{\theta}, \boldsymbol{\Theta})], \quad (4.10)$$

$$T \in \mathbb{R}, \quad 0 \leq Y \leq 1, \quad 0 \leq \Theta_i \leq \pi \quad (i = 1, \dots, d-2), \quad 0 \leq \Theta_{d-1} < 2\pi, \quad (4.11)$$

where $\lambda = (T, Y, \boldsymbol{\Theta})$ are $d+1$ parameters that we regard as the bulk coordinate. The domains of definition (4.11) should be determined by the range where the bulk-point singularity occurs. Note that the difference between R and Y is just a choice of the bulk local coordinate.

Because it is difficult to solve directly the tangential conditions (4.2), we instead use mathematical induction on d and the symmetry of (4.10). Leaving the details of the computation of the conformal metric from the cut (4.10) to App. A.1, here we describe the outline:

- ii-1) Reconstruct the conformal metric in $d = 2$ using (4.3).
- ii-2) Assuming that the conformal metric in $d = D-1$ ($D \geq 3$) has already been reconstructed, reconstruct the conformal metric in $d = D$ around the equator $\Theta_i = \pi/2$.
- ii-3) Rotate the coordinate in the above result to obtain the conformal metric for any $\boldsymbol{\Theta}$.

Following these three steps, we obtain the conformal metric as

$$ds^2 = e^{2\sigma(\lambda)} \left[-(1 - Y^2) dT^2 + L^2 dY^2 + L^2 Y^2 (1 - Y^2) d\Omega_{d-1}^2 \right], \quad (4.12)$$

where $\sigma(\lambda)$ is the undetermined conformal factor. Substituting $Y = R/\sqrt{R^2 + L^2}$ into (4.12), we can confirm this conformal metric reproduces the conformal structure of the true metric (4.4).

4.2. Holographic inverse scattering

4.1.3 Summary and comments

Here we comment on some advantages and feasibility of the lightcone cuts method.

- This method consists of two steps:
 - (i) First, we should reconstruct a set of lightcone cuts from the bulk-point singularity of CFT correlators.
 - (ii) Then, we can recover the causal structure of the bulk geometry from the cuts.
- This method is formulated covariantly and applicable to asymptotically AdS spacetimes with an arbitrary dimension. It has a wide range of applications [37–39, 145–147].
- There are several attempts to identify the conformal factor of the bulk metric [36–39]. For example, our work [39] revealed that the causality in the boundary CFT can be used for the identification in some special cases. However, a general prescription has not been known yet.
- To obtain cuts using the bulk-point singularity, we must (repeatedly) calculate $(d+3)$ -point correlators in CFT_d . It is too costly to perform practically in general. Even in CFT_2 , we should calculate five-point functions, which cannot be determined by the conformal symmetry.
- In addition, since we use boundary-to-boundary correlators, all bulk points we can reconstruct are in $I^+(\partial M) \cap I^-(\partial M)$, that is, points that can connect to the boundary in both the chronological future and past. For example, the bulk points inside the black hole horizon are not contained in $I^-(\partial M)$ and cannot be reconstructed. Also, we cannot reconstruct any point in the BTZ black hole spacetime, because all the null geodesics shot from the boundary are absorbed by the black hole and never come back again (this means that there is no photon sphere around the BTZ black hole).
- The existence of the bulk-point singularity is assured only in the global patch, where the boundary $\mathbb{R} \times \mathbb{S}^{d-1}$ is closed. For example, in the pure Poincaré AdS, any null geodesic shot from the boundary has a momentum directed away from the boundary, so the momentum conservation necessary for the singularity cannot be satisfied.

4.2 Holographic inverse scattering

Although the lightcone cuts method has a very wide range of applications, it requires intractable $(d+3)$ -point correlators in CFT_d . In this section, we describe a more restricted but more practical reconstruction method by Fan and Yang [46, 47], which is based on inverse scattering theory in quantum mechanics. In Sec. 4.2.1, we give a brief review of inverse scattering theory. Then, in Sec. 4.2.2, this framework is applied to the reconstruction of a static and planar symmetric AdS black brane. In this method, what we need is just a retarded Green function in the dual CFT. The reconstruction of the BTZ black hole is demonstrated in Sec. 4.2.3. Sec. 4.2.4 is for the summary of this method.

4.2. Holographic inverse scattering

4.2.1 Inverse scattering theory

We summarize some notations and related results of inverse scattering theory in quantum mechanics (see [148] for details). We address the radial Schrödinger equation

$$\left(\frac{d^2}{dr^2} + \omega^2 - V(r) - \frac{l(l+1)}{r^2}\right)\psi(r) = 0, \quad (4.13)$$

where the potential $V(r)$ is assumed to satisfy the regularity condition

$$\int_b^\infty dr r |V(r)| < \infty, \quad \forall b \geq 0. \quad (4.14)$$

This condition roughly means that the potential diverges at most $\mathcal{O}(r^{-2})$ at $r = 0$ and decreases at least $\mathcal{O}(r^{-2})$ at $r \rightarrow \infty$.

The equation (4.13) has two independent solutions. In the free case $V(r) \equiv 0$, the solutions are given by

$$u_l(\omega r) = \sqrt{\frac{\pi\omega r}{2}} J_{l+1/2}(\omega r), \quad (4.15)$$

$$w_l(\omega r) = ie^{i\pi l} \sqrt{\frac{\pi\omega r}{2}} H_{l+1/2}^{(1)}(\omega r), \quad (4.16)$$

where J_ν and $H_\nu^{(1)}$ are the first kind of Bessel function and Hankel function. The former solution behaves at $r \simeq 0$ as

$$u_l(\omega r \simeq 0) \simeq \frac{(\omega r)^{l+1}}{(2l+1)!!}. \quad (4.17)$$

Referring to this, we define the regular solution φ_l of (4.13) for general $V(r)$ by

$$\lim_{r \rightarrow 0} (2l+1)!! r^{-(l+1)} \varphi_l(\omega, r) = 1. \quad (4.18)$$

On the other hand, the latter solution is singular at $r = 0$

$$w_l(\omega r \simeq 0) \simeq (-1)^l (2l-1)!! (\omega r)^{-l} \quad (4.19)$$

and behaves at $r \rightarrow \infty$ as

$$w_l(\omega r) \underset{r \rightarrow \infty}{\simeq} e^{i\pi l/2} e^{i\omega r}. \quad (4.20)$$

Corresponding to this, the Jost solution f_l of (4.13) for general $V(r)$ is defined by

$$\lim_{r \rightarrow \infty} e^{-i\pi l/2} e^{-i\omega r} f_l(\omega, r) = 1. \quad (4.21)$$

Note that these solutions satisfy

$$\varphi_l(\omega, r)^* = \varphi_l(-\omega, r) = \varphi_l(\omega, r), \quad (4.22)$$

$$\left[e^{-i\pi l/2} f_l(\omega, r) \right]^* = e^{-i\pi l/2} f_l(-\omega, r) \quad (4.23)$$

for $\omega \in \mathbb{R}$ by definitions.

4.2. Holographic inverse scattering

When $\omega \neq 0$, the two solutions $f_l(\pm\omega, r)$ are independent of each other, because their Wronskian can be calculated as

$$W\{f_l(\omega, r), f_l(-\omega, r)\} = (-1)^{l+1} 2i\omega. \quad (4.24)$$

Hence, the regular solution (4.18) can be expanded as a linear combination of these two Jost solutions

$$\varphi_l(\omega, r) = \frac{i}{2\omega^{l+1}} \left[F_l(\omega) f_l(-\omega, r) - (-1)^l F_l(-\omega) f_l(\omega, r) \right]. \quad (4.25)$$

The coefficient function F_l , which is defined by

$$F_l(\omega) := (-\omega)^l W\{f_l(\omega, r), \varphi_l(\omega, r)\}, \quad (4.26)$$

is called the Jost function, and it plays an essential role in inverse scattering theory. Moreover, in the application to the bulk reconstruction, it encodes CFT data. The Jost function is defined in the upper half of the complex ω -plane $\text{Im } \omega \geq 0$. It is known that $F_l(\omega)$ has simple zeros $\omega = i\gamma_i$ on the imaginary ω -axis, which correspond to the bound states. Also, by its integral representation

$$F_l(\omega) = 1 + (-\omega)^l \int_0^\infty dr w_l(\omega r) V(r) \varphi_l(\omega, r), \quad (4.27)$$

it behaves in the high energy limit $|\omega| \rightarrow \infty$ as

$$F_l(\omega) \xrightarrow{|\omega| \rightarrow \infty} 1. \quad (4.28)$$

A complete set of the system consists of the scattering states $\{\varphi_l(\omega, r) | \omega^2 \geq 0\}$ and the bound states $\{\varphi_l(i\gamma_i, r) | \omega_i^2 < 0, \omega_i = i\gamma_i\}$. We express them collectively $\varphi_l(E, r)$ with $E := \omega^2$. Their completeness relation can be written as

$$\int_{-\infty}^\infty d\mu_l(E) \varphi_l(E, r) \varphi_l(E, r') = \delta(r - r'), \quad (4.29)$$

where the integral measure (spectral function) is defined by

$$\frac{d\mu_l(E)}{dE} := \begin{cases} \frac{1}{\pi} E^{l+1/2} \left| F_l(\sqrt{E}) \right|^{-2} & (E \geq 0), \\ \sum_i C_i \delta(E - E_i) & (E < 0) \end{cases} \quad (4.30)$$

with appropriate normalization constants $C_i > 0$ of the bound states. Since $F_l(\omega) \equiv 1$ holds in the free case $V(r) \equiv 0$ from (4.27), the difference of the spectral function from the free case becomes

$$\frac{d\nu_l(E)}{dE} := \frac{d\mu_l(E)}{dE} - \frac{d\mu_l^{(0)}(E)}{dE} = \begin{cases} \frac{1}{\pi} E^{l+1/2} \left[\left| F_l(\sqrt{E}) \right|^{-2} - 1 \right] & (E \geq 0), \\ \sum_i C_i \delta(E - E_i) & (E < 0). \end{cases} \quad (4.31)$$

4.2. Holographic inverse scattering

Note that there are no bound states in the free case.

The central issue of inverse scattering theory is to recover the potential $V(r)$ from the scattering data, which are actually encoded in the Jost function $F_l(\omega)$. Once the Jost function is obtained, we can reconstruct the potential by the following procedure: first, we calculate the spectral function (4.31). Then, we calculate the kernel $\Omega_l(r, s)$ defined by

$$\Omega_l(r, s) := \int_{-\infty}^{\infty} d\nu_l(E) \varphi_l^{(0)}(E, r) \varphi_l^{(0)}(E, s), \quad (4.32)$$

where $\varphi_l^{(0)}$ is the free regular solution

$$\varphi_l^{(0)}(\omega, r) := \omega^{-(l+1)} u_l(\omega r). \quad (4.33)$$

This kernel is used in the Gel'fand-Levitan equation

$$K_l(r, s) + \Omega_l(r, s) + \int_0^r dv K_l(r, v) \Omega_l(v, s) = 0 \quad (4.34)$$

to obtain the transformation kernel $K_l(r, s)$. Finally, it is known that the potential is given by

$$V(r) = 2 \frac{d}{dr} K_l(r, r). \quad (4.35)$$

So far, we have used the free case as the reference system. However, we can start from a given potential V_1 to construct the target potential V_2 that both satisfy the regularity condition (4.14). The modifications are quite simple. The Gel'fand-Levitan equation (4.34) is replaced by

$$\bar{K}_l(r, s) + \bar{\Omega}_l(r, s) + \int_0^r dv \bar{K}_l(r, v) \bar{\Omega}_l(v, s) = 0, \quad (4.36)$$

where

$$\bar{\Omega}_l(r, s) := \int_{-\infty}^{\infty} d\bar{\nu}_l(E) \varphi_l^{(1)}(\omega, r) \varphi_l^{(1)}(\omega, s), \quad (4.37)$$

$$\frac{d\bar{\nu}_l(E)}{dE} := \frac{d\mu_l^{(2)}(E)}{dE} - \frac{d\mu_l^{(1)}(E)}{dE} \quad (4.38)$$

$$= \begin{cases} \frac{1}{\pi} E^{l+1/2} \left[\left| F_l^{(2)}(\sqrt{E}) \right|^{-2} - \left| F_l^{(1)}(\sqrt{E}) \right|^{-2} \right] & (E \geq 0), \\ \sum_{i_2} C_{i_2}^{(2)} \delta(E - E_{i_2}^{(2)}) - \sum_{i_1} C_{i_1}^{(1)} \delta(E - E_{i_1}^{(1)}) & (E < 0). \end{cases} \quad (4.39)$$

Here the superscript (j) means the quantities defined under V_j . Then, the target potential V_2 is given by

$$V_2(r) - V_1(r) = 2 \frac{d}{dr} K_l(r, r). \quad (4.40)$$

4.2. Holographic inverse scattering

4.2.2 CFT correlator as scattering data

Let us apply the above framework to a scalar field in a static and planar symmetric AdS black brane. The metric of the AdS_{d+1} black brane can be written in the form

$$ds^2 = \frac{R^2}{z(\rho)^2} [h(\rho)(-dt^2 + d\rho^2) + d\mathbf{x}_{d-1}^2], \quad (4.41)$$

where $\rho \in \mathbb{R}_{\geq 0}$ corresponds to the tortoise coordinate. The functions $h(\rho)$ and $z(\rho)$ satisfy at the AdS boundary $\rho = 0$

$$h(0) = 1, \quad z(0) = 0, \quad z'(0) = 1, \quad (4.42)$$

and at the horizon $\rho \rightarrow \infty$

$$h(\infty) = 0, \quad z(\infty) = z_h, \quad (4.43)$$

The Klein-Gordon equation for the scalar field $\Phi(t, \rho, \mathbf{x}) = e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}} \psi(\rho) z(\rho)^{(d-1)/2}$ with mass μ on (4.41) reads

$$\left(\frac{d^2}{d\rho^2} + \omega^2 - V_k(\rho) - \frac{\nu^2 - 1/4}{\rho^2} \right) \psi(\rho) = 0, \quad (4.44)$$

where $\nu = \sqrt{\mu^2 R^2 + d^2/4}$, and the potential is given by

$$V_k(\rho) = -\frac{(d-1)z''}{2z} + \frac{d^2-1}{4} \frac{z'^2}{z^2} + \left(\nu^2 - \frac{d^2}{4} \right) \frac{h}{z^2} - \frac{\nu^2 - 1/4}{\rho^2} + \mathbf{k}^2 h. \quad (4.45)$$

Note that the ρ^{-2} term is intentionally extracted from the potential to ensure its regularity at $\rho = 0$.

This potential has enough information about the bulk metric. Once the potential is obtained for two different momenta $\mathbf{k}_1$ and $\mathbf{k}_2$, a metric component $h(\rho)$ can be constructed by

$$h(\rho) = \frac{V_{k_1}(\rho) - V_{k_2}(\rho)}{\mathbf{k}_1^2 - \mathbf{k}_2^2}. \quad (4.46)$$

Then, the other component $z(\rho)$ can be obtained by solving (4.45) as the differential equation for $z(\rho)$ with the given functions $V_{k_1}(\rho)$ and $h(\rho)$. In solving that, we should impose the boundary conditions⁴

$$\lim_{\rho \rightarrow +0} \frac{z(\rho)}{\rho} = 1, \quad z'(\infty) = 0. \quad (4.47)$$

The potential $V_k(\rho)$ can be constructed by solving the Gel'fand-Levitan equation. Comparing the radial Schrödinger equation (4.13) and (4.44), we can formally set $l = \nu - 1/2$ to match these systems. Then, the regular solution (4.33) in the free case becomes

$$\varphi_{\nu-1/2}^{(0)}(\omega, \rho) = \omega^{-(\nu+1/2)} \sqrt{\frac{\pi\omega\rho}{2}} J_\nu(\omega\rho), \quad (4.48)$$

⁴In [46, 47], $z(\infty) = z_h$ is imposed as a boundary condition. However, from the viewpoint of the boundary CFT, the horizon radius z_h cannot be identified before reconstructing the bulk metric. Hence, here we propose to impose $z'(\infty) = 0$ instead.

4.2. Holographic inverse scattering

and the kernel $\Omega_k(\rho, \sigma) := \Omega_{\nu-1/2}(\rho, \sigma)$ (4.32) can be written as

$$\Omega_k(\rho, \sigma) = \int_0^\infty d\omega \hat{J}_\nu(\omega\rho) \hat{J}_\nu(\omega\sigma) \left[\frac{1}{|F_k(\omega)|^2} - 1 \right], \quad (4.49)$$

where $\hat{J}_\nu(z) := zJ_\nu(z)$ and $F_k := F_{\nu-1/2}$. Here, it is assumed that there are no bound states. Because a bound state $\omega^2 < 0$ corresponds to an unstable QNM $\text{Im}\omega > 0$,⁵ this assumption means that the background black brane (or, equivalently, the thermal state in the boundary CFT) is stable against a perturbation.

Now, we want to interpret the Jost function $F_k(\omega)$ as CFT data. Similarly to the free case (4.17) and (4.19), the equation (4.44) has the basis around the AdS boundary $\rho = 0$, which behaves as

$$\varphi_k^{(\pm)}(\omega, \rho \simeq 0) \simeq \rho^{1/2 \pm \nu} (1 + \dots), \quad (4.50)$$

since $V_k(\rho \simeq 0) \simeq \mathbf{k}^2$. Here the plus one is the regular solution:

$$\varphi_k^{(+)}(\omega, \rho) = \frac{2^{\nu+1/2} \Gamma(\nu+1)}{\sqrt{\pi}} \varphi_{\nu-1/2}(\omega, \rho). \quad (4.51)$$

Because the asymptotic behaviors (4.50) of $\varphi_k^{(\pm)}$ are translated into that of Φ as $\Phi(\rho \simeq 0) \simeq z^{\Delta_\pm}$, they are exactly the normalizable and non-normalizable modes appearing in (2.15). On the other hand, the basis around $\rho \rightarrow \infty$ consists of the Jost functions $f_k(\pm\omega, \rho) := f_{\nu-1/2}(\pm\omega, \rho)$, which are an outgoing ($+\omega$) and ingoing ($-\omega$) mode at the horizon by the definitions (4.21). Therefore, expanding the outgoing mode $f_k(\omega, \rho)$ in terms of $\varphi_k^{(\pm)}$ as

$$f_k(\omega, \rho) = \Phi_k^{(-)}(\omega) \varphi_k^{(-)}(\omega, \rho) + \Phi_k^{(+)}(\omega) \varphi_k^{(+)}(\omega, \rho), \quad (4.52)$$

the ratio of the coefficients coincides the retarded Green function G_R of the boundary CFT

$$G_R(\omega, \mathbf{k}) \propto \frac{\Phi_k^{(+)}(\omega)}{\Phi_k^{(-)}(\omega)}, \quad (4.53)$$

according to the Son-Starinets prescription (see Sec. 2.2). Combining this with the definition (4.26) of the Jost function F_k , we get

$$|F_k(\omega)|^2 = \frac{\pi\nu|\omega/2|^{2\nu}}{\Gamma(\nu+1)^2 \text{Im} G_R(\omega, \mathbf{k})}. \quad (4.54)$$

The proportional constant in (4.53) can be fixed by the asymptotic behavior (4.28) of the Jost function. Note that the values of G_R only in $\omega \in [0, \infty)$ contribute to the integral (4.49), and the stable QNMs $\text{Im}\omega < 0$ do not. Eventually, the CFT two-point function G_R (or, the spectral function $\rho(\omega) = \text{Im} G_R(\omega)$ (2.73)) plays a role of the scattering data, $|F_k|^2$.

⁵By (4.52), a QNM (3.6) is a Jost solution $f_k(\omega, \rho) = \Phi_k^{(+)} \varphi_k^{(+)}(\omega, \rho)$. This vanishes at $\rho = 0$ by definition (4.50). In addition, if this QNM is unstable $\text{Im}\omega > 0$, it also vanishes at $\rho = \infty$: $f_k(\omega, \rho) \sim e^{i\omega\rho} \sim e^{-(\text{Im}\omega)\rho} \rightarrow 0$. Hence, an unstable QNM is square integrable and corresponds to a bound state $\omega^2 < 0$, and vice versa.

4.2. Holographic inverse scattering

4.2.3 BTZ black hole

As an example, we demonstrate the reconstruction of the BTZ black hole with holographic inverse scattering. While the BTZ black hole considered in Sec. 2.2.3 had the spherical horizon, the one with the planar horizon has the same form of the metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 dx^2, \quad f(r) = \frac{r^2 - r_h^2}{R^2}, \quad (4.55)$$

where $x \in \mathbb{R}$ is dimensionless. Introducing the tortoise coordinate

$$\rho := \int_r^\infty \frac{ds}{f(s)} = \frac{R^2}{r_h} \tanh^{-1} \left(\frac{r_h}{r} \right), \quad (4.56)$$

the metric becomes

$$ds^2 = \frac{r_h^2}{R^2 \tanh^2(r_h \rho / R^2)} \left[\frac{1}{\cosh^2(r_h \rho / R^2)} (-dt^2 + d\rho^2) + R^2 dx^2 \right]. \quad (4.57)$$

Hence, the metric functions are

$$z(\rho) = \frac{R^2}{r_h} \tanh \left(\frac{r_h}{R^2} \rho \right), \quad h(\rho) = \frac{1}{\cosh^2(r_h \rho / R^2)}. \quad (4.58)$$

The retarded Green function in a thermal CFT dual to this geometry can be calculated as

$$G_R(\omega, k) = C \frac{r_h^{2\nu} \Gamma(-\nu) \Gamma\left(\frac{\Delta}{2} + \frac{i(k-\omega)}{4\pi T}\right) \Gamma\left(\frac{\Delta}{2} - \frac{i(k+\omega)}{4\pi T}\right)}{\Gamma(\nu) \Gamma\left(1 - \frac{\Delta}{2} + \frac{i(k-\omega)}{4\pi T}\right) \Gamma\left(1 - \frac{\Delta}{2} - \frac{i(k+\omega)}{4\pi T}\right)}, \quad (4.59)$$

where C is an overall constant depending on the normalization of the bulk action of the scalar field Φ . Its derivation is completely parallel to that in Sec. 2.2.3, except that the angular momentum l is replaced by the spatial momentum kR along the x -direction. Because, according to the Stirling's formula, G_R has the asymptotic form

$$\text{Im } G_R(\omega \rightarrow \infty, k) \simeq C \frac{\pi \nu (\omega/2)^{2\nu}}{\Gamma(\nu+1)^2} R^{4\nu}, \quad (4.60)$$

we assume $C = R^{-4\nu}$ so that the Jost function (4.54) satisfies the appropriate asymptotic behavior (4.28). Then G_R becomes independent of the AdS radius R .

To numerically solve the Gel'fand-Levitan equation (4.34), the Fourier-Jacobi series representation for the kernel $K_k := K_{\nu-1/2}$ is used in [46]. The kernel can be expanded as

$$K_k(\rho, \sigma) = \sum_{n=0}^M \frac{\beta_n(\rho)}{\rho^{\nu+3/2}} \sigma^{\nu+1/2} P_n^{(\nu,0)} \left(1 - \frac{2\sigma^2}{\rho^2} \right) \quad (\text{with } M \rightarrow \infty), \quad (4.61)$$

where $P_n^{(\alpha,\beta)}$ is the Jacobi polynomial (see Theorem 3.1. in [149]; Note that our potential (4.45) is $V_k(\rho) \in L^1[0, b]$). Their coefficients, especially β_0 , carry information about the potential V_k . It is known that β_0 can be written as

$$\beta_0(\rho) = 2(\nu+1) \left(\frac{\varphi_k^{(+)}(0, \rho)}{\rho^{\nu+1/2}} - 1 \right) \quad (4.62)$$

4.2. Holographic inverse scattering

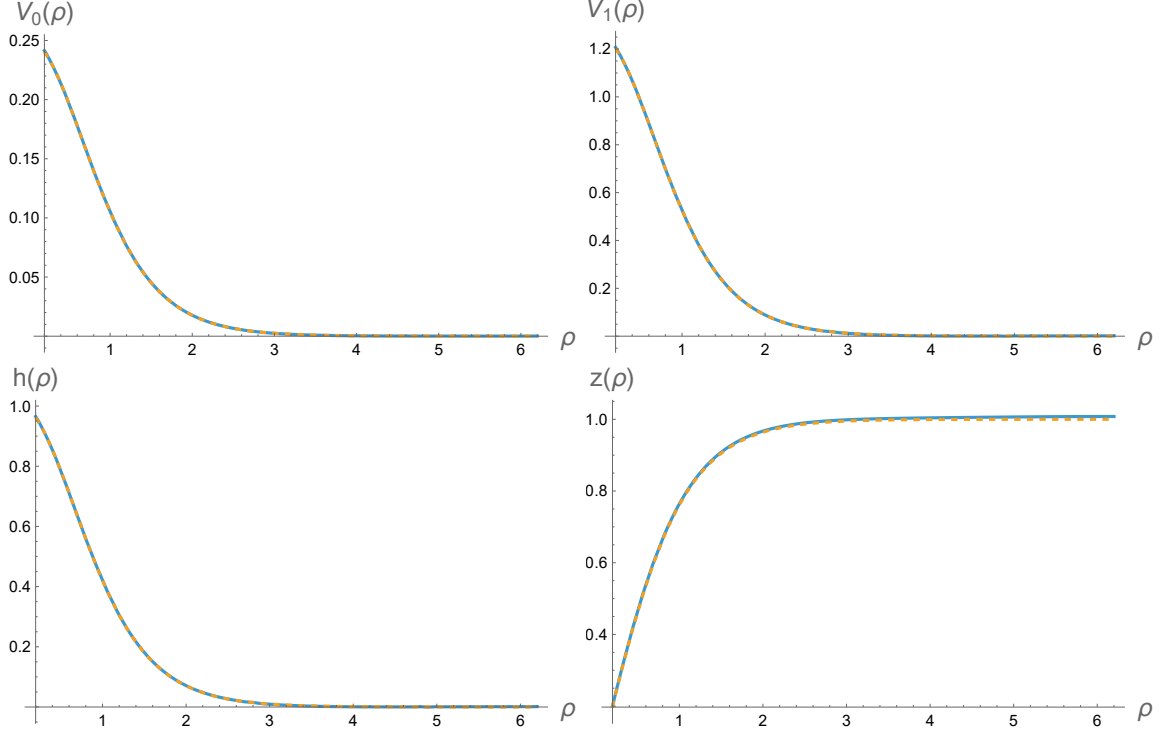


Figure 4.2: Numerical results of the holographic inverse scattering reconstruction for the BTZ black hole. The blue curves represent numerical results, while the orange dashed curves represent exact functions. (Top left panel) The potential $V_{k=0}(\rho)$ reconstructed from (4.63). (Top right panel) The potential $V_{k=1}(\rho)$. (Bottom left panel) The metric function $h(\rho)$ reconstructed from $V_{0,1}(\rho)$ by (4.46). (Bottom right panel) the metric function $z(\rho)$ reconstructed by solving (4.45) with given $V_{0,1}(\rho)$ and $h(\rho)$. For this figure, we took $\Delta = 3/2$, $2\pi T = 1$, and $M = 1$. We imposed the boundary conditions (4.47) for (4.45) at $\rho = 0.2$ and 6.2 , instead of $\rho = 0$ and ∞ , respectively. Also, we replace the interval $[0, \infty)$ of the ω -integration in (4.49) with $[1/100, 100]$.

in terms of the regular solution (4.50) of (4.44) with $\omega = 0$ (see Sec. 7 of [150]), and thus the potential V_k can be obtained from β_0 as

$$V_k(\rho) = \frac{\rho\beta_0''(\rho) + (2\nu + 1)\beta_0'(\rho)}{\rho[\beta_0(\rho) + 2(\nu + 1)]}. \quad (4.63)$$

On the other hand, introducing a matrix A_{mn} and a vector B_n as

$$A_{mn}(\rho) := \int_0^\infty d\omega \frac{J_{\nu+1+2m}(\omega\rho)J_{\nu+1+2n}(\omega\rho)}{\omega\rho} \left[\frac{1}{|F_k(\omega)|^2} - 1 \right], \quad (4.64)$$

$$B_n(\omega\rho) := - \int_0^\infty d\omega J_\nu(\omega\rho)J_{\nu+1+2n}(\omega\rho) \left[\frac{1}{|F_k(\omega)|^2} - 1 \right], \quad (4.65)$$

the Gel'fand-Levitan equation (4.34) becomes the infinite system of linear equations for the coefficients β_n in (4.61)

$$\frac{\beta_m(\rho)}{2(\nu + 1 + 2m)\rho} + \sum_{n=0}^M A_{mn}(\rho)\beta_n(\rho) = B_m(\rho), \quad (4.66)$$

4.2. Holographic inverse scattering

for all $m = 1, 2, \dots, M$ (see Theorem 4.1. in [149]; Here, the assumption of no bound states is used; In fact, there are no QNMs with $\text{Im}\omega > 0$ in the BTZ case). These equations are used to determine β_0 .

The numerical results of the reconstruction are shown in Fig. 4.2. We find that the expansion (4.61) works quite well even if truncated at $M = 1$.

4.2.4 Summary and comments

Here we comment on some advantages and generalizations of the holographic inverse scattering method.

- This method consists of two steps:
 - (i) First, we should reconstruct the Jost function (4.54) from the retarded Green function in a CFT.
 - (ii) Then, the potential V_k is obtained by solving the Gel'fand-Levitan equation (4.34) (or (4.66)), and we can reconstruct the metric functions by V_{k_1} and V_{k_2} .
- This method is based on the well-established inverse scattering theory in quantum mechanics, and we can apply its techniques, like the Fourier-Jacobi series expansion, to solve the Gel'fand-Levitan equation.
- The interior of the black brane cannot be reconstructed, since it is not covered by the tortoise coordinate ρ .
- We used the potential of a single field with different momenta $\mathbf{k}$. Instead, we can use the potential $V_{0,\mu}$ of two fields with different masses μ and the same momentum $\mathbf{k} = \mathbf{0}$ [46]. The metrics function $z(\rho)$ is determined by solving

$$\frac{\mu_2^2 V_{0,\mu_1} - \mu_1^2 V_{0,\mu_2}}{\mu_2^2 - \mu_1^2} = -\frac{(d-1)z''}{2z} + \frac{d^2-1}{4} \left(\frac{z'^2}{z^2} - \frac{1}{\rho^2} \right) \quad (4.67)$$

under the boundary conditions (2.6), and then $h(\rho)$ is obtained as

$$h(\rho) = \frac{V_{0,\mu_1} - V_{0,\mu_2}}{\mu_1^2 - \mu_2^2} z^2 + \frac{z^2}{\rho^2}. \quad (4.68)$$

In the AdS/CFT context, this needs G_R 's for two different operators with conformal dimension $\Delta_i = d/2 + \sqrt{\mu_i^2 R^2 + d^2/4}$.

- While we described the formulation for a scalar field on a static and planar symmetric AdS black brane, this method was generalized to a gauge field and an anisotropic background in [47].
- With experimental applications in mind, it is notable that the reconstructions using different $\mathbf{k}$, different Δ , and different spins work as a consistency check for each other.

Chapter 5

Photon-sphere subsector: an existence condition of holographic duals

In this chapter, we describe our work [1]. In Sec. 5.1, we examine the existence condition of the photon sphere in the Schwarzschild AdS black hole spacetime. Then, in Sec. 5.2, we relate the QNM field equation with the Schrödinger equation of a quantum mechanics with a potential hill, and find an explicit relation between the QNM spectrum at a large angular momentum and the quantum mechanical energy. In Sec. 5.3, we solve the spectrum of solvable quantum models analogous to the one given in Sec. 5.2, and obtain a peculiar pattern of the QNM spectrum that consists of long-lived modes and their sudden decay. This spectrum in the large angular momentum subsector emerges from the existence of the AdS boundary and the photon sphere. Through the AdS/CFT dictionary, it predicts that the “photon-sphere subsector” universally exists in the spectrum of holographic CFTs on a sphere.

5.1 Photon sphere in AdS black hole

In this section, the photon sphere in the Schwarzschild AdS black hole spacetime is described. We demonstrate that a test field feels a photon sphere for any Hawking temperature T , taking a sufficiently large angular momentum $l > l_{\min}$, where $l_{\min}$ is a T -dependent minimum. Then, the T -dependence of $l_{\min}$ is estimated.

In the cases of the asymptotically flat Schwarzschild black hole, nonzero angular momentum always produces a photon sphere. However, in the asymptotic AdS cases, the centrifugal potential is buried in the AdS curvature and the photon sphere disappears for small l .

Let us consider a scalar field Φ with mass μ in the $(d+1)$ -dimensional Schwarzschild AdS spacetime

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-1}^2, \quad f(r) = 1 + \frac{r^2}{R^2} - \left(\frac{r_0}{r}\right)^{d-2}, \quad (5.1)$$

where $d \geq 3$, R is the AdS radius, and r_0 corresponds to the black hole temperature T .¹ Decomposing the scalar field to the spherical harmonics as $\Phi = e^{-i\Omega t} Y_{lm}(\text{angles}) \psi(r) r^{(1-d)/2}$,

¹The horizon radius r_h is related to r_0 as $r_0 = r_h(1 + r_h^2/R^2)^{1/(d-2)}$. This is a monotonic function of r_h . The black hole temperature T , which is equal to the dual CFT temperature, is given by $T = [d(r_h/R)^2 + (d-2)]/4\pi r_h$.

5.1. Photon sphere in AdS black hole

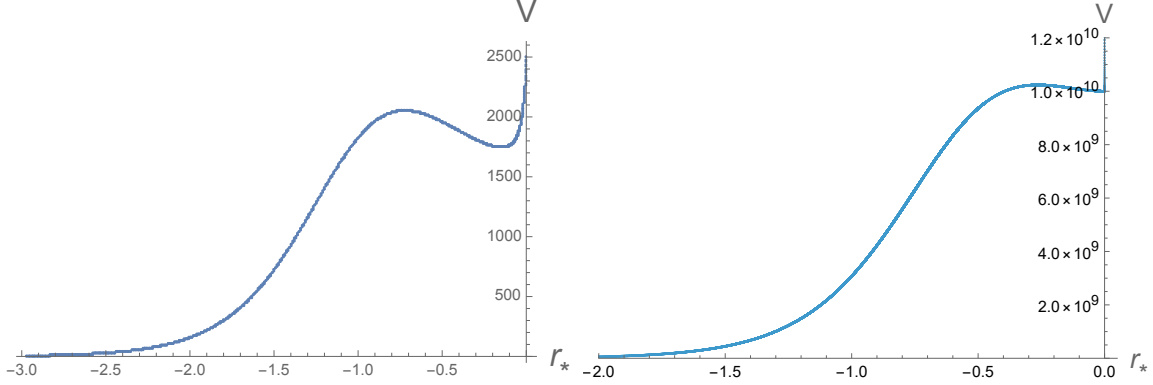


Figure 5.1: The effective potential V in the tortoise coordinate r_* . For the left panel, we took $\mu = 0$, $d = 3$, $R = 1$, $r_0 = 4\sqrt{3}/9$, and $l = 40$, while $r_0 = 2.5$ and $l = 10^5$ for the right panel. The local maximum and $r_* = 0$ represent the photon sphere and the AdS boundary, respectively. Note that the structure of the large l potential characterized by them is universal both above and below the critical temperature $r_0 = 2^{1/(d-2)}R$.

the Klein-Gordon equation reduces to a Schrödinger-like equation [151]

$$\left(\frac{d^2}{dr_*^2} + \Omega^2 - V(r)\right)\psi(r) = 0, \quad (5.2)$$

where

$$r_* := \int_{\infty}^r \frac{ds}{f(s)} \quad (5.3)$$

is the tortoise coordinate (the AdS boundary is at $r_* = 0$), and V is the effective potential defined by

$$V(r) = f \left[\frac{(2l + d - 1)(2l + d - 3)}{4r^2} + \frac{(d - 1)(d - 3)}{4} \frac{f - 1}{r^2} + \frac{d - 1}{2} \frac{f'}{r} + \mu^2 \right]. \quad (5.4)$$

Here we define the prime as the derivative with respect to r . The potential in the tortoise coordinate r_* is illustrated in Fig. 5.1.

The potential (5.4) is found to be decomposed to the potential in the flat case and some corrections to it:

$$\begin{aligned} V(r) = & \left[1 - \left(\frac{r_0}{r}\right)^{d-2} \right] \left[\frac{(2l + d - 1)(2l + d - 3)}{4r^2} + \left(\frac{d - 1}{2}\right)^2 \frac{r_0^{d-2}}{r^d} + \mu_{\text{eff}}^2 \right] \\ & + \frac{1}{R^2} \left[\mu_{\text{eff}}^2 r^2 + \left(\frac{d - 1}{2}\right)^2 \left(\frac{r_0}{r}\right)^{d-2} + \left(l + \frac{d - 1}{2}\right) \left(l + \frac{d - 3}{2}\right) \right], \end{aligned} \quad (5.5)$$

where the first line is the potential of the scalar field with mass μ_{eff} in the Schwarzschild space-time. The effect of the AdS curvature consists of 1) the effective mass $\mu_{\text{eff}}^2 := \mu^2 + (d^2 - 1)/4R^2$, 2) the r^2 term typical of AdS, and 3) the additional attractive term from the black hole; all of these tend to make the photon sphere vanish.

At high temperature (large black holes, $r_h \gg R$), we find $T \simeq dr_h/4\pi R^2$ and $r_0 \simeq r_h(r_h/R)^{2/(d-2)}$, thus $r_0 \simeq R(4\pi RT/d)^{d/(d-2)}$. This means that the parameter r_0 is a monotonic function of T at high temperature.

5.1. Photon sphere in AdS black hole

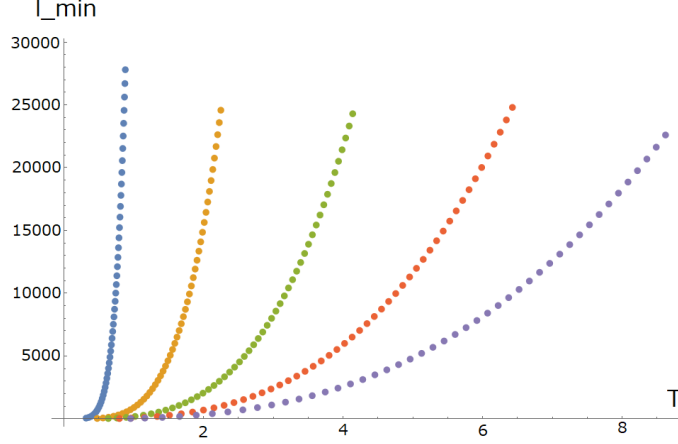


Figure 5.2: The T -dependence of the minimum angular momenta $l_{\min}$ to generate a photon sphere in various spatial dimensions d . From the blue curve to the purple, d is 3 to 7. For this figure we took $\mu = 0$.

If the angular momentum l is large enough, the effective potential (5.4) of the scalar field reduces to the potential of a null geodesic

$$V(r) \simeq f(r) \frac{l^2}{r^2}. \quad (5.6)$$

Because the AdS effect in (5.6) causes only the constant shift to the flat potential, the photon sphere always exists. Denoting the location of the photon sphere as $r = r_{\text{PS}}$, we find the location in the large l limit

$$r_{\text{PS}} \simeq \left(\frac{d}{2}\right)^{1/(d-2)} r_0 \quad (=:\tilde{r}). \quad (5.7)$$

Therefore, the photon sphere must exist for any values of the parameters d and r_0 , taking a sufficiently large but finite value of l .

Indeed, we can numerically confirm that there is a minimum angular momentum $l_{\min}$ such that the photon sphere exists for any $l > l_{\min}$ (Fig. 5.2). The value of $l_{\min}$ increases as the Hawking temperature T does. Now we note that T grows monotonically with respect to r_0 when T is above the Hawking-Page transition temperature $T_c = (d-1)/2\pi R$, that is, $r_0 > 2^{1/(d-2)}R$.

We can estimate the expression of $l_{\min} = l_{\min}(r_0)$ with an analytic calculation. In [95], the critical value $l_{\min}$ of the angular momentum has been found as a solution of $V'(r_{\text{PS}}) = V''(r_{\text{PS}}) = 0$ and obtained in a large μ limit. Now, instead, we provide the expression for $\mu = 0$, at a large l . It is expected that the location of the photon sphere r_{PS} approaches the geodesic one (5.7) as l increases. The difference can be estimated from $V'(r_{\text{PS}}) = 0$ in the large l limit as

$$r_{\text{PS}} = \tilde{r} - \frac{V'(\tilde{r})}{V''(\tilde{r})} + \mathcal{O}(l^{-3}). \quad (5.8)$$

Regarding the condition that the photon sphere exists, which is expressed as $V''(r_{\text{PS}}) < 0$, as the restriction for l , the value of l is bounded from below by a bound given by the function of r_0 . If we naively use the expression of V'' and (5.8), and truncate them at the first sub-leading order

5.2. From quasinormal modes to quantum mechanics

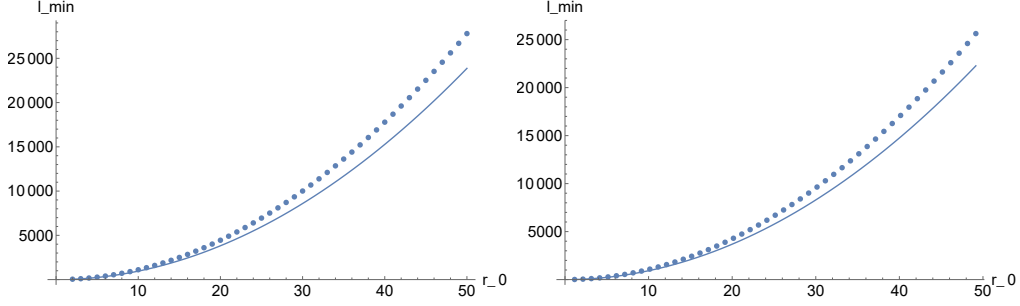


Figure 5.3: The r_0 -dependence of $l_{\min}$ in $d = 3$ (left panel) and $d = 7$ (right panel). The dotted curves and the solid curves represent the numerical calculation and the analytic estimation (5.9), respectively.

in the large l expansion, we find the following expression for the lower bounds, for example:

$$\begin{aligned} d = 3 : \quad l^2 &> \frac{729}{8} \left(\frac{r_0}{R}\right)^4 + \frac{9}{2} \left(\frac{r_0}{R}\right)^2 - \frac{4}{3}, \\ d = 7 : \quad l^2 &> \frac{78}{5} 2^{\frac{1}{5}} 7^{\frac{4}{5}} \left(\frac{r_0}{R}\right)^4 + 9 \left(\frac{2}{7}\right)^{\frac{3}{5}} \left(\frac{r_0}{R}\right)^2 - \frac{690}{49}. \end{aligned} \quad (5.9)$$

Fig. 5.3 is the plot of these analytic bounds (solid curves) and the actual lower bound found numerically (dots). The qualitative behavior of $l_{\min}$ is well captured.² From the analytic estimate, we find that the minimum angular momentum $l_{\min}$ would grow as $\mathcal{O}(r_0^2)$, that is, as $\mathcal{O}(T^{2d/(d-2)})$.

5.2 From quasinormal modes to quantum mechanics

In this section we derive a quantum mechanical model which provides the quasinormal mode spectra, in the large angular momentum approximation. The goal is, from the equation (5.2), in the large l limit, to derive a potential problem of the form

$$\left(\frac{d^2}{dx^2} + E - \tilde{V}(x) \right) \psi(x) = 0, \quad (5.10)$$

where the potential $\tilde{V}(x)$ is approximated by $\tilde{V}(x) \simeq -x^2/4$ around the top of the potential hill $x = 0$.

5.2.1 Potential hill relates to the quantum-mechanical model

When the photon sphere exists, we can always expand the potential around the photon sphere. First, let us evaluate the potential $V(r)$ given in (5.4) around the top of the potential hill as

$$V(r) = V(r_{\text{PS}}) + \frac{1}{2} V''(r_{\text{PS}}) (r - r_{\text{PS}})^2 + \dots \quad (5.11)$$

²The difference between the analytic estimation and the numerical one grows as r_0 (or T) increases. In the derivation of the bound (5.9), we adopt a naive truncation at the first sub-leading order, while the leading $\mathcal{O}(l^2)$ term and the sub-leading $\mathcal{O}(l^0)$ term are comparable to each other. It means that the expression (5.9) is not compatible with the large l expansion. We do not locate the reason why nevertheless (5.9) looks a good approximation in Fig. 5.3.

5.2. From quasinormal modes to quantum mechanics

Note that here $V''(r_{\text{PS}})$ is negative. Converting the coordinate r to the tortoise coordinate r_* near the top, and denoting the difference from the top as δr_* in the tortoise coordinate, we obtain

$$V = V(r_{\text{PS}}) + \frac{1}{2}V''(r_{\text{PS}})f(r_{\text{PS}})^2(\delta r_*)^2 + \dots \quad (5.12)$$

Then we perform the following coordinate transformation

$$x := [-2V''(r_{\text{PS}})f(r_{\text{PS}})^2]^{1/4} \delta r_*. \quad (5.13)$$

Together with the definition

$$E := \frac{1}{\sqrt{-2V''(r_{\text{PS}})f(r_{\text{PS}})}} (\Omega^2 - V(r_{\text{PS}})), \quad (5.14)$$

we arrive at the equation (5.10) with

$$\tilde{V}(x) = -\frac{x^2}{4} + \mathcal{O}(x^3). \quad (5.15)$$

Next, we shall evaluate the potential $\tilde{V}(x)$ and the relation (5.14) in various limits, including the large l expansion and the high temperature limit $r_0 \gg R$.

5.2.2 Large l expansion

QNM spectrum and quantum mechanical spectrum

We can evaluate the coefficients in (5.14), in the large l approximation. First, we redefine the frequency as

$$\Omega =: \sqrt{V(r_{\text{PS}})} + \tilde{\Omega}. \quad (5.16)$$

At large l , this amounts to looking closely at the energy region near the photon sphere, since we regard $\tilde{\Omega}$ to be $\mathcal{O}(l^0)$, that is, parametrically smaller compared to $\sqrt{V(r_{\text{PS}})}$. Using the definition (5.16), the relation (5.14) is rewritten as

$$\Omega = \sqrt{V(r_{\text{PS}})} + f(r_{\text{PS}}) \sqrt{\frac{-V''(r_{\text{PS}})}{2V(r_{\text{PS}})}} E - \frac{1}{2\sqrt{V(r_{\text{PS}})}} \tilde{\Omega}^2. \quad (5.17)$$

In the large l expansion, the first term is $\mathcal{O}(l)$, and the second term is $\mathcal{O}(l^0)$, while the third term is $\mathcal{O}(l^{-1})$. This means that, in the large l limit, we find

$$\Omega = a + bE + \mathcal{O}(l^{-1}), \quad (5.18)$$

where

$$a := \sqrt{V(r_{\text{PS}})}, \quad b := f(r_{\text{PS}}) \sqrt{\frac{-V''(r_{\text{PS}})}{2V(r_{\text{PS}})}}. \quad (5.19)$$

An explicit large l expansion leads to

$$a = \sqrt{\left[\frac{1}{R^2} + \frac{1}{r_0^2} \frac{d-2}{d} \left(\frac{2}{d} \right)^{2/(d-2)} \right] \left(l + \frac{d-1}{2} \right) \left(l + \frac{d-3}{2} \right) + \mathcal{O}(l^{-1})}, \quad (5.20)$$

5.2. From quasinormal modes to quantum mechanics

$$b = \sqrt{(d-2) \left[\frac{1}{R^2} + \frac{1}{r_0^2} \frac{d-2}{d} \left(\frac{2}{d} \right)^{2/(d-2)} \right]} + \mathcal{O}(l^{-2}). \quad (5.21)$$

So, in summary, we find a formula for the relation between the QNM spectra Ω and the quantum mechanical energy E of (5.10) in the large l approximation as

$$\Omega = \sqrt{\frac{1}{R^2} + \frac{1}{r_0^2} \frac{d-2}{d} \left(\frac{2}{d} \right)^{2/(d-2)}} \left[\sqrt{\left(l + \frac{d-1}{2} \right) \left(l + \frac{d-3}{2} \right)} + \sqrt{d-2} E \right] + \mathcal{O}(l^{-1}). \quad (5.22)$$

Once one solves the quantum mechanical model (5.10), substituting E into the equation above derives the QNM spectra Ω in the large angular momentum limit.

On the validity of inverted harmonic oscillator

While the energy spectra E to be substituted into (5.22) will be evaluated in the following section, we here study the shape of the potential $V(x)$. It is an inverted harmonic potential around the top of the potential hill, and we will see how good the approximation is at large l .

Near the photon sphere at $r = r_{\text{PS}}$, the potential is expanded with the displacement $\delta r \ll r_{\text{PS}}$ as

$$V(r_{\text{PS}} + \delta r) = V(r_{\text{PS}}) + \frac{1}{2} V''(r_{\text{PS}}) (\delta r)^2 + \frac{1}{3!} V^{(3)}(r_{\text{PS}}) (\delta r)^3 + \dots \quad (5.23)$$

This is approximated by an inverted harmonic oscillator when δr satisfies

$$\delta r \ll \left| \frac{3V''(r_{\text{PS}})}{V^{(3)}(r_{\text{PS}})} \right|. \quad (5.24)$$

This means that the approximation by the inverted harmonic oscillator is valid in the range

$$V(r_{\text{PS}}) \geq V(r) \gg V(r_{\text{PS}}) - \frac{9}{2} \frac{(-V''(r_{\text{PS}}))^3}{V^{(3)}(r_{\text{PS}})^2}. \quad (5.25)$$

The potential range $\Delta V = |9(-V''(r_{\text{PS}}))^3/2V^{(3)}(r_{\text{PS}})^2|$ is of $\mathcal{O}(l^2)$ and is wide enough for a large l . Therefore, as long as the energy Ω^2 of the quasinormal mode is in this range, one expects a safe use of the inverted harmonic potential. However, as we will discuss below, we argue that the spectra are not determined solely by the potential shape near the top.

AdS boundary is a hard wall

Let us turn to the shape of the potential near the AdS boundary. We will show that the AdS boundary can be regarded as a hard-wall potential, where a Dirichlet condition is imposed on the field.

Near the boundary $r \rightarrow \infty$, the tortoise coordinate (5.3) is well-approximated as

$$r_* \simeq -\frac{R^2}{r}, \quad (5.26)$$

because the effect of the black hole disappears and the geometry is almost equal to that of the pure AdS. Using this expression, the potential $V(r)$ in (5.4) is approximated as

$$V \simeq \frac{1}{R^2} \left(l + \frac{d-1}{2} \right) \left(l + \frac{d-3}{2} \right) + \frac{1}{r_*^2} \left(\frac{d^2-1}{4} + \mu^2 R^2 \right). \quad (5.27)$$

5.2. From quasinormal modes to quantum mechanics

The second term diverges at the AdS boundary $r_* = 0$. So, it works as a wall by which any wave is bounced back.

The reason why this is a hard wall is as follows. In the large l limit, the first term in (5.27) is pushed up as a constant and being very high. The effect of the second term appears only when these two terms are comparable,

$$r_* \simeq -\frac{R}{l} \sqrt{\frac{d^2 - 1}{4} + \mu^2 R^2}, \quad (5.28)$$

at a large l . This point is very close to the AdS boundary $r_* = 0$, and in the large l limit, it coincides with the AdS boundary. Therefore, the AdS boundary behaves as a hard wall at a large l .

The existence of the hard wall is crucial in the analysis, as we will find in the next section. In fact, the difference between the asymptotically flat case and the AdS case was already noted in [78].

5.2.3 Analytic potential at high temperature

In the above, we saw that the potential is approximated by the inverted harmonic oscillator around the top, and there exists a hard wall at the AdS boundary. In the next section, we evaluate E in systems which possess these characteristics. In this subsection, we consider the case in high temperature and will find an analytic shape of the potential $\tilde{V}(x)$ which actually possesses the features described above.³

First, we evaluate the tortoise coordinate (5.3) as

$$r_* = \int_{\infty}^r ds \left[1 + \frac{s^2}{R^2} - \frac{r_0^{d-2}}{s^{d-2}} \right]^{-1} = r_0 \int_{\infty}^{r/r_0} du \left[1 + \frac{r_0^2}{R^2} u^2 - \frac{1}{u^{d-2}} \right]^{-1}, \quad (5.29)$$

where we made a change of the variable $u := s/r_0$. It is understood that in the high temperature case $r_0 \gg R$, we find an approximation

$$r_* \simeq r_0 \int_{\infty}^{r/r_0} du \left[\frac{r_0^2}{R^2} u^2 \right]^{-1} = -\frac{R^2}{r} \quad (5.30)$$

as long as

$$r \gg r_0^{\frac{d-2}{d}} R^{\frac{2}{d}}. \quad (5.31)$$

Note that the latter condition is satisfied for the photon sphere $r = r_{\text{PS}}$ given in (5.7) at large l , under the high temperature condition $r_0 \gg R$. Thus, although (5.30) is exactly the same as the near-boundary expression (5.26), we can use it even near the top of the potential hill, at high temperature.

Using (5.30), the large l potential (5.6) is written as

$$\begin{cases} V \simeq \frac{l^2}{R^2} \left[1 + \frac{r_*^2}{R^2} - \frac{r_0^{d-2}}{R^{2d-2}} (-r_*)^d \right] & \left(r_* < 0, \quad |r_*| \ll r_0^{-\frac{d-2}{d}} R^{\frac{2(d-1)}{d}} \right), \\ V = \infty & (r_* \geq 0). \end{cases} \quad (5.32)$$

$$(r_* \geq 0). \quad (5.33)$$

³Note that we take the limit $l \rightarrow \infty$ first and then a high temperature limit, otherwise the potential hill would disappear as we described in Sec. 5.1.

5.2. From quasinormal modes to quantum mechanics

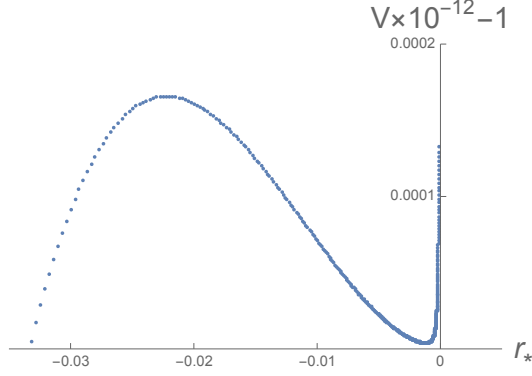


Figure 5.4: The effective potential V in the tortoise coordinate r_* , at high temperature. For this figure we took $d = 3$, $\mu = 0$, $r_0 = 30$, $l = 10^6$, in the unit $R = 1$.

The latter is to make sure that there exists a hard wall at $r_* = 0$, as explained in the previous subsection. Note that the potential (5.32) is the same as the WKB potential (3.48) in Chap. 3.

With the linear redefinition (5.13) of r_* , we can rewrite this potential into the form (5.10). A simple calculation leads to the quantum mechanical potential $\tilde{V}(x)$ whose hill top is located at $x = 0$ with the hard wall located at $x = L$,

$$\tilde{V}(x) = \frac{1}{4(d-2)} \left[(L-x)^2 - \alpha^{d-2}(L-x)^d - L^2 \left(1 - (\alpha L)^{d-2} \right) \right], \quad (5.34)$$

where

$$\alpha := (d-2)^{-1/4} 2^{-1/2} \frac{1}{\sqrt{l}} \frac{r_0}{R}, \quad L := \alpha^{-1} \left(\frac{2}{d} \right)^{1/(d-2)}. \quad (5.35)$$

Note that in these expressions we keep only leading terms at the high temperature $r_0 \gg R$ and in the large l , and we have adjusted the constant term of the potential so that the height of the potential hill is tuned to $\tilde{V} = 0$, for our later purpose.

To illustrate this potential, we consider the case of $d = 3$. We obtain

$$\tilde{V}(x) = -\frac{1}{4} x^2 + 2^{-5/2} \frac{1}{\sqrt{l}} \frac{r_0}{R} x^3, \quad (5.36)$$

and the wall position is

$$L = \frac{2^{3/2}}{3} \sqrt{l} \frac{R}{r_0}. \quad (5.37)$$

The system is defined in the region $x \leq L$, and all waves are reflected at the AdS boundary hard wall $x = L$. The approximated potential $\tilde{V}$ is not trustable for large negative values of x . In Fig. 5.4, we plot the potential (5.4) numerically evaluated at high temperature $r_0/R = 30 \gg 1$. We can confirm the potential shape (5.34), with the hard wall at the AdS boundary.

How far is this wall from the top of the hill? The existence condition of the photon sphere (5.9) shows that, at high temperature $r_0/R \gg 1$, we have the condition $l^2 \gg \mathcal{O}(r_0/R)^4$. The wall position (5.37) then implies $L \gg \mathcal{O}(r_0/R)$. We conclude that the wall is located very far from the potential hill, in the metric of the x -coordinate. On the other hand, near the wall, we find a bottom of the potential well, $V'(L) = 0$, which means that the whole potential cannot be exactly equal to the inverted harmonic potential for any choice of the wall position.

5.3 Solving quasinormal modes respecting AdS boundary

In this section, we analyze the QNM spectrum by solving quantum mechanical models, through the relation studied in Sec. 5.2. As mentioned in [66], it is demanding to numerically calculate the spectrum with large l . Thus, instead, we use solvable models, the inverted harmonic oscillator and the Pöschl-Teller model, which share the same shape of the potential as that of (5.10) but have different potential tails. We show that the discrete spectrum is located on a peculiar curve in the complex energy E -plane, affected by the potential hill at the photon sphere and the hard wall at the AdS boundary. The effect of the boundary does not vanish even if the boundary is sent to spatial infinity, because decaying modes $\text{Im } E < 0$ grow near the boundary. The solvable models result in the unique pattern of the spectrum, and in particular, the curve is rather insensitive to the shape of the potential tail. This universality leads us to state that the pattern of the QNM spectrum is well captured by that of the solvable models.

5.3.1 Inverted harmonic oscillator

Recall that the Klein-Gordon equation on the Schwarzschild AdS background reduces to a quantum mechanical model (5.10). The potential has a hill $\tilde{V}(x) \simeq -x^2/4$ whose top corresponds to the photon sphere. The energy E is measured from the top of the hill.

In this subsection we replace the potential with the exact inverted harmonic potential

$$\tilde{V}(x) \longrightarrow -\frac{x^2}{4}, \quad (5.38)$$

and solve the Schrödinger equation

$$\left(\frac{d^2}{dx^2} + E + \frac{x^2}{4} \right) \psi(x) = 0. \quad (5.39)$$

We impose the Dirichlet boundary condition,

$$\psi(L) = 0, \quad L \gg 1. \quad (5.40)$$

We also assume that the wave function $\psi(x)$ has only an outgoing mode at $x \rightarrow -\infty$. The potential hill at $x = 0$ represents the photon sphere, while the Dirichlet boundary at $x = L$ represents the AdS boundary. The outgoing mode at $x \rightarrow -\infty$ corresponds to a scalar mode absorbed by the black hole at the horizon.

The Schrödinger equation is solvable,⁴ and its solution is called the parabolic cylinder function. Changing the two variables x, E into z, ν as

$$z := e^{i\pi/4}x, \quad E := i \left(\nu + \frac{1}{2} \right), \quad (5.41)$$

the general form of the solution is written as

$$\psi = A D_\nu(z) + B D_{-\nu-1}(iz), \quad (5.42)$$

where A and B are some constants.

⁴See App. B.2.1 for more detailed analysis on the spectrum.

5.3. Solving quasinormal modes respecting AdS boundary

The parabolic cylinder function $D_\nu(z)$ has an asymptotic form

$$D_\nu(z) \simeq \begin{cases} e^{-z^2/4} z^\nu - \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{\pm i\pi\nu} e^{z^2/4} z^{-\nu-1} & \left(\text{for } \frac{\pi}{4} < \pm \arg z < \frac{5\pi}{4} \right), \\ e^{-z^2/4} z^\nu & \left(\text{for } -\frac{3\pi}{4} < \arg z < \frac{3\pi}{4} \right). \end{cases} \quad (5.43)$$

For our later convenience, we introduce a function

$$\vartheta_\pm(\nu; z) := -\frac{iz^2}{2} + i(2\nu + 1) \ln z \pm \pi\nu - i \ln \frac{\sqrt{2\pi}}{\Gamma(-\nu)} \quad (5.44)$$

so that the location of zeros of the wave function is easily spotted,

$$D_\nu(z) \simeq e^{-z^2/4} z^\nu \left(1 - e^{i\vartheta_\pm(\nu; z)} \right) \quad \left(\text{for } \frac{\pi}{4} < \pm \arg z < \frac{5\pi}{4} \right). \quad (5.45)$$

By using the form of the asymptotic expansion, we find that the parabolic cylinder function $D_\nu(z)$ has an ingoing mode at $x \rightarrow +\infty$ and in/out-going modes at $x \rightarrow -\infty$, whereas $D_{-\nu-1}(iz)$ has in/out-going modes at $x \rightarrow +\infty$ and an outgoing mode at $x \rightarrow -\infty$. Thus, the appropriate choice of the solution is

$$\psi = B D_{-\nu-1}(iz). \quad (5.46)$$

Near the boundary $x \simeq L \gg 1$, the wave function takes the form of

$$\psi \simeq e^{-(iz)^2/4} (iz)^{-\nu-1} \left(1 - e^{i\vartheta_+(-\nu-1; iz)} \right). \quad (5.47)$$

Hence the Dirichlet boundary condition $\psi(L) = 0$ yields the quantization condition

$$\vartheta_+(-\nu-1; ie^{i\pi/4}L) \in 2\pi\mathbb{Z}. \quad (5.48)$$

That is, more explicitly,

$$\frac{\pi}{2} + \frac{L^2}{2} + \left(2 \ln L + \frac{i\pi}{2} \right) E + i \ln \frac{\sqrt{2\pi}}{\Gamma(1/2 - iE)} \in 2\pi\mathbb{Z}. \quad (5.49)$$

The spectrum in the complex E -plane is plotted in Fig. 5.5. The blue/orange curves are for $\text{Im } \vartheta_+ = 0$ and $\text{Re } \vartheta_+ \in 2\pi\mathbb{Z}$, respectively. The intersection points of the two curves are the solutions of the quantization condition.

The solutions in the region $\text{Re } E < 0$ have almost vanishing imaginary part, and correspond to extremely long-lived QNMs localized in the region between the photon sphere and the AdS boundary. As we saw in Sec. 3.2.3, the existence of these modes was pointed out by [95], focusing on the bottom of the potential well, not the photon sphere.

The spectrum in the region $\text{Re } E > 0$ and $\text{Im } E < 0$ corresponds to short-lived QNMs above the top of the potential hill of the photon sphere. The whole spectrum is located on the following peculiar curve in the complex E -plane,

$$0 = \text{Im} \left[\left(2 \ln L + \frac{i\pi}{2} \right) E + i \ln \frac{\sqrt{2\pi}}{\Gamma(1/2 - iE)} \right]. \quad (5.50)$$

5.3. Solving quasinormal modes respecting AdS boundary

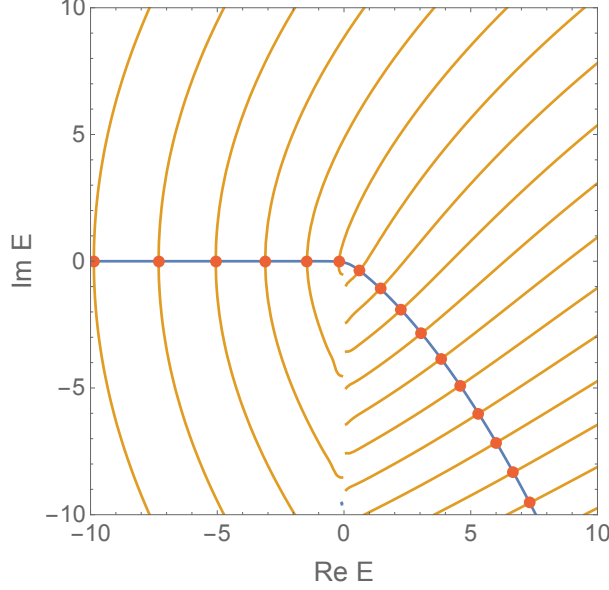


Figure 5.5: The spectrum of the inverted harmonic oscillator $\tilde{V}(x) = -x^2/4$ (indicated as red blobs). We have imposed the Dirichlet boundary condition at $x = L = 10$, and have assumed that there is only an outgoing mode at $x \rightarrow -\infty$. The blue curve represents $\text{Im } \vartheta_+ = 0$, while the orange curve represents $\text{Re } \vartheta_+ \in 2\pi\mathbb{Z}$. The intersection points of the two curves represent the quantized energy spectrum.

This is a curve whose negative imaginary part of E grows when the real part of E grows, as shown in Fig. 5.5.

Let us consider the large angular momentum limit $l \rightarrow \infty$. As seen in (5.35), it corresponds to the large L limit in the original quantum mechanical model. When the boundary is sent to spatial infinity, $L \rightarrow +\infty$, the slope of the curve (5.50) in the E -plane decreases and the curve approaches the positive real axis. In fact, expanding the gamma function $\Gamma(1/2 - iE)$ around $E = 0$, we find

$$E \simeq \frac{\pi}{\ln L} \left[n - \left(\frac{L^2}{4\pi} - \left\lfloor \frac{L^2}{4\pi} \right\rfloor \right) - \frac{1}{4} - \frac{i \ln 2}{4\pi} \right] \quad (n = 1, 2, \dots \ll \ln L). \quad (5.51)$$

The quantum number n counts the QNM, and the imaginary part of the spectrum E becomes a constant negative value independent of n .

Before closing this subsection, it will be worthwhile to compare this spectrum to the one with different boundary conditions. Suppose that we allow only outgoing modes on both sides of the potential hill, $x \rightarrow \pm\infty$, instead of the Dirichlet condition. The spectrum is obtained by killing the ingoing mode of $\psi = B D_{-\nu-1}(iz)$ at $x \rightarrow +\infty$ by setting

$$\frac{\sqrt{2\pi}}{\Gamma(1+\nu)} e^{+i\pi(-\nu-1)} = 0. \quad (5.52)$$

The gamma function has poles at non-positive integers, thus,

$$E = -i \left(n + \frac{1}{2} \right) \quad (n = 0, 1, \dots). \quad (5.53)$$

5.3. Solving quasinormal modes respecting AdS boundary

The apparent mismatch between (5.53) and (5.51) in the $L \rightarrow \infty$ limit is not a contradiction. In our case of (5.51), for a decaying solution $\text{Im } E < 0$, the outgoing mode of $D_{-\nu-1}(iz)$ grows in power as x approaches the boundary, whereas the ingoing mode decreases in power. Thus we needed to add an ingoing mode with a large amplitude coefficient to realize the Dirichlet boundary condition. This is different from just having the outgoing mode. Therefore, the existence of the hard wall, even placed at the spatial infinity, discontinuously changes the spectrum.

5.3.2 Pöschl-Teller model

In this subsection, we demonstrate that the curve (5.50) is almost insensitive to the shape of the potential tail in a solvable model. We replace the potential of the quantum mechanical model (5.10) with that of the Pöschl-Teller type,

$$\tilde{V}(x) \longrightarrow V_0 \left(\frac{1}{\cosh^2 \alpha x} - 1 \right). \quad (5.54)$$

The top of the hill in this potential is approximated by the inverted harmonic potential, while the potential has a potential tail which resembles the actual shape of the potential near the AdS boundary (5.34). Let us introduce a variable

$$\omega^2 := V_0 + E \quad (5.55)$$

and consider the Schrödinger equation with the Pöschl-Teller potential⁵

$$\left(\frac{d^2}{dx^2} + \omega^2 - \frac{V_0}{\cosh^2 \alpha x} \right) \psi(x) = 0, \quad (5.56)$$

where

$$V_0 \equiv \alpha^2 \lambda(1 - \lambda), \quad (5.57)$$

$$\alpha > 0, \quad \lambda \equiv \frac{1}{2} + i\nu, \quad \nu > 0. \quad (5.58)$$

In a manner similar to the inverted harmonic oscillator, we shall solve this Schrödinger equation under the Dirichlet boundary condition

$$\psi(L) = 0, \quad L \gg 1. \quad (5.59)$$

We also assume that the wave function $\psi(x)$ has only an outgoing mode at $x \rightarrow -\infty$.

The Schrödinger equation has two independent solutions (see Eq. (9) in [152])

$$f_+(x) = e^{i\omega x} F \left(\begin{matrix} \lambda, & 1 - \lambda \\ 1 + i\omega/\alpha \end{matrix}; \frac{e^{\alpha x}}{e^{\alpha x} + e^{-\alpha x}} \right), \quad (5.60)$$

$$f_-(x) = (e^{\alpha x} + e^{-\alpha x})^{i\omega/\alpha} F \left(\begin{matrix} \lambda - i\omega/\alpha, & 1 - \lambda - i\omega/\alpha \\ 1 - i\omega/\alpha \end{matrix}; \frac{e^{\alpha x}}{e^{\alpha x} + e^{-\alpha x}} \right). \quad (5.61)$$

Here F is the hypergeometric function. Transforming variables, we find

$$\begin{aligned} f_+(x) = & e^{i\omega x} \frac{\Gamma(1 + i\omega/\alpha) \Gamma(i\omega/\alpha)}{\Gamma(\lambda + i\omega/\alpha) \Gamma(1 - \lambda + i\omega/\alpha)} F \left(\begin{matrix} \lambda, & 1 - \lambda \\ 1 - i\omega/\alpha \end{matrix}; \frac{1}{e^{2\alpha x} + 1} \right) \\ & + (e^{\alpha x} + e^{-\alpha x})^{-i\omega/\alpha} \frac{\Gamma(1 + i\omega/\alpha) \Gamma(-i\omega/\alpha)}{\Gamma(\lambda) \Gamma(1 - \lambda)} F \left(\begin{matrix} \lambda + i\omega/\alpha, & 1 - \lambda + i\omega/\alpha \\ 1 + i\omega/\alpha \end{matrix}; \frac{1}{e^{2\alpha x} + 1} \right) \end{aligned}$$

⁵See App. B.2.2 for more detailed analysis on the spectrum.

5.3. Solving quasinormal modes respecting AdS boundary

$$\simeq \begin{cases} e^{i\omega x} & (\text{Im } \omega > 0, x \rightarrow -\infty) \\ e^{i\omega x} \frac{\Gamma(1+i\omega/\alpha)\Gamma(i\omega/\alpha)}{\Gamma(\lambda+i\omega/\alpha)\Gamma(1-\lambda+i\omega/\alpha)} \\ + e^{-i\omega x} \frac{\Gamma(1+i\omega/\alpha)\Gamma(-i\omega/\alpha)}{\Gamma(\lambda)\Gamma(1-\lambda)} & (\text{Im } \omega > 0, x \rightarrow +\infty) \end{cases} \quad (5.62)$$

$$\begin{aligned} f_-(x) = & (e^{\alpha x} + e^{-\alpha x})^{i\omega/\alpha} \frac{\Gamma(1-i\omega/\alpha)\Gamma(i\omega/\alpha)}{\Gamma(\lambda)\Gamma(1-\lambda)} F\left(\begin{matrix} \lambda - i\omega/\alpha, & 1 - \lambda - i\omega/\alpha \\ & 1 - i\omega/\alpha \end{matrix}; \frac{1}{e^{2\alpha x} + 1}\right) \\ & + e^{-i\omega x} \frac{\Gamma(1-i\omega/\alpha)\Gamma(-i\omega/\alpha)}{\Gamma(\lambda-i\omega/\alpha)\Gamma(1-\lambda-i\omega/\alpha)} F\left(\begin{matrix} \lambda, & 1 - \lambda \\ & 1 + i\omega/\alpha \end{matrix}; \frac{1}{e^{2\alpha x} + 1}\right) \\ \simeq & \begin{cases} e^{-i\omega x} & (\text{Im } \omega < 0, x \rightarrow -\infty) \\ e^{i\omega x} \frac{\Gamma(1-i\omega/\alpha)\Gamma(i\omega/\alpha)}{\Gamma(\lambda)\Gamma(1-\lambda)} \\ + e^{-i\omega x} \frac{\Gamma(1-i\omega/\alpha)\Gamma(-i\omega/\alpha)}{\Gamma(\lambda-i\omega/\alpha)\Gamma(1-\lambda-i\omega/\alpha)} & (\text{Im } \omega < 0, x \rightarrow +\infty) \end{cases} \quad (5.63) \end{aligned}$$

Thus, the appropriate choice of the solution is

$$\psi \propto f_-(x). \quad (5.64)$$

Near the boundary $x \simeq L \gg 1$, the wave function takes the following form

$$\psi \simeq e^{i\omega x} \frac{\Gamma(1-i\omega/\alpha)\Gamma(i\omega/\alpha)}{\Gamma(\lambda)\Gamma(1-\lambda)} \left(1 - e^{i\vartheta(\omega;x)}\right), \quad (5.65)$$

where

$$\vartheta(\omega;x) := \pi - 2\omega x - i \ln \frac{\Gamma(-i\omega/\alpha)}{\Gamma(i\omega/\alpha)} \frac{\Gamma(\lambda)\Gamma(1-\lambda)}{\Gamma(\lambda-i\omega/\alpha)\Gamma(1-\lambda-i\omega/\alpha)}. \quad (5.66)$$

The Dirichlet boundary condition $\psi(L) = 0$ yields the quantization condition

$$\vartheta(\omega; L) \in 2\pi\mathbb{Z}. \quad (5.67)$$

Its spectrum is plotted in the left panel of Fig. 5.6. The blue/orange curves are for $\text{Im } \vartheta = 0$ and $\text{Re } \vartheta \in 2\pi\mathbb{Z}$, respectively. Their intersection points are the solutions of the quantization condition. The parameters in the potential are chosen so that $\tilde{V}(x) \simeq -x^2/4$ around the top of the potential hill, to be compared with the result of the inverted harmonic oscillator, Fig. 5.5.

As in the case of the inverted harmonic oscillator, the spectrum in the region $\text{Re } E < 0$ has almost vanishing imaginary part, and corresponds to extremely long-lived QNMs living in the spatial region between the photon sphere and the AdS boundary. The spectrum in the region $\text{Re } E > 0$ and $\text{Im } E < 0$ corresponds to short-lived modes above the top of the potential hill of the photon sphere. Therefore, we conclude that the spectral pattern of this model shares the same characteristic with that of the inverted harmonic potential.

In the right panel of Fig. 5.6, we plot the spectrum in the complex ω -plane. In the ω -plane, the spectra appear on both sides of the positive/negative real axis since there are two branches $\omega = \pm\sqrt{E + V_0}$. One of the blue horizontal curves in the ω -plane is mapped to the blue curve on the E -plane.⁶

⁶As a remark aside, one observes other series of complex solutions in the ω -plane: the intersection points of two blue vertical curves and series of orange curves around the origin (see the central part of the right panel of Fig. 5.6). Such a series of solutions is mapped to the region $\text{Re } E < -V_0 = -10$, which is below the bottom of the potential defined by the potential tail. Thus, it is natural to regard it as an unphysical series of solutions, and we ignored it. It would be interesting if this unphysical series of solutions allows some holographic interpretation in quantum path-integral.

5.3. Solving quasinormal modes respecting AdS boundary

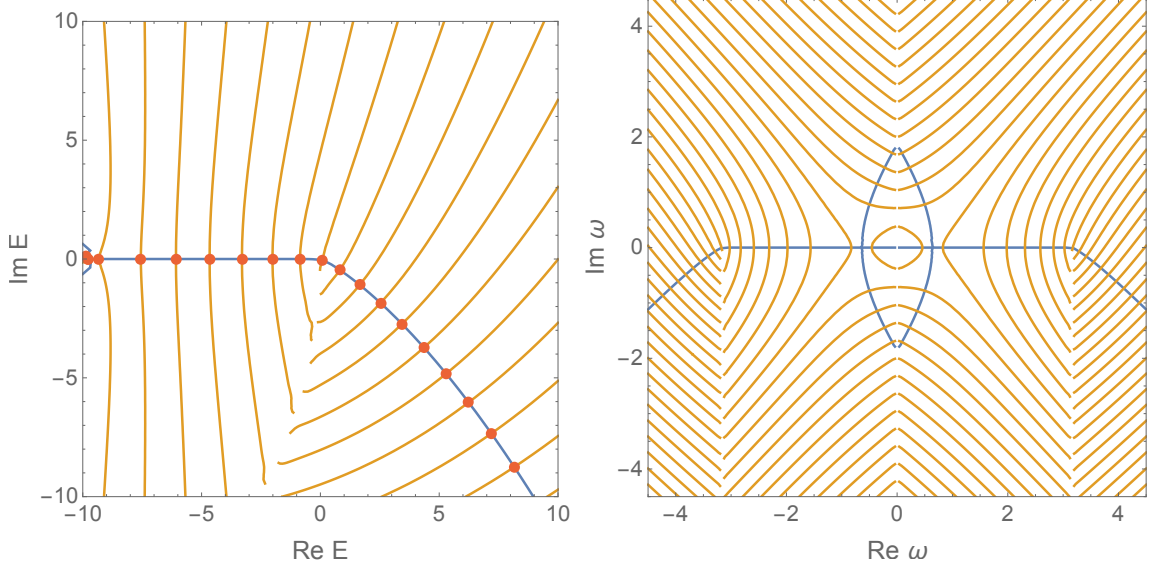


Figure 5.6: The spectrum of the Pöschl-Teller type potential $\tilde{V}(x) = V_0 \left(\frac{1}{\cosh^2 \alpha x} - 1 \right)$ with $V_0 = 10$ (indicated as red blobs). We have imposed the Dirichlet boundary condition at $x = L = 10$, and have assumed that there is only an outgoing mode at $x \rightarrow -\infty$. Other parameters are tuned as $\alpha = 1/\sqrt{4V_0}$ and $\nu = \sqrt{4V_0^2 - 1}/4$ so that we keep $\tilde{V}(x) \simeq -x^2/4$ around the top of the potential hill, to be compared with Fig. 5.5. The blue curve represents $\text{Im } \vartheta = 0$, while the orange curve represents $\text{Re } \vartheta \in 2\pi\mathbb{Z}$. Their intersection points represent the quantized energy spectrum. The left (right) panel is the spectrum in the complex E -plane (ω -plane).

We can show that the equation for the blue curve $\text{Im } \vartheta = 0$ approaches that of the inverted harmonic oscillator when the height of the potential V_0 is sent to the infinity. Let us fix the parameters as

$$\alpha = \frac{1}{\sqrt{4V_0}}, \quad \nu = \sqrt{4V_0^2 - 1}/4 \quad (5.68)$$

so that $\tilde{V}(x) \simeq -x^2/4$ near the top of the potential hill, then send the potential height V_0 to a large value. Some straightforward calculations of (5.66) with the Stirling's formula show that

$$\vartheta(\omega; x) \simeq (\text{real const.}) - \left(2 \ln L_{\text{eff}} + \frac{i\pi}{2} \right) E - i \ln \frac{\sqrt{2\pi}}{\Gamma(1/2 - iE)}, \quad (5.69)$$

where $2 \ln L_{\text{eff}} := \frac{L}{\sqrt{V_0}} + \ln V_0$. Hence, the curve on which QNMs are located is

$$0 = \text{Im} \left[\left(2 \ln L_{\text{eff}} + \frac{i\pi}{2} \right) E + i \ln \frac{\sqrt{2\pi}}{\Gamma(1/2 - iE)} \right], \quad (5.70)$$

which is exactly the same form as (5.50).

Our results in the inverted harmonic oscillator and in the Pöschl-Teller type potential imply that QNMs near the photon sphere appear with a peculiar spectral pattern in the large angular momentum approximation. The pattern is shown in Fig. 5.5 and Fig. 5.6, and the spectrum is on a curve of the form given by (5.50). This QNM spectrum is translated by the AdS/CFT dictionary to the CFT spectrum in a large angular momentum subsector, which we call the “photon-sphere subsector”.

Chapter 6

Conclusion and discussion

In this thesis, we studied the AdS/CFT dictionary by classifying it into three classes:

1. What type of QFTs can be dual to gravity theory? Are there any typical characteristics that distinguish them from non-holographic QFTs? What are the existence conditions of holographic duals?
2. From a holographic QFT, how do the constituents of bulk gravity theory (metric of background geometry, field components, and their dynamics) emerge from boundary data?
3. Suppose that all ingredients of the theories are given, what are the relations of physical quantities in the holographically dual two theories?

The problems in class 3 were treated in Chap. 2 and Chap. 3, while those in class 2 were in Chap. 4. The main subject of this thesis belongs to class 1, and was described in Chap. 5. There, we studied an existence condition of holographic duals and found that the photon sphere works as an indicator of dual black holes, based on our work [1]. In this chapter, we give a summary and discuss some directions for future work.

Conclusion

In Chap. 2, we reviewed some bulk-to-boundary parts of the AdS/CFT dictionary. We explained that the top-down $\text{AdS}_5/\text{CFT}_4$ correspondence, which is derived from the observation in string theory, is a duality between classical gravity theory and strong-coupling gauge theory. We emphasized, in order to extend this correspondence to quantum gravity theory and general QFTs, the necessity of accumulating bottom-up evidence of the $\text{AdS}_{d+1}/\text{CFT}_d$ correspondence. Then, the GKPW relation (2.7), or the equivalence of the spectrum (2.8), was overviewed. We also described the Son-Starinets prescription, by which the retarded Green function $G_R(\Omega)$ in a CFT can be calculated from the asymptotic behavior of the bulk field (2.15) near the AdS boundary as (2.21). We demonstrated the calculation in three bulk geometries: the pure AdS_{d+1} spacetime in the Poincaré patch (2.32) and the global patch (2.48), and the three-dimensional BTZ black hole (2.70). In this calculation, the outgoing boundary condition at the Poincaré/black hole horizon is essentially important. Then, the difference of the analytic structure among the three correlators was compared as shown in Fig. 2.1. It is a noteworthy fact that $G_R(\Omega)$ calculated from the BTZ black hole, which is dual to a thermal CFT_2 , has the poles (2.77) with a negative imaginary part.

In Chap. 3, we reviewed the concept of quasinormal modes and their calculation. In an asymptotically AdS black hole, QNMs are defined by the outgoing boundary condition at the black hole horizon (3.2) and the Dirichlet condition at the AdS boundary (3.3). In general, the energy spectrum Ω is quantized by these boundary conditions and allows a negative imaginary part, as exemplified in the BTZ case. According to the AdS/CFT dictionary, the decay of the QNM describes the relaxation to thermal equilibrium in the dual CFT, and its imaginary part corresponds to the timescale of thermalization (3.8). Then, as one of the semi-analytic approaches to solve QNM spectra, the WKB approximation is elaborated. After the notion of WKB solutions (3.14), turning points (3.17), and Stokes curves (3.18) was introduced, the connection formula (3.25) for the analytic continuation across a Stokes curve was presented. Through the analysis of an inverted harmonic oscillator, we explicitly showed that different boundary conditions induce different spectra (3.33) and (3.38). The validity and practical limit of the WKB method were also demonstrated. Then, the WKB approximation was applied to a scalar field around the Schwarzschild AdS black hole. In a large angular momentum p limit (3.46), the Klein-Gordon equation was cast into the WKB formulation (3.47). Under the assumption that the Stokes geometry of the AdS black hole can be approximated by that of the inverted harmonic oscillator, the QNM boundary condition led to the quantization conditions (3.60) and (3.61). Although the WKB solution is expected to contain the information for any complex Ω in principle, the calculation of the phase integral $\mathcal{Z}$ (3.16) cannot be performed without a suitable approximation. This was why we cannot avoid focusing on $\Omega^2 \simeq p^2/R^2$. In this regime, we obtained the analytic expression of the QNM frequencies (3.63) and (3.65).

In Chap. 4, we reviewed a boundary-to-bulk part of the AdS/CFT dictionary, especially the bulk reconstruction of spacetime metrics from CFT correlators. In principle, the causal structure of an arbitrary asymptotically AdS spacetime can be recovered from a set of lightcone cuts (4.1) by solving (4.2) and (4.3), as we demonstrated for the pure global AdS $_{d+1}$ geometry. However, the reconstruction of lightcone cuts requires $(d+3)$ -point functions in CFT $_d$ and is hardly feasible in general. So, as a more restricted but comparatively practical method, we also described the holographic inverse scattering method. It is based on well-established inverse scattering theory in quantum mechanics, whose subject is to construct the potential V of the radial Schrödinger equation (4.13) from scattering data encoded in the Jost function (4.26). This is realized by solving the Gel'fand-Levitan equation (4.34) for the kernel K_l , which is connected to the potential through (4.35). Or, more practically, by introducing the Fourier-Jacobi series representation (4.61), we can solve a system of the linear equations (4.66) and obtain the potential by (4.63). In its application to bulk reconstruction, the retarded Green function in a boundary CFT is translated into the Jost function as (4.54), and the bulk metric can be reconstructed by (4.46) and (4.45), using a single CFT operator with two different spatial momenta. This procedure was performed numerically for the BTZ black hole and accurately reproduced its metric, as shown in Fig. 4.2.

Chap. 5 is the main part of this thesis. We calculated the spectrum Ω of the scalar quasinormal modes associated with the photon sphere in the Schwarzschild AdS black hole, and found that the photon sphere can be interpreted as an existence condition of holographic duals. In the large l limit where l is the angular momentum of the QNMs, we found that Ω is related to the energy spectrum E of the quantum mechanics (QM) (5.10), as given in (5.22). At high temperature, the QM potential was analytically calculated as (5.34). The QM is not solvable, so we instead analytically solved the inverted harmonic oscillator with a hard wall as (5.49), to obtain the feature of the QM spectrum E . The resultant E is a discrete series along a curve on the complex E -plane. As we saw in Fig. 5.5, $\text{Im } E$ (which should be negative for decaying

modes) grows in its magnitude as $\text{Re } E > 0$ grows, while $\text{Im } E$ almost vanishes for $\text{Re } E < 0$. Quite a similar pattern of the spectrum is universally obtained in the Pöschl-Teller model with a hard wall, as shown in Fig. 5.6. It suggests that the original QM has the same spectral feature, which is translated to that of the QNMs through the relation (5.22). According to the AdS/CFT dictionary, this spectrum should be equal to that of a thermal holographic CFT on a sphere whose radius is R , the bulk AdS radius, at the large angular momentum l .

An interesting feature of the obtained QNM spectrum, (5.22) with (5.49) or (5.67), is the fact that the spectrum whose real part is larger than the photon sphere energy always accompanies a non-vanishing imaginary part which is not directly related to the temperature. Normally the imaginary part of QNMs is due to the boundary condition at the black hole horizon, thus is a direct consequence of the temperature. In the present case, the imaginary part is dictated by the photon sphere, which is determined mainly by the angular momentum. The precise spectrum (5.22) is a prediction for the thermal holographic CFT. In addition, because of the typicality of photon spheres, we expect that such a “photon-sphere subsector” exists in the spectrum of other holographic QFTs.

Here we also note that the spectrum whose real part is smaller than the photon sphere energy ($\text{Re } E < 0$) has almost vanishing imaginary part, meaning that they are extremely long-lived. These modes are confined in the radial region bounded by the photon sphere and the AdS boundary. As we saw in Chap. 3, the existence of these modes was pointed out in [95], focusing on the bottom of the potential well, not the photon sphere.

Discussion for Chapter 3

Let us reconsider the WKB analysis of QNMs in Sec. 3.2.3 to clarify the positioning of our work in Chap. 5. There, we replaced the entire Stokes geometry with that of an inverted harmonic oscillator and focused on $u^2 \simeq 1/R^2$, where the phase integral $\mathcal{Z}$ can be evaluated. However, in (5.25), we showed that the quadratic approximation of the potential is valid in the range of

$$\Delta V = \left| \frac{9(-V''(r_{\text{PS}}))^3}{2(V^{(3)}(r_{\text{PS}}))^2} \right| \quad (6.1)$$

from the top of the photon sphere. For the WKB potential V_0 (3.48), this range can be calculated explicitly and found

$$\Delta V_0 < V_0(r_{\text{max}}) - V_0(\infty), \quad (6.2)$$

where $r_{\text{max}} = \tilde{r}$ (5.7) is the location of the potential hill and $V_0(\infty) = 1/R^2$ is the height of the local minimum, where the WKB frequency was evaluated. This means that the bottom of the potential well is actually out of the validity range ΔV_0 . Nevertheless, we could obtain the QNM frequencies as (3.64). Recalling the exponentially small $\text{Im } \Omega$ (3.65), the field is regarded to be confined near the AdS boundary $0 < z < z_a \ll 1$. So, the result that the QNMs coincide with those of the pure global AdS (2.74) is natural.

At this point, we infer that what led us to the correct $\text{Re } \Omega$ is the harmonic oscillator approximation around the AdS boundary $z = 0$, where $V'(z = 0) = 0$ but $V''(z = 0) > 0$. In fact, if we consider the WKB analysis of the harmonic oscillator around $z = 0$, $\text{Re } \Omega$ will be precisely reproduced. On the other hand, the imaginary part (3.65) is qualitatively consistent with some numerical studies, but its quantitative behavior is not [81, 84]. So, the assumption for the Stokes geometry would collapse, at least, around z_b .

After the work of Festuccia and Liu [95], it was found in [79] that the Stokes geometry in the Schwarzschild AdS for asymptotic QNMs, $|\Omega| \rightarrow \infty$, has a non-trivial structure. More recently, in the context of the asymptotically flat Schwarzschild black hole, the significance of logarithmic spirals of the Stokes curves emerging from the horizon was found in [142]. It would be possible that the rough approximation assumed in Sec. 3.2.3 overlooked such a global structure of the Stokes geometry contributing to $\text{Im } \Omega$. By contrast, the approximation would be valid around the top of the photon sphere, but the phase integral $\mathcal{Z}$ is not simplified for $u^2 \simeq V_0(r_{\text{max}})$. This is why the WKB analysis could not find the photon-sphere subsector.

Discussion for Chapter 4

Next, we comment on bulk reconstruction with the holographic inverse scattering method. Its validity is explicitly restricted by the regularity condition (4.14) for the potential $V(\rho)$. This condition is maintained in the cases of AdS black branes by the existence of the horizon at $\rho = \infty$ and the AdS boundary at $\rho = 0$. While the potential vanishes at the horizon, the ρ^{-2} behavior typical of AdS at the origin is taken out from the potential and treated as the external centrifugal term $(\nu^2 - 1/4)/\rho^2$.

Hence, there are two patterns for which this method fails. First, this method would not be able to reconstruct spacetimes with no horizon. For example, the metric of the pure Poincaré AdS spacetime (2.9) has the form of (4.41) with $h(\rho) = 1$ and $z(\rho) = \rho$, which does not satisfy the boundary condition (4.43). As a consequence, the potential becomes $V_k(\rho) = \mathbf{k}^2$ and violates the regularity condition at $\rho = \infty$. We would avoid this obstacle if we could make use of two operators with distinct conformal dimensions Δ and could somehow know an appropriate boundary condition instead of the second equation in (4.47).

Second, this method would not be valid for spacetimes with different asymptotic structure in general. Since the Minkowski spacetime is reproduced by $h(\rho) = 1$ and $z(\rho) = R$, the potential in an asymptotically flat black hole would asymptote to $V_k(\rho \simeq 0) \simeq (\nu^2 - 1/4)/R^2 + \mathbf{k}^2 - (\nu^2 - 1/4)/\rho^2$, which violates the regularity condition at $\rho = 0$. So, even if the Jost function was obtained, the method would need some modification.

A more subtle and worthwhile example is the bulk geometry dual to a Lifshitz field theory, which is a QFT invariant under the Lifshitz scaling

$$t \rightarrow \lambda^z t, \quad x_i \rightarrow \lambda x_i. \quad (6.3)$$

The parameter $z \in \mathbb{R}$ is called the dynamical exponent, and of course, a CFT is invariant under the $z = 1$ scaling. An anisotropic scaling $z > 1$ appears in quantum critical points in non-relativistic systems. So, this class of QFTs plays an important role in the application of the holographic principle to condensed matter physics.

The Lifshitz symmetry can be realized in the following bulk geometry [153–156]:

$$ds^2 = \frac{R^2}{\rho^2} (-dt^2 + d\rho^2) + \frac{R^{2/z}}{\rho^{2/z}} d\mathbf{x}_{d-1}^2, \quad (6.4)$$

which corresponds to $h(\rho) = (\rho/R)^{2/z-2}$ and $z(\rho) = R(\rho/R)^{1/z}$. While the $z = 1$ case is actually the pure Poincaré AdS, this is not asymptotically AdS for $z > 1$. However, the holographic dictionary, especially the Son-Starinets prescription, can be applied to the $z > 1$ cases parallel with the AdS case, except that the conformal dimension depends on the dynamical exponent z .

as

$$\Delta_{\pm} = \frac{d-1+z}{2z} \pm \nu, \quad \nu = \sqrt{\mu^2 R^2 + \left(\frac{d-1+z}{2z}\right)^2}, \quad (6.5)$$

where μ is the mass of the scalar field. The scalar field $\Phi = e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}} \psi(\rho) z(\rho)^{(d-1)/2}$, which still behaves near the boundary as $\Phi(\rho \simeq 0) \sim \rho^{\Delta_{\pm}}$, is governed by the potential

$$V_k(\rho) = \mathbf{k}^2 \left(\frac{R}{\rho}\right)^{2-2/z}. \quad (6.6)$$

For $z > 1$, this potential is always regular at $\rho = 0$, but is not at $\rho = \infty$.

Nevertheless, the potential, and consequently the bulk metric, can be reconstructed by holographic inverse scattering, up to a certain depth of the bulk spacetime [157]. For $z = 2$, the Klein-Gordon equation on the Lifshitz geometry (6.4) can be solved using the Whittaker function, which is a subspecies of the confluent hypergeometric function. Then, through the Son-Starinets prescription, the retarded Green function in the Lifshitz field theory is obtained as

$$G_R(\omega, \mathbf{k}) = \frac{\Gamma(-\nu)}{\Gamma(\nu)} e^{-i\pi\nu} \frac{|\omega^2|^\nu}{2^{2\nu}} \frac{\Gamma(\frac{1}{2} - \nu) \Gamma(\frac{1}{2} + \nu + \frac{i\mathbf{k}^2 R}{2|\omega|})}{\Gamma(\frac{1}{2} + \nu) \Gamma(\frac{1}{2} - \nu + \frac{i\mathbf{k}^2 R}{2|\omega|})}. \quad (6.7)$$

This has no poles (QNMs), and the Jost function made of this G_R satisfies the correct asymptotic behavior (4.28). Therefore, we can formally attempt to solve the Gel'fand-Levitan equation (4.66), and to reconstruct the metric functions under the boundary conditions

$$z(\rho \simeq 0) \simeq \sqrt{R\rho}, \quad z'(\rho \simeq 0) \simeq \frac{1}{2} \sqrt{\frac{R}{\rho}}. \quad (6.8)$$

Then we find that the numerically reconstructed metric functions agree with the exact ones up to a certain depth of the bulk, but they deviate as we probe deeper.

A lesson from this example is that we typically confront method-induced errors in bulk reconstruction. When we try to explore an unknown gravitational system dual to a given QFT, it is difficult to verify a priori whether the bulk potential is under the regularity condition (4.14). Then, how can we justify the obtained metric? One possible approach is to refer to the result of another reconstruction method. However, such an approximation error generally exists in other methods. For example, in the holographic inverse scattering method, errors are induced not only by the regularity condition but also by a cutoff in the ω -integrations (4.65) and a truncation in the series expansion (4.61), which are the kinds of approximations also needed in other reconstruction methods. Hence, it is possible that the cross-reference of the obtained metrics does not work.

This presents an issue in the following situation. Suppose that we theoretically or experimentally obtained a matter system that is possible to have a gravitational dual, and would like to verify its existence. We can reconstruct the bulk metric and decide whether it truly originates from emergent gravity or is just an architecture. Here, the inconsistency due to the method-induced errors would mislead us into concluding that this system is not holographic, even if it actually was. This is why we emphasize the necessity of an existence condition of holographic duals formulated in an analytically tractable manner.

Discussion for Chapter 5

Finally, we discuss some features of the photon-sphere subsector of the QNMs and its dual CFT interpretation. In our analysis, the spectrum consists of (a) QNMs at $\text{Re } E > 0$ with a nonzero and growing imaginary part and (b) those at $\text{Re } E < 0$ with an almost vanishing imaginary part.

The latter QNMs (b) are localized in the spherical shell region between the AdS boundary and the photon sphere and rotate along the boundary sphere. They correspond to the long-lived states in the CFT. We saw that their energy Ω is given by

$$\Omega \lesssim \sqrt{V(r_{\text{PS}})} \simeq \frac{l}{R} \sqrt{1 + \frac{R^2}{r_0^2} \frac{d-2}{d} \left(\frac{2}{d}\right)^{2/(d-2)}}. \quad (6.9)$$

in the large l limit (see (5.22)), where l is the angular momentum of the states and R is the radius of the sphere where the CFT lives. On the other hand, as we showed in Sec. 3.2.3, these long-lived states are found in [95] around the bottom of the potential well $\Omega \gtrsim l/R$. Combining these two results suggests that there are a number of long-lived quasiparticles of $\mathcal{O}(l)$ near the AdS boundary.

Recalling that, in the original $\text{AdS}_5/\text{CFT}_4$ correspondence, the bulk scalar (dilaton) is dual to the gluon condensate $\text{tr } F^2$, our large angular momentum QNMs should correspond to CFT operators of the form $\text{tr } [FDDD \cdots DDF]$, where $D = D_\theta$ is the covariant derivative along the equator of the sphere. This type of operator carries a large angular momentum that is equal to the number of D_θ 's in the operator. Starting from the bottom of the potential well $\Omega = l/R$, where the QNM would correspond to the CFT state $\text{tr } [FD^l F]$, the more impurity operators we add, the more the energy of the state grows. These are reminiscent of the strings on the pp-wave geometry and the BMN operators [158], which are stable and dictated by a long spin chain.¹ Further studies on these states would be interesting.

On the other hand, what we studied in detail in Chap. 5 is the sector (a) whose real part of the energy Ω is beyond the top of the potential hill (photon sphere), $\text{Re } \Omega \gtrsim \sqrt{V(r_{\text{PS}})}$. The QNM develops a negative imaginary part. The corresponding operator on the CFT side is of the same form, $\text{tr } [FDDD \cdots DDF]$, but with very many impurity operators inserted. When the number of the impurities exceeds the photon sphere bound $\sim l \cdot R^2/r_0^2$, which amounts to the energy difference between the top of the photon sphere and the bottom at the AdS boundary, the state suddenly becomes unstable and begins to decay. In other words, the state of the form $\text{tr } [FDDD \cdots DDF]$ with a large angular momentum could be quite stable against a large addition of impurities whose number can go up to $\sim l \cdot R^2/r_0^2$. This universal behavior of the CFT states, expected through the AdS/CFT dictionary, needs a more intuitive explanation in the CFT language.

We should notice that the existence of this photon-sphere subsector in a QFT would be sufficient to confirm the emergence of a holographic black hole, but not necessary. One simple counterexample is a CFT_2 and its dual BTZ black hole. It has no photon sphere, and as seen in Sec. 2.2.4, there are no long-lived QNMs, regardless of the topology of the boundary. Also, in contrast to the case of a CFT on a sphere, if the boundary $\text{CFT}_{d \geq 3}$ lives on the flat spacetime, the dual AdS_{d+1} black brane is given by

$$ds^2 = \frac{R^2}{z^2} \left[-f(z) dt^2 + \frac{dz^2}{f(z)} + d\mathbf{x}_{d-1}^2 \right], \quad f(z) = 1 - \left(\frac{z}{z_h} \right)^d, \quad (6.10)$$

¹Whereas our QNMs have a spatial angular momentum along AdS_{d+1} , the bulk dual of the BMN operators lives on $\text{AdS}_5 \times \mathbb{S}^5$ and has a large angular momentum along $\mathbb{S}^5$, which corresponds to the internal R-symmetry in the boundary CFT_4 .

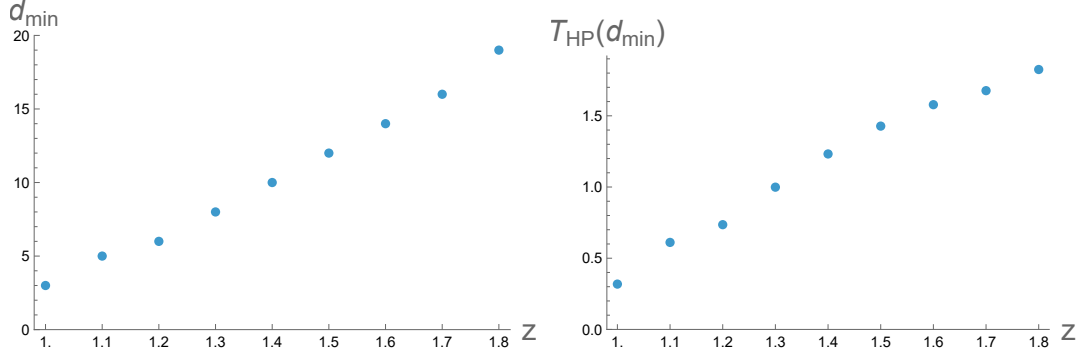


Figure 6.1: (Left panel) Minima of the bulk spatial dimension d so that the large l potential $V_0 = f(r)/r^2$ has a photon sphere in the Lifshitz black hole (6.12) with a dynamical exponent z . The black hole temperature is fixed as the Hawking-Page transition temperature (6.14) $T = T_{\text{HP}}$. (Right panel) The values of T_{HP} at d_{min} . Qualitatively, increasing z tends to make the photon sphere vanish, while increasing d opposes. For this figure, we took $R = 1$.

and the photon sphere (potential hill) does not exist in this geometry. This suggests that the finite size effect would be important for the long-lived excitations.

Another interesting example is a finite-temperature extension of the Lifshitz geometry (6.4). The metric of a Lifshitz black hole [159] is given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega_{d-1}^2, \quad (6.11)$$

$$f(r) = \frac{r^{2z}}{R^{2z}} b(r), \quad g(r) = \frac{r^2}{R^2} b(r), \quad b(r) = \left(\frac{d-2}{d+z-3} \right)^2 \frac{R^2}{r^2} + 1 - \frac{m}{r^{d+z-1}}, \quad (6.12)$$

which coincides with the Schwarzschild AdS (5.1) for $z = 1$. When we consider a scalar field on this background, we find that the effective potential

$$V(r) = f \left[\frac{l(l+d-2)}{r^2} + \frac{(d-1)(d-3)}{4} \frac{g}{r^2} + \mu^2 \right] + \frac{d-1}{4r} (f'g + fg') \quad (6.13)$$

diverges at the boundary $r \rightarrow \infty$ and a hard wall like the AdS boundary exists. On the other hand, this potential has a photon sphere only for the certain values of d above a threshold d_{min} determined by z and the temperature T (Fig. 6.1).² For the left panel of Fig. 6.1, the temperature is fixed as the Hawking-Page transition temperature

$$T_{\text{HP}} = \frac{d-1}{2\pi(2-z)R} \left[\frac{(2-z)(d-2)^2}{z(d+z-3)^2} \right]^{z/2}, \quad (6.14)$$

that is, the minimum temperature of the black hole.³ If $T > T_{\text{HP}}$, d_{min} rapidly grows as z increases. For $d \geq d_{\text{min}}(z, T)$, the photon-sphere subsector is expected to exist. However, as the spatial dimension d increases, a fluctuation in the thermal QFT would be able to dissipate into more directions. This seems to lessen the finite size effect and to make the lifetime $t \sim |\text{Im } \Omega|^{-1}$

²Note that there would be no photon sphere for $z = 2$, because the large l potential $V_0(r)$ is monotonic.

³In the Lifshitz black hole geometry (6.12) with $1 \leq z \leq 2$, there is a reminiscent of the Hawking-Page transition between the black hole geometry and the pure Lifshitz geometry. For $z > 2$, instead, the black hole geometry always dominates the thermal ensemble [159].

shorter, that is, to prevent the photon sphere from forming. Increasing T also affects in the same direction (see Fig. 5.2). To investigate why, contrary to these intuitions, the photon sphere requires a large d for $z > 1$ would be intriguing.

The QFT structure underlying the photon-sphere subsector would be directly connected with the fundamental mechanism of the holographic principle itself. Its robust stability against thermal fluctuation is an interesting phenomenon, also from a purely field theoretic point of view. In searching for this, its large angular momentum is expected to simplify the QFT analysis, just as the WKB approximation and the quadratic approximation are available on the bulk side. Since we worked with just the solvable models in Chap. 5, it is important to confirm in numerical calculations the existence of the photon-sphere subsector both in the QNM spectra in AdS black holes and in QFT spectra.

In the following, we present several discussions on how our work may be related to some other aspects of the AdS/CFT correspondence and holography.

- First, let us discuss the relation between our spectra and those found in the asymptotically flat black hole [97]. As we have emphasized in Sec. 5.3, our AdS case has a hard wall at the AdS boundary, while the asymptotically flat case does not. This difference was crucial in determining the spectrum. Now, suppose that we add a flat spacetime joined to the AdS boundary, as was done in [160]. Then the wall could “disappear”, and the QNM in the AdS bulk can escape to the joined flat spacetime. Since the AdS plus the joined flat spacetime may allow a holographic interpretation as a CFT joined with a higher dimensional CFT, the flat spectrum found in [97], which is dictated by the $SL(2; \mathbb{R})$ symmetry, may be realized in the AdS/CFT setup in this manner. It would be interesting to find the AdS/CFT interpretation of the $SL(2; \mathbb{R})$ subsector observed in [97], in view of the issues concerning evaporation of black holes and the Page curve.
- As mentioned in Chap. 1, Einstein rings can be obtained in a holographic CFT [57, 58] by using an imaging transform on a CFT one-point function with a point-like source.⁴ Thus, the holographic imaging of the Einstein rings is a direct consequence of two-point functions in the holographic CFT. Since the QNM spectrum near the photon sphere should characterize Einstein rings in general, it would be interesting to find a relation between the image of the holographic Einstein rings and the photon-sphere subsector of the CFT spectrum.
- In the context of the AdS/CFT correspondence, Lyapunov exponents have been studied in connection to the saturation of the chaos bound [54], since the bound is related to the black hole surface gravity [53, 169, 170]. The QNMs associated with the photon sphere naturally define the Lyapunov exponent there [78], thus their relevance to the chaos bound would be interesting if exists. Namely, the chaos bound could work as a bound for the QNM spectra. This research direction would also contribute to some general discussion on the black hole instability, such as [171].

Further interplay between photon spheres and boundary QFTs will reveal the mystery of the holographic principle and quantum gravity.

⁴The imaging in holography and string theory has been further developed for various purposes (see [62, 161–168]).

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Appendix A

Appendix for Chapter 4

A.1 Calculation details in the lightcone cuts method

In this appendix, we provide the details of the calculation in (ii) of Sec. 4.1.2 to reconstruct the metric of the global pure AdS_{d+1} spacetime up to a conformal factor. Although the calculation below is slightly technical, what we will do is just a mathematical induction on the space dimension d :

- ii-1) Reconstruct the conformal metric in $d = 2$.
- ii-2) Assuming that the conformal metric in $d = D - 1$ ($D \geq 3$) has already been reconstructed, reconstruct the conformal metric in $d = D$ around the equator $\Theta_i = \pi/2$.
- ii-3) Rotate the coordinate in the above result to obtain the conformal metric for any Θ .

ii-1) Reconstruct the conformal metric in $d = 2$.

In the $d = 2$ case, from the lightcone cuts (4.10), the tangential conditions (4.2) are given as

$$0 = \delta T - \frac{L[\delta Y \cos(\theta - \Theta) + Y \delta \Theta \sin(\theta - \Theta)]}{\sqrt{1 - Y^2 \cos^2(\theta - \Theta)}}, \quad (\text{A.1})$$

$$0 = \delta Y \sin(\theta - \Theta) - Y(1 - Y^2) \delta \Theta \cos(\theta - \Theta), \quad (\text{A.2})$$

up to $\mathcal{O}(\delta \lambda^\mu)$. Solving these equations gives a bulk null vector

$$\delta \lambda = -LY \sqrt{1 - Y^2 \cos^2(\theta - \Theta)} \partial_T - Y(1 - Y^2) \cos(\theta - \Theta) \partial_Y - \sin(\theta - \Theta) \partial_\Theta. \quad (\text{A.3})$$

Then the null condition (4.3) of $\delta \lambda$ becomes

$$\begin{aligned} 0 = & g_{TT} L^2 Y^2 [1 - Y^2 \cos^2(\theta - \Theta)] + 2g_{TY} LY^2 (1 - Y^2) \sqrt{1 - Y^2 \cos^2(\theta - \Theta)} \\ & + 2g_{T\Theta} LY \sin(\theta - \Theta) \sqrt{1 - Y^2 \cos^2(\theta - \Theta)} + g_{YY} Y^2 (1 - Y^2) \cos^2(\theta - \Theta) \\ & + 2g_{Y\Theta} Y (1 - Y^2) \sin(\theta - \Theta) \cos(\theta - \Theta) + g_{\Theta\Theta} \sin^2(\theta - \Theta), \end{aligned} \quad (\text{A.4})$$

and expanding it around $\theta = \Theta$ gives

$$0 = g_{TT} L^2 Y^2 (1 - Y^2) + 2g_{TY} LY^2 (1 - Y^2)^{\frac{3}{2}} - g_{YY} Y^2 (1 - Y^2)^2$$

A.1. Calculation details in the lightcone cuts method

$$\begin{aligned}
& + \left[2g_{T\Theta}LY\sqrt{1-Y^2} + 2g_{Y\Theta}Y(1-Y^2) \right] (\theta - \Theta) \\
& + \left[g_{\Theta\Theta} + g_{TT}L^2Y^4 + g_{YY}Y^2(1-Y^2)^2 - g_{TY}LY^2\sqrt{1-Y^2}(1-2Y^2) \right] (\theta - \Theta)^2 \\
& + \left[-\frac{4}{3}g_{Y\Theta}Y(1-Y^2) - g_{T\Theta}\frac{LY(1-4Y^2)}{3\sqrt{1-Y^2}} \right] (\theta - \Theta)^3 \\
& + \frac{1}{3} \left[-g_{\Theta\Theta} - g_{TT}L^2Y^4 + g_{YY}Y^2(1-Y^2)^2 + g_{TY}\frac{LY^2(1-12Y^2+8Y^4)}{4\sqrt{1-Y^2}} \right] (\theta - \Theta)^4 \\
& + \mathcal{O}(\theta - \Theta)^5.
\end{aligned} \tag{A.5}$$

Requiring that the above equation should hold for any θ , we have

$$g_{TT} = -\frac{1-Y^2}{L^2}g_{YY}, \quad g_{\Theta\Theta} = Y^2(1-Y^2)g_{YY}, \quad g_{TY} = g_{T\Theta} = g_{Y\Theta} = 0. \tag{A.6}$$

This is also sufficient for (A.4). Therefore, the conformal metric of the bulk geometry in the $d = 2$ case is determined as

$$ds^2 \propto -(1-Y^2)dT^2 + L^2dY^2 + L^2Y^2(1-Y^2)d\Theta^2 \quad (d = 2). \tag{A.7}$$

ii-2) Reconstruct the $d = D$ metric around the equatorial plane.

On the equatorial plane ($\Theta_i = \pi/2$ for $i = 1, \dots, D-1$), the tangential conditions (4.2) become

$$0 = \delta T - \frac{\mathcal{S}_{D-1}}{\sqrt{1-Y^2\mathcal{S}_{D-1}^2}}\delta Y + \sum_{k=1}^{D-1} \frac{Y\partial_k\mathcal{S}_k}{\sqrt{1-Y^2\mathcal{S}_{D-1}^2}}, \tag{A.8}$$

$$\begin{aligned}
0 = & -\delta Y\mathcal{S}_{D-1}\cot\theta_k + \sum_{i=1}^{k-1} Y^3(\partial_i\mathcal{S}_i)\mathcal{S}_{D-1}^2\delta\Theta_i\cot\theta_k - Y\mathcal{S}_k\left(1 - \frac{Y^2}{\sin^2\theta_k}\mathcal{S}_{D-1}^2\right)\delta\Theta_k \\
& + \sum_{i=k+1}^{D-1} Y(\partial_i\mathcal{S}_i)\delta\Theta_i\cot\theta_k \quad (k = 1, \dots, D-1),
\end{aligned} \tag{A.9}$$

where $\mathcal{S}_k := \sin\theta_1 \cdots \sin\theta_k$, $\mathcal{S}_0 := 1$, and $\partial_k := \partial/\partial\theta_k$. Note that, expressing the geodesic distance between $(\theta_k, \dots, \theta_{D-1})$ and $(\Theta_k, \dots, \Theta_{D-1})$ on the unit $\mathbb{S}^{d-k}$ as $\Omega_{d-k}(\boldsymbol{\theta}, \boldsymbol{\Theta})$, the geodesic distance in (4.9) obeys a recurrence formula,

$$\cos\Omega_{d-k+1}(\boldsymbol{\theta}, \boldsymbol{\Theta}) = \cos\theta_k\cos\Theta_k + \sin\theta_k\sin\Theta_k\cos\Omega_{d-k}(\boldsymbol{\theta}, \boldsymbol{\Theta}). \tag{A.10}$$

This can be straightforwardly shown by considering the $\text{SO}(d)$ rotation of $\mathbb{S}^{d-1}$.

First, when we set $\delta\Theta_{D-1} = 0$ and $\theta_{D-1} = \pi/2$, the condition (A.9) for $k = D-1$ is trivially satisfied and the others are the same as in the $d = D-1$ case. According to the assumption of the mathematical induction that the metric of the pure global AdS_{D-1} be reconstructed from the (A.8), (A.9), the $D \times D$ components of the metric are determined as

$$g_{TT} = -\frac{1-Y^2}{L^2}g_{YY}, \tag{A.11}$$

$$g_{\Theta_i\Theta_i} = Y^2(1-Y^2)g_{YY} \quad (i = 1, \dots, D-2), \tag{A.12}$$

$$g_{TY} = g_{T\Theta_i} = g_{Y\Theta_i} = g_{\Theta_i\Theta_j} = 0 \quad (i, j = 1, \dots, D-2, i \neq j). \tag{A.13}$$

A.1. Calculation details in the lightcone cuts method

Second, when we set $\delta\Theta_i = 0$ and $\theta_i = \pi/2$ for $i = 1, \dots, D-2$, the nontrivial ones among the conditions (A.8), (A.9) become

$$0 = \delta T - \frac{L[\delta Y \sin \theta_{D-1} - Y \delta \Theta_{D-1} \cos \theta_{D-1}]}{\sqrt{1 - Y^2 \sin^2 \theta_{D-1}}}, \quad (\text{A.14})$$

$$0 = \delta Y \cos \theta_{D-1} + Y(1 - Y^2) \delta \Theta_{D-1} \sin \theta_{D-1}. \quad (\text{A.15})$$

Since this is the same as the $d = 2$ case (A.1), (A.2) with $\Theta = \pi/2$, some components of the metric are determined as

$$g_{\Theta_{D-1}\Theta_{D-1}} = Y^2(1 - Y^2)g_{YY}, \quad g_{T\Theta_{D-1}} = g_{Y\Theta_{D-1}} = 0. \quad (\text{A.16})$$

Finally, when we set $\delta\Theta_k = 0$, $\theta_k = \pi/2$ for $k \neq i, D-1$ and $\theta_i = \theta_{D-1} = \pi/4$, the nontrivial ones among the conditions (A.8), (A.9) become

$$0 = \delta T - \frac{L[\delta Y - Y(\sqrt{2}\delta\Theta_i + \delta\Theta_{D-1})]}{2\sqrt{1 - Y^2/4}}, \quad (\text{A.17})$$

$$0 = \delta Y + \frac{Y}{\sqrt{2}}(2 - Y^2)\delta\Theta_i - Y\delta\Theta_{D-1}, \quad (\text{A.18})$$

$$0 = 2\delta Y - \frac{Y^3}{\sqrt{2}}\delta\Theta_i + Y(2 - Y^2)\delta\Theta_{D-1}. \quad (\text{A.19})$$

Their solution reads

$$\delta\lambda = -L\sqrt{4 - Y^2}\partial_T - (1 - Y^2)\partial_Y + \frac{\sqrt{2}}{Y}\partial_{\Theta_i} + \frac{1}{Y}\partial_{\Theta_{D-1}}, \quad (\text{A.20})$$

and substituting this into the null condition (4.3) gives $g_{\Theta_i\Theta_{D-1}} = 0$ for $i = 1, \dots, D-2$. Therefore, the conformal metric around the equatorial plane is determined as

$$ds^2 \propto -(1 - Y^2)dT^2 + L^2dY^2 + L^2Y^2(1 - Y^2)\sum_{i=1}^{D-1}(d\Theta_i)^2 \quad \left(\Theta_i \simeq \frac{\pi}{2}\right). \quad (\text{A.21})$$

ii-3) Reconstruct the metric at any Θ .

The lightcone cuts (4.10) have the $\text{SO}(d)$ symmetry, which act as

$$C(T, Y, \Theta_O; \theta_O) = C(T, Y, \Theta; \theta), \quad (\text{A.22})$$

where the subscript O is used to express the coordinates of the points rotated by $O \in \text{SO}(d)$. Now we use this symmetry to rotate an arbitrary point Θ to the point $(\Theta_O)_i = \pi/2$ ($i = 1, \dots, d-1$), which satisfies

$$\sum_{i=1}^{d-1}(d(\Theta_O)_i)^2 = \sum_{i=1}^{d-1}\sin^2 \Theta_1 \cdots \sin^2 \Theta_{i-1}(d\Theta_i)^2 \equiv d\Omega_{d-1}^2. \quad (\text{A.23})$$

Since the cuts are unchanged under this rotation, the metric at the general point (T, Y, Θ) is obtained simply by substituting (A.23) into the metric at (T, Y, Θ_O) of the form (A.21):

$$ds^2 \propto -(1 - Y^2)dT^2 + L^2dY^2 + L^2Y^2(1 - Y^2)d\Omega_{d-1}^2. \quad (\text{A.24})$$

Appendix B

Appendix for Chapter 5

In this appendix, we present supplemental materials for our analysis in Chap. 5. In Sec. B.1, we summarize the representation theory of $SL(2; \mathbb{R})$. Sec. B.2 is for detailed information on the solvable models used in Sec. 5.3. We also illustrate that the $SL(2; \mathbb{R})$ symmetry is broken by the AdS boundary, in contrast to the case in the asymptotically flat black holes [96, 97].

B.1 Representation of $SL(2; \mathbb{R})$

B.1.1 Relation between $SO(2, 1)$, $SU(1, 1)$, and $SL(2; \mathbb{R})$

There are three kinds of non-compact Lie algebras $\mathfrak{so}(2, 1) \cong \mathfrak{su}(1, 1) \cong \mathfrak{sl}(2; \mathbb{R})$ that are isomorphic one another as described in the Fig. B.1.

The non-compact group $SL(2; \mathbb{R})$ is the linear group consisting of matrices $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ ($\alpha, \beta, \gamma, \delta \in \mathbb{R}$) with a relation $\alpha\delta - \beta\gamma = 1$. The $SU(1, 1)$ is the complex linear group generated by $g' = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$ ($a, b \in \mathbb{C}$) with a constraint $|a|^2 - |b|^2 = 1$.

This $SU(1, 1)$ is isomorphic to $SL(2; \mathbb{R})$ under a similarity transformation $g = h^{-1}g'h$ for $g \in SU(1, 1)$ and $g' \in SL(2; \mathbb{R})$ with $h := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$. That leads to relations of components of matrices as $a = \frac{\alpha+\delta}{2} + i\frac{\beta-\gamma}{2}$, $b = \frac{\beta+\gamma}{2} + i\frac{\alpha-\delta}{2}$. As concrete examples, we can consider one-parameter subgroups of $SL(2; \mathbb{R})$ and their correspondents as $SU(1, 1)$ elements:

$$\text{elliptic} : \begin{pmatrix} \cos \frac{t}{2} & -\sin \frac{t}{2} \\ \sin \frac{t}{2} & \cos \frac{t}{2} \end{pmatrix} \leftrightarrow \begin{pmatrix} e^{-it/2} & 0 \\ 0 & e^{+it/2} \end{pmatrix}, \quad (\text{B.1})$$

$$\text{hyperbolic} : \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix} \leftrightarrow \begin{pmatrix} \cosh \frac{t}{2} & -i \sinh \frac{t}{2} \\ i \sinh \frac{t}{2} & \cosh \frac{t}{2} \end{pmatrix}, \quad (\text{B.2})$$

$$\text{parabolic} : \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 - i\frac{t}{2} & \frac{t}{2} \\ \frac{t}{2} & 1 + i\frac{t}{2} \end{pmatrix}. \quad (\text{B.3})$$

The $SU(1, 1)$ is a covering space of $SO(2, 1)$. The latter leaves the bilinear form $x_0^2 - x_1^2 - x_2^2$ invariant and is a subgroup of the Lorentz group $SO(3, 1)$. The center of $SU(1, 1)$ is $\mathbb{Z}_2$, and $SU(1, 1)/\mathbb{Z}_2$ is locally isomorphic to $SO(2, 1)$. Indeed $SU(1, 1)$ is a simply connected group and is the covering group of $SO(2, 1)$.

B.1. Representation of $\text{SL}(2; \mathbb{R})$

$$\boxed{\begin{array}{ccccc} \text{SO}(3, 1) & \leftrightarrow & \text{SL}(2; \mathbb{C}) & & \\ \downarrow & & \downarrow & & \\ \text{SO}(2, 1) & \leftrightarrow & \text{SU}(1, 1) & \leftrightarrow & \text{SL}(2; \mathbb{R}) \end{array}}$$

Figure B.1: Relation of the non-compact groups. $\text{SO}(2, 1)$ is constructed by a reduction from $\text{SO}(3, 1)$. $\text{SU}(1, 1)$ is obtained from $\text{SL}(2; \mathbb{C})$ by a suitable restriction. $\text{SU}(1, 1)$ is also related to $\text{SL}(2; \mathbb{R})$ by a similarity transformation.

$\text{SO}(3, 1)$ has a covering space $\text{SL}(2; \mathbb{C})$, and we can write down relations among components of $\text{SU}(1, 1)$ and $\text{SL}(2; \mathbb{C})$ by using a reduction of $\text{SO}(3, 1)$ to $\text{SO}(2, 1)$.

The group $\text{SO}(3, 1)$ acts on a set of coordinates $x^\mu = (x^0, x^1, x^2, x^3)$ and x^μ transforms into $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ ($\Lambda \in \text{SO}(3, 1)$). By introducing the Pauli matrices, one can represent this transformation in a matrix form $\mathbf{x} = x^\mu \sigma_\mu$. The $\text{SL}(2; \mathbb{C})$ can act on this matrix as left/right multiplications by an element $\hat{g} \in \text{SL}(2; \mathbb{C})$:

$$x^\mu = (x^0, x^1, x^2, x^3), \quad \sigma_\mu = (1, \sigma_1, \sigma_2, \sigma_3), \quad (\text{B.4})$$

$$\mathbf{x} := x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}, \quad (\text{B.5})$$

$$\hat{g} = \exp \left(i\vec{\theta} \cdot \frac{\vec{\sigma}}{2} - \vec{\omega} \cdot \frac{\vec{\sigma}}{2} \right), \quad \det \hat{g} = 1, \quad (\text{B.6})$$

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \quad \mathbf{x}' = \hat{g} \mathbf{x} \hat{g}^\dagger. \quad (\text{B.7})$$

We can construct $\text{SO}(2, 1)$ group by putting $x^3 = 0$ and restricting $\text{SO}(3, 1)$ into $\text{SO}(2, 1)$. Then the covering space $\text{SL}(2; \mathbb{C})$ is reduced to $\text{SU}(1, 1)$. But one has to impose a constraint $\hat{g} \sigma_3 \hat{g}^\dagger = \sigma_3$ for $\hat{g} \in \text{SL}(2; \mathbb{C})$ to remove mixing between the x^3 part and the remaining three-dimensional parts. This condition leads to a restriction of components of $\hat{g} \in \text{SL}(2; \mathbb{C})$ and induces an element of $g' \in \text{SU}(1, 1)$, which is parameterized by two complex parameters a and b with a constraint $|a|^2 - |b|^2 = 1$,

$$\text{SL}(2; \mathbb{C}) \ni \hat{g} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{C}, \quad (\text{B.8})$$

$$\text{SU}(1, 1) \ni g' = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \quad |a|^2 - |b|^2 = 1, \quad a, b \in \mathbb{C}. \quad (\text{B.9})$$

Every element $g' \in \text{SU}(1, 1)$ satisfies a constraint $g' \sigma_3 g'^\dagger = \sigma_3$. It induces a natural reduction of $\text{SL}(2; \mathbb{C})$ to $\text{SU}(1, 1)$ associated with the reduction of $\text{SO}(3, 1)$ to $\text{SO}(2, 1)$. Then we can write down the corresponding element $\Lambda^\mu{}_\nu$ of $\text{SO}(2, 1)$ as

$$\text{SO}(2, 1) \ni \Lambda^\mu{}_\nu = \begin{pmatrix} |a|^2 + |b|^2 & 2\text{Re}(a\bar{b}) & 2\text{Im}(a\bar{b}) \\ 2\text{Re}(ab) & \text{Re}(a^2 + b^2) & \text{Im}(a^2 - b^2) \\ -2\text{Im}(ab) & -\text{Im}(a^2 + b^2) & \text{Re}(a^2 - b^2) \end{pmatrix}, \quad |a|^2 - |b|^2 = 1. \quad (\text{B.10})$$

Recalling the relations between $\text{SU}(1, 1)$ and $\text{SL}(2; \mathbb{R})$, we can show an element $g \in \text{SL}(2; \mathbb{R})$ in this parameterization as

$$\text{SL}(2; \mathbb{R}) \ni g = \begin{pmatrix} \text{Re } a + \text{Im } b & \text{Re } b + \text{Im } a \\ \text{Re } b - \text{Im } a & \text{Re } a - \text{Im } b \end{pmatrix}, \quad |a|^2 - |b|^2 = 1. \quad (\text{B.11})$$

B.1. Representation of $\text{SL}(2; \mathbb{R})$

B.1.2 Representation of $\mathfrak{su}(1, 1)$

For the universal covering group associated with locally isomorphic Lie groups, all its irreducible representations are single-valued. Keeping this in mind, we study representations of the Lie algebra $\mathfrak{su}(1, 1)$.

The Lie algebra $\mathfrak{su}(1, 1)$ is generated by a set of J_a 's ($a = 1, 2, 3$). The commutation relations are expressed in the form

$$[J_1, J_2] = -iJ_3, \quad [J_2, J_3] = iJ_1, \quad [J_3, J_1] = iJ_2, \quad (\text{B.12})$$

$$J_a^\dagger = J_a. \quad (\text{B.13})$$

By introducing $J_\pm$, we rewrite the commutators of the generators as

$$J_\pm = J_1 \pm iJ_2, \quad J_\pm^\dagger = J_\mp, \quad (\text{B.14})$$

$$[J_+, J_-] = -2J_3, \quad [J_3, J_\pm] = \pm J_\pm, \quad (\text{B.15})$$

and a Casimir invariant is represented as a quadratic form

$$J^2 = J_1^2 + J_2^2 - J_3^2 = \frac{1}{2}(J_+J_- + J_-J_+) - J_3^2 = J_\pm J_\mp \pm J_3 - J_3^2. \quad (\text{B.16})$$

When we take a redefinition $L_0 = -J_3$ and $L_{\pm 1} = J_\pm$, we can express the Lie algebra $\mathfrak{sl}(2; \mathbb{R})$ with generators L_n ($n = 0, \pm 1$). Their commutation relations are written as $[L_{+1}, L_{-1}] = 2L_0$ and $[L_0, L_{\pm 1}] = \mp L_{\pm 1}$.

Now we shall consider representations of $\mathfrak{su}(1, 1)$. The operators J^2 and J_3 are hermitian operators and their eigenvalues are real numbers $\Lambda \in \mathbb{R}$ and $a \in \mathbb{R}$, respectively. The states of the Lie algebra are labeled by this set of eigenvalues (Λ, a) ,

$$J^2|\Lambda, a\rangle = \Lambda|\Lambda, a\rangle, \quad J_3|\Lambda, a\rangle = a|\Lambda, a\rangle. \quad (\text{B.17})$$

By using $J_\mp^\dagger J_\mp = J_\pm J_\mp = J^2 + J_3^2 \mp J_3$, a useful condition is obtained as

$$J_\pm J_\mp |\Lambda, a\rangle = [\Lambda + a(a \mp 1)]|\Lambda, a\rangle, \quad (\text{B.18})$$

$$\Lambda + a(a \mp 1) \geq 0. \quad (\text{B.19})$$

Here we assumed that the inner product is positive definite. It is the natural assumption in the unitary theories.

We can also show that the generator J_+ raises the eigenvalue a by one (when it satisfies $\langle \Lambda, a' | J_+ | \Lambda, a \rangle \neq 0$), by considering the following equation

$$\langle \Lambda, a' | J_+ | \Lambda, a \rangle = \langle \Lambda, a' | [J_3, J_+] | \Lambda, a \rangle = (a' - a) \langle \Lambda, a' | J_+ | \Lambda, a \rangle. \quad (\text{B.20})$$

Using these two properties, we study unitary representations of $\mathfrak{su}(1, 1)$. The representation is labeled by Λ of the Casimir invariant. We parametrize $\Lambda = -j(j+1)$ by $j \in \mathbb{C}$. The reality of Λ leads to an important constraint that Λ is expressed in either of the following two series,

$$\Lambda \in \mathbb{R} \quad \Rightarrow \quad \begin{cases} j = \alpha & (\alpha \in \mathbb{R}) \\ j = -\frac{1}{2} + i\beta & (\beta \in \mathbb{R}). \end{cases} \quad (\text{B.21})$$

Each series is parametrized by a set of real numbers, α and β , respectively. One can show by a direct calculation that only these two are possible as follows. If $j = \alpha + i\beta$ ($\alpha, \beta \in \mathbb{R}$),

$$\Lambda = -(\alpha + i\beta)[(\alpha + 1) + i\beta]$$

B.1. Representation of $SL(2; \mathbb{R})$

$\Lambda \leq 0$	$(j \geq 0)$
$0 < \Lambda \leq \frac{1}{4}$	$(-\frac{1}{2} \leq j < 0)$
$0 \leq \Lambda < \frac{1}{4}$	$(-1 \leq j < -\frac{1}{2})$
$\Lambda < 0$	$(j < -1)$

Figure B.2: The values of Λ and the range of real j . Λ is negative or zero for the cases $j \geq 0$ or $j < -1$. On the other hand, Λ is positive but is less than or equal to $\frac{1}{4}$ for the case $-1 \leq j < 0$.

$$= [-\alpha(\alpha + 1) + \beta^2] - i\beta(2\alpha + 1) \in \mathbb{R}. \quad (\text{B.22})$$

In the first case, $\beta = 0$ and $j = \alpha \in \mathbb{R}$, the states are labeled by a set of real numbers (j, a) . Λ is also found to be less than or equal to $\frac{1}{4}$ from the relation $\Lambda = \frac{1}{4} - (j + \frac{1}{2})^2 \leq \frac{1}{4}$. There are four regions of j according to the value of Λ , as described in Fig. B.2. However, $\Lambda = -j(j + 1)$ is invariant under the transformation $j \rightarrow -j - 1$, so we can restrict j to be negative, $j < 0$. Then, in summary, the states are labeled by a set of real numbers (j, a) with $j < 0$.

In the second case, $2\alpha + 1 = 0$ and $j = -\frac{1}{2} + i\beta$ ($0 \neq \beta \in \mathbb{R}$), the relation $\Lambda = \frac{1}{4} + \beta^2 > \frac{1}{4}$ holds, and Λ is greater than $\frac{1}{4}$. These two cases are complementary to each other in view of the values of Λ .

Next, we turn to the states $|\Lambda, a\rangle$ in the first case and investigate the properties of the states. The semi-positivity condition (B.19) for the operator $J_{\pm}^{\dagger} J_{\pm}$ will give allowed regions in the (j, a) -plane. Recalling the equations

$$J_+ J_- |\Lambda, a\rangle = (a - j - 1)(a + j) |\Lambda, a\rangle, \quad (\text{B.23})$$

$$J_- J_+ |\Lambda, a\rangle = (a + j + 1)(a - j) |\Lambda, a\rangle, \quad (\text{B.24})$$

we can write down constraints for (j, a) ,

$$\begin{cases} (a - j - 1)(a + j) \geq 0 \\ (a + j + 1)(a - j) \geq 0. \end{cases} \quad (\text{B.25})$$

The allowed region is illustrated in Fig. B.3 by shaded regions.

There are three possible series for $j < 0$. First, if there exists a state $|\Lambda, a_0\rangle$ such that $a_0 = -j$, this state vanishes by acting J_- . On the other hand, the states obtained by the action of J_+ 's from $|\Lambda, a_0\rangle$ are in the allowed region. So, by applying J_+ to $|\Lambda, a_0\rangle$ repeatedly, we obtain a discrete series $D^+(j)$ of the states, whose eigenvalues are bounded from below and written as $a = a_0, a_0 + 1, a_0 + 2, \dots$

Second, we have another discrete series $D^-(j)$ of the states by considering $a_0 = j$. For the states in this series, the values of a 's are bounded from above and $a = a_0, a_0 - 1, a_0 - 2, \dots$

Third, we have continuous supplementary series $D_s(\Lambda, a_0)$. The states in this representation have eigenvalues a 's that take their values in discrete numbers $a = a_0, a_0 \pm 1, a_0 \pm 2, \dots$. They are neither bounded from above nor from below. These three representations are shown in Fig. B.3.

If we consider a complex $j = -\frac{1}{2} + i\beta$ ($\beta \in \mathbb{R}$), we have another continuous representation $D_p(\Lambda, a_0)$, which is called the continuous principal series. The states are characterized by the eigenvalues a 's, and their values take $a = a_0, a_0 \pm 1, a_0 \pm 2, \dots$

We summarize these four representations below. There are four types of representations. Three of them are associated with $j < 0$ and the allowed regions in Fig. B.3. The remainder

B.1. Representation of $\mathrm{SL}(2; \mathbb{R})$

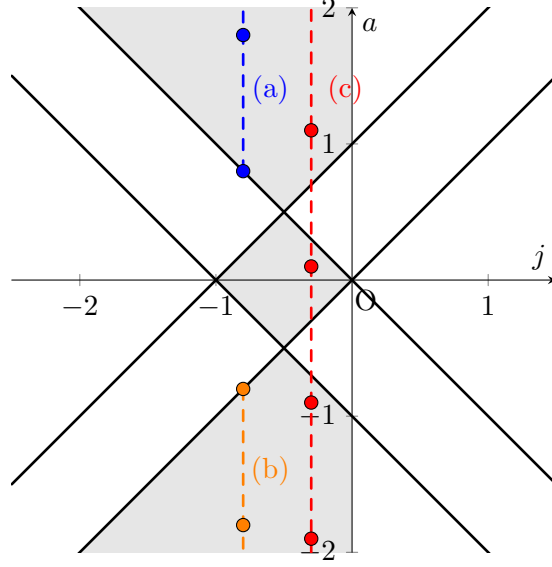


Figure B.3: Representations of $\mathfrak{su}(1, 1)$. The allowed regions for a combination (j, a) are illustrated by shaded regions. The blue blobs belong to (a) discrete series $D^+(j)$ bounded from below, the orange ones belong to (b) discrete series $D^-(j)$ bounded from above, and the red ones belong to (c) continuous supplementary series $D_s(\Lambda, a_0)$, which is unbounded.

corresponds to the case of a complex $j = -\frac{1}{2} + i\beta$ ($\beta \in \mathbb{R}$). The states are represented by $|\Lambda, a\rangle$ with $\Lambda = -j(j+1)$:

(a) Discrete series $D^+(j)$; bounded from below; $J_-|\Lambda, a_0\rangle = 0$,

$$a_0 := -j \quad (j < 0), \quad (\text{B.26})$$

$$a = a_0, a_0 + 1, a_0 + 2, \dots \quad (\text{B.27})$$

(b) Discrete series $D^-(j)$; bounded from above; $J_+|\Lambda, a_0\rangle = 0$,

$$a_0 := j \quad (j < 0), \quad (\text{B.28})$$

$$a = a_0, a_0 - 1, a_0 - 2, \dots \quad (\text{B.29})$$

(c) Continuous supplementary series $D_s(\Lambda, a_0)$; unbounded;

$$\left| j + \frac{1}{2} \right| < \frac{1}{2} - |a_0|, \quad (\text{B.30})$$

$$a = a_0, a_0 \pm 1, a_0 \pm 2, \dots \quad (\text{B.31})$$

(d) Continuous principal series $D_p(\Lambda, a_0)$; unbounded;

$$j = -\frac{1}{2} + i\beta \quad (\beta > 0) \quad (\text{B.32})$$

$$a = a_0, a_0 \pm 1, a_0 \pm 2, \dots \quad (\text{B.33})$$

B.2 Solvable models and symmetry

In this appendix, we summarize the solvable models used in Sec. 5.3 and discuss their spectra from the viewpoint of the symmetry studied in Sec. B.1.

B.2.1 Inverted harmonic oscillator

SL(2; $\mathbb{R}$) symmetry and spectra

Using the representation theory described in Sec. B.1, we can obtain the spectra of a harmonic oscillator or an inverted harmonic oscillator. This is because the Hamiltonians of these systems in fact satisfy the algebra.

Let us consider a set of differential operators J_a 's ($a = 1, 2, 3$) as an example of a realization of this symmetry algebra $\mathfrak{su}(1, 1)$:

$$J_1 = \frac{1}{2} \left(\frac{d^2}{dx^2} + \frac{\tilde{a}}{x^2} + \frac{x^2}{4} \right), \quad (\text{B.34})$$

$$J_2 = \frac{-i}{2} \left(x \frac{d}{dx} + \frac{1}{2} \right), \quad (\text{B.35})$$

$$J_3 = \frac{1}{2} \left(\frac{d^2}{dx^2} + \frac{\tilde{a}}{x^2} - \frac{x^2}{4} \right), \quad (\text{B.36})$$

$$J_a^\dagger = J_a \quad (a = 1, 2, 3). \quad (\text{B.37})$$

The Casimir invariant is constant in this model, which is specified by the parameter $\tilde{a}$.

$$J^2 \equiv J_1^2 + J_2^2 - J_3^2 = \frac{\tilde{a}}{4} + \frac{3}{16} = -j(j+1). \quad (\text{B.38})$$

Then j is expressed using $\tilde{a}$ by $j = -\frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{1}{4} - \tilde{a}}$.

Now we shall study the hidden conformal symmetry proposed in [96] by using these differential operators with $\tilde{a} = 0$. We follow the notation of [96] and redefine the currents as

$$\hat{J}_3 := -iJ_1 = -\frac{i}{2} \left(\frac{d^2}{dx^2} + \frac{x^2}{4} \right) =: \frac{i}{2}H, \quad (\text{B.39})$$

$$\hat{J}_1 = J_2 := -\frac{i}{2} \left(x \frac{d}{dx} + \frac{1}{2} \right), \quad (\text{B.40})$$

$$\hat{J}_2 := iJ_3 = \frac{i}{2} \left(\frac{d^2}{dx^2} - \frac{x^2}{4} \right) =: -\frac{i}{2}H_{\text{ho}}, \quad (\text{B.41})$$

$$\hat{J}_\pm := \pm i\hat{J}_1 - \hat{J}_2 = \pm iJ_2 - iJ_3, \quad (\text{B.42})$$

$$[\hat{J}_+, \hat{J}_-] = -2\hat{J}_3, \quad [\hat{J}_3, \hat{J}_\pm] = \pm \hat{J}_\pm. \quad (\text{B.43})$$

This system is the inverted harmonic oscillator in one dimension with the Hamiltonian H . The spectrum of this model is represented by the eigenvalues of $H = -2i\hat{J}_3$. In addition, the Casimir invariant is defined as

$$\hat{J}^2 = \hat{J}_3^2 - \hat{J}_1^2 - \hat{J}_2^2 = -J^2, \quad (\text{B.44})$$

$$\hat{J}^2 = \hat{J}_3^2 - \frac{1}{2}(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+) = \hat{J}_3^2 \pm \hat{J}_3 - \hat{J}_\mp\hat{J}_\pm, \quad (\text{B.45})$$

B.2. Solvable models and symmetry

$$J^2 \cong \Lambda = -j(j+1), \quad (\text{B.46})$$

$$\hat{J}^2 \cong \hat{\Lambda} = -\Lambda = j(j+1) = \hat{j}(\hat{j}-1), \quad (\text{B.47})$$

$$\hat{j} := j+1. \quad (\text{B.48})$$

The states are labeled by a set of integers $(\hat{j}, m)$ with

$$\hat{J}^2 \cong \hat{\Lambda} = \hat{j}(\hat{j}-1), \quad (\text{B.49})$$

$$\hat{J}_3 \cong m. \quad (\text{B.50})$$

By using the explicit form of the differential operators, one can obtain a relation

$$\hat{J}^2 = -\frac{3}{16}. \quad (\text{B.51})$$

It determines the values of $\hat{j}$ as $\hat{j} = \frac{1}{4}, \frac{3}{4}$. We introduce a useful notation to label these two eigenvalues as

$$\hat{j}_\alpha := \frac{2\alpha-1}{4} \quad (\alpha = 1, 2), \quad (\text{B.52})$$

$$\hat{J}^2 \cong \hat{j}_\alpha(\hat{j}_\alpha-1), \quad (\text{B.53})$$

$$\hat{J}_3 \cong m. \quad (\text{B.54})$$

We can change the eigenvalue m by a unit, by acting the raising $\hat{J}_+$ and the lowering $\hat{J}_-$ operators on a state, to find

$$\begin{aligned} \hat{J}_\mp \hat{J}_\pm |\hat{j}, m\rangle &= (m \pm \hat{j}) \left[m \pm (1 - \hat{j}) \right] |\hat{j}, m\rangle \\ &= \left(m \pm \frac{1}{4} \right) \left(m \pm \frac{3}{4} \right) |\hat{j}, m\rangle. \end{aligned} \quad (\text{B.55})$$

Let us take the following two representations of this algebra and evaluate the energy spectrum $H = -2i\hat{J}_3$.

1. Spectrum bounded from below

$$\hat{J}_- |\hat{j}_\alpha, m_0\rangle = 0; \quad m_0 = +\hat{j}_\alpha, \quad (\text{B.56})$$

$$\hat{J}_+^k |\hat{j}_\alpha, +\hat{j}_\alpha\rangle \implies H \cong -2i \left(k + \hat{j}_\alpha \right) = -i \left(n + \frac{1}{2} \right), \quad (\text{B.57})$$

$$n = 2k + \alpha - 1. \quad (\text{B.58})$$

2. Spectrum bounded from above

$$\hat{J}_+ |\hat{j}_\alpha, m_0\rangle = 0; \quad m_0 = -\hat{j}_\alpha, \quad (\text{B.59})$$

$$\hat{J}_-^k |\hat{j}_\alpha, -\hat{j}_\alpha\rangle \implies H \cong -2i \left(-k - \hat{j}_\alpha \right) = +i \left(n + \frac{1}{2} \right), \quad (\text{B.60})$$

$$n = 2k + \alpha - 1. \quad (\text{B.61})$$

B.2. Solvable models and symmetry

Then, one can construct eigenstates $|n, \pm\rangle$ ($n = 0, 1, \dots$) and associated eigenfunctions $\tilde{\Phi}_n^{(\pm)}(x)$ as

$$|n, \pm\rangle := \hat{J}_\pm^k |\hat{j}_\alpha, \pm \hat{j}_\alpha\rangle, \quad (\text{B.62})$$

$$n = 2k + \alpha - 1, \quad \hat{j}_\alpha = \frac{2\alpha - 1}{4} \quad (\alpha = 1, 2), \quad (\text{B.63})$$

and

$$\tilde{\Phi}_n^{(\pm)}(x) \equiv \tilde{\Phi}_{k,\alpha}^{(\pm)}(x) = \langle x | n, \pm \rangle, \quad (\text{B.64})$$

$$\tilde{\Phi}_{0,\alpha}^{(\pm)}(x) = x^{\alpha-1} e^{\pm i x^2/4} \quad (\alpha = 1, 2), \quad (\text{B.65})$$

$$\tilde{\Phi}_{k,\alpha}^{(\pm)}(x) = e^{\pm i x^2/4} H_n \left(e^{\mp i \pi/4} \frac{x}{\sqrt{2}} \right), \quad (\text{B.66})$$

where

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \quad (\text{B.67})$$

is the Hermitian polynomial. This system describes the inverted harmonic oscillator, and its spectra match the eigenvalues of its Schrödinger equation. We will see the details of the wave functions in Sec. B.2.1.

The spectra are also related to those of the usual harmonic oscillator by a change of variables: by considering a rotation in the 2, 3-plane, we can transform the equation of the harmonic oscillator $H_{\text{ho}} \tilde{\Phi}_{\text{ho}} = E_{\text{ho}} \tilde{\Phi}_{\text{ho}}$ into that of the inverted potential case $H \tilde{\Phi} = E \tilde{\Phi}$, with $D_\pm := \exp\left(\pm \frac{\pi}{2} \hat{J}_1\right)$, as

$$D_\pm H D_\pm^{-1} = \pm i H_{\text{ho}}, \quad (\text{B.68})$$

$$\tilde{\Phi}(x) = D_\pm^{-1} \tilde{\Phi}_{\text{ho}}(x) = e^{\pm i \pi/8} \tilde{\Phi}_{\text{ho}}(e^{\pm i \pi/4} x), \quad (\text{B.69})$$

$$E = \pm i E_{\text{ho}}. \quad (\text{B.70})$$

Wave functions and spectra

Here, let us turn to the Schrödinger equation of the inverted harmonic oscillator in one dimension, and look at the solutions to identify what kind of boundary conditions are implicitly imposed in the spectrum obtained by the symmetry argument in Sec. B.2.1.

The Schrödinger equation determines the energy eigenvalue E and the associated wave function $\psi(x)$

$$\left(\frac{d^2}{dx^2} + E + \frac{1}{4} x^2 \right) \psi = 0. \quad (\text{B.71})$$

By changing variables $z := e^{i \pi/4} x$ and $\nu := -\frac{1}{2} - iE$, one can rewrite this equation into a differential equation

$$\left(\frac{d^2}{dz^2} + \nu + \frac{1}{2} - \frac{1}{4} z^2 \right) \psi = 0. \quad (\text{B.72})$$

This has two independent solutions expressed by the parabolic cylinder functions $D_\nu(z)$ and $D_{-\nu-1}(iz)$, and $\psi(x)$ is described as a linear combination of them

$$\psi(z) = A D_\nu(z) + B D_{-\nu-1}(iz). \quad (\text{B.73})$$

B.2. Solvable models and symmetry

Here we summarize several useful formulas for the parabolic cylinder function. The function $D_\nu(z)$ is represented by the confluent hypergeometric function F as

$$D_\nu(z) = 2^{\nu/2} \sqrt{\pi} e^{-z^2/4} \left[\frac{1}{\Gamma(\frac{1-\nu}{2})} F\left(-\frac{\nu}{2}; \frac{z^2}{2}\right) - \frac{\sqrt{2}}{\Gamma(-\frac{\nu}{2})} z F\left(\frac{1-\nu}{2}; \frac{z^2}{2}\right) \right]. \quad (\text{B.74})$$

The differential equation (B.72) has four types of solutions $D_\nu(z)$, $D_{-\nu}(z)$, $D_{-\nu-1}(iz)$, and $D_{-\nu-1}(-iz)$. They are related to each another by the following identities

$$D_\nu(z) = \frac{\Gamma(\nu+1)}{\sqrt{2\pi}} \left[e^{+i\pi\nu/2} D_{-\nu-1}(iz) + e^{-i\pi\nu/2} D_{-\nu-1}(-iz) \right], \quad (\text{B.75})$$

$$D_\nu(z) = e^{\pm i\pi\nu} D_\nu(-z) + \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{\pm i\pi(\nu+1)/2} D_{-\nu-1}(\mp iz). \quad (\text{B.76})$$

The function $D_\nu(z)$ has an asymptotic form in the region $|z| \gg 1$ ($\nu \notin \mathbb{Z}_{\geq 0}$) depending on the argument of z

$$D_\nu(z) \simeq e^{-z^2/4} z^\nu - \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{+i\pi\nu} e^{+z^2/4} z^{-\nu-1} \quad \left(\text{for } \frac{\pi}{4} < \arg z < \frac{5\pi}{4} \right), \quad (\text{B.77})$$

$$D_\nu(z) \simeq e^{-z^2/4} z^\nu - \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{-i\pi\nu} e^{+z^2/4} z^{-\nu-1} \quad \left(\text{for } -\frac{5\pi}{4} < \arg z < -\frac{\pi}{4} \right), \quad (\text{B.78})$$

$$D_\nu(z) \simeq e^{-z^2/4} z^\nu \quad \left(\text{for } -\frac{3\pi}{4} < \arg z < \frac{3\pi}{4} \right). \quad (\text{B.79})$$

For $\nu = n \in \mathbb{Z}_{\geq 0}$, $D_\nu(z)$ is represented by the Hermitian polynomial $H_n(z)$ as

$$D_n(z) = (-1)^n e^{z^2/4} \frac{d^n}{dz^n} e^{-z^2/2} = 2^{-n/2} e^{-z^2/4} H_n\left(\frac{z}{\sqrt{2}}\right). \quad (\text{B.80})$$

When there is the inverted harmonic potential, the waves are scattered by this repulsive potential. In the region $x > 0$ ($z \equiv e^{i\pi/4}x$), the wave function $\psi(x)$ behaves at $x \rightarrow +\infty$ as

$$\psi(z) \simeq A e^{-z^2/4} z^\nu + B e^{+z^2/4} (iz)^{-\nu-1}. \quad (\text{B.81})$$

If the system time evolves according to the factor $e^{-i\Omega t}$ in the wave function, one can see that the first term $e^{-z^2/4} = e^{-ix^2/4}$ is related to a left-moving wave, or an ingoing mode (as seen from the potential hill). On the other hand, the second term $e^{+z^2/4} = e^{+ix^2/4}$ represents a right-moving wave, or an outgoing mode.

In order to analyze the scattering process, we want to know the behavior in the region $x \rightarrow -\infty$. The two solutions can be transformed by analytic continuations with $\tilde{x} := -x$ ($\tilde{x} > 0$)

$$\tilde{z} := e^{i\pi/4} \tilde{x}, \quad (\text{B.82})$$

$$\psi(z) = A D_\nu(-\tilde{z}) + B D_{-\nu-1}(-i\tilde{z}) \quad (\text{B.83})$$

$$= \psi(-\tilde{z}) = A' D_\nu(\tilde{z}) + B' D_{-\nu-1}(i\tilde{z}). \quad (\text{B.84})$$

Now we have two sets of coefficients (A, B) and (A', B') , which encode the information of the scattering. There are relations between those:

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}, \quad (\text{B.85})$$

B.2. Solvable models and symmetry

where

$$T_{11} = e^{i\pi\nu}, \quad T_{21} = \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{i\pi(\nu+1)/2}, \quad T_{12} = \frac{\sqrt{2\pi}}{\Gamma(\nu+1)} e^{i\pi\nu/2}, \quad T_{22} = e^{i\pi(\nu+1)}. \quad (\text{B.86})$$

In the region $x \rightarrow -\infty$, the first term $e^{-z^2/4} = e^{-ix^2/4}$ with A' is a right-moving wave (ingoing mode), while the second term $e^{+z^2/4} = e^{+ix^2/4}$ with B' represents a left-moving wave (outgoing mode).

The boundary conditions for the wave function determine the coefficients (A, B) and (A', B') . To illustrate how the condition on the coefficients can determine the spectrum, let us consider quasinormal modes, which satisfies

$$A = A' = 0. \quad (\text{B.87})$$

This condition means that there are purely outgoing modes on both sides of the potential hill. We have an equation $T_{12} = \frac{\sqrt{2\pi}}{\Gamma(\nu+1)} e^{i\pi\nu/2} = 0$ from $A' = T_{11}A + T_{12}B = T_{12}B = 0$. The poles of $\Gamma(\nu+1)$ contribute to this equation and we have a series of solutions

$$\nu = -n - 1 \quad (n = 0, 1, \dots). \quad (\text{B.88})$$

In this discrete series, the associated wave function can be described by the Hermitian polynomial as

$$\psi(z) = B D_{-\nu-1}(iz) = B D_n(iz), \quad (\text{B.89})$$

$$D_n(z) = (-1)^n e^{z^2/4} \frac{d^n}{dz^n} e^{-z^2/2} = 2^{-n/2} e^{-z^2/4} H_n\left(\frac{z}{\sqrt{2}}\right). \quad (\text{B.90})$$

Now we shall relate this spectral formula (B.88) to the quasinormal mode spectrum of the black hole. When one considers a wave equation of the scalar field Φ , the radial part $\psi(r)$ is described by the Schrödinger equation

$$\left(\frac{d^2}{dr_*^2} + \Omega^2 - V(r) \right) \psi = 0, \quad (\text{B.91})$$

$$V = V(r_{\text{PS}}) + \frac{1}{2} (f(r_{\text{PS}})^2 V''(r_{\text{PS}})) \delta r_*^2, \quad (\text{B.92})$$

$$\delta r_* := r_* - (r_*)_{\text{PS}}. \quad (\text{B.93})$$

When we associate x and E with δr_* and Ω as

$$x = [-2f(r_{\text{PS}})^2 V''(r_{\text{PS}})]^{1/4} \delta r_*, \quad (\text{B.94})$$

$$E = \frac{\Omega^2 - V(r_{\text{PS}})}{[-2f(r_{\text{PS}})^2 V''(r_{\text{PS}})]^{1/2}}, \quad (\text{B.95})$$

this system is reduced to the inverted harmonic oscillator

$$\left(\frac{d^2}{dx^2} + E + \frac{1}{4}x^2 \right) \psi = 0. \quad (\text{B.96})$$

Recalling the above analysis, we can write down the values of E and Ω^2 , which are characterized by a integer n ($n = 0, 1, \dots$)

$$E = i\left(\nu + \frac{1}{2}\right) = -i\left(n + \frac{1}{2}\right) \quad (n = 0, 1, \dots), \quad (\text{B.97})$$

B.2. Solvable models and symmetry

$$\Omega^2 = V(r_{\text{PS}}) + [-2f(r_{\text{PS}})^2 V''(r_{\text{PS}})]^{1/2} E \quad (\text{B.98})$$

$$= V(r_{\text{PS}}) + 2\sqrt{V(r_{\text{PS}})}\gamma_L \left[-i \left(n + \frac{1}{2} \right) \right], \quad (\text{B.99})$$

where

$$\gamma_L := f(r_{\text{PS}}) \sqrt{\frac{-V''(r_{\text{PS}})}{2V(r_{\text{PS}})}}. \quad (\text{B.100})$$

If a condition $\frac{2\gamma_L}{\sqrt{V(r_{\text{PS}})}}(n + \frac{1}{2}) \ll 1$ is satisfied, we have a formula for the quasinormal mode spectrum Ω

$$\Omega = \sqrt{V(r_{\text{PS}})} - i\gamma_L \left(n + \frac{1}{2} \right). \quad (\text{B.101})$$

In the case of the $d = 4$ asymptotically flat Schwarzschild black hole, one can evaluate $\sqrt{V(r_{\text{PS}})} \simeq l \frac{\sqrt{f(r_{\text{PS}})}}{r_{\text{PS}}}$ and $\gamma_L \simeq \frac{\sqrt{f(r_{\text{PS}})}}{r_{\text{PS}}}$ in the large l limit. Then, we obtain a restriction $(2n + 1)l^{-1} \ll 1$. This is the spectrum obtained in [97]. Therefore, the spectrum generated by the symmetry is equivalent to the spectrum subject to the boundary condition that requires the purely outgoing modes on both sides of the potential hill.

In our case of the Schwarzschild AdS black hole studied in Chap. 5, the existence of the AdS boundary as a hard wall actually breaks the $\text{SL}(2; \mathbb{R})$ symmetry. This is because there is a scaling operator J_2 (B.35) as one of the symmetry generators, and a concrete choice of the position of the hard wall $x = L$ ruins this generator. That is why our quasinormal mode spectrum in the Schwarzschild AdS black hole does not follow the $\text{SL}(2; \mathbb{R})$ symmetry.

B.2.2 Pöschl-Teller model

Here we summarize properties of the Pöschl-Teller model and the related algebra studied in [152].

Wave functions and spectra

The model is the integrable system in one dimension with a potential

$$V(x) = \frac{V_0}{\cosh^2 \alpha x}, \quad (\text{B.102})$$

where

$$V_0 = -\alpha^2 \lambda(\lambda - 1). \quad (\text{B.103})$$

The parameter $\alpha > 0$ characterizes the width of the potential in the x -direction, and λ specifies the height of the peak V_0 of the potential. In this appendix, we consider the repulsive case $V_0 > 0$ and study the scattering process by the potential hill generated by $V(x)$. The energy ω^2 and the eigenfunction $\psi(x)$ obey the Schrödinger equation

$$\left(\frac{d^2}{dx^2} + \omega^2 - \frac{V_0}{\cosh^2 \alpha x} \right) \psi(x) = 0. \quad (\text{B.104})$$

B.2. Solvable models and symmetry

The wave function is expressed as a linear combination of two independent solutions $f_{\pm}(x)$ as

$$\psi(x) = A f_+(x) + B f_-(x), \quad (\text{B.105})$$

$$f_+(x) := (e^{\alpha x})^{i\omega/\alpha} F\left(\lambda, \frac{1-\lambda}{1+\frac{i\omega}{\alpha}}; \frac{e^{\alpha x}}{e^{\alpha x} + e^{-\alpha x}}\right), \quad (\text{B.106})$$

$$f_-(x) := (e^{\alpha x} + e^{-\alpha x})^{i\omega/\alpha} F\left(\lambda - \frac{i\omega}{\alpha}, \frac{1-\lambda - \frac{i\omega}{\alpha}}{1 - \frac{i\omega}{\alpha}}; \frac{e^{\alpha x}}{e^{\alpha x} + e^{-\alpha x}}\right). \quad (\text{B.107})$$

In the large negative region $x \rightarrow -\infty$, we can evaluate the asymptotic behavior of $\psi(x)$ as

$$f_+(x) \simeq e^{i\omega x}, \quad f_-(x) \simeq e^{-i\omega x}, \quad (\text{B.108})$$

$$\psi(x) \simeq A e^{i\omega x} + B e^{-i\omega x}. \quad (\text{B.109})$$

In order to analyze the scattering, we want to know the asymptotic behavior of $\psi(x)$ in the large positive region $x \rightarrow +\infty$. The two solutions $f_{\pm}(x)$ can be rewritten by analytic continuations

$$\begin{aligned} f_+(x) &= T_{11}(\omega) e^{+i\omega x} F\left(\lambda, \frac{1-\lambda}{1-\frac{i\omega}{\alpha}}; \frac{e^{-\alpha x}}{e^{\alpha x} + e^{-\alpha x}}\right) \\ &\quad + T_{21}(\omega) (e^{\alpha x} + e^{-\alpha x})^{-i\omega/\alpha} F\left(\lambda + \frac{i\omega}{\alpha}, \frac{1-\lambda + \frac{i\omega}{\alpha}}{1+\frac{i\omega}{\alpha}}; \frac{e^{-\alpha x}}{e^{\alpha x} + e^{-\alpha x}}\right), \end{aligned} \quad (\text{B.110})$$

$$\begin{aligned} f_-(x) &= T_{22}(\omega) e^{-i\omega x} F\left(\lambda, \frac{1-\lambda}{1+\frac{i\omega}{\alpha}}; \frac{e^{-\alpha x}}{e^{\alpha x} + e^{-\alpha x}}\right) \\ &\quad + T_{12}(\omega) (e^{\alpha x} + e^{-\alpha x})^{+i\omega/\alpha} F\left(\lambda - \frac{i\omega}{\alpha}, \frac{1-\lambda - \frac{i\omega}{\alpha}}{1 - \frac{i\omega}{\alpha}}; \frac{e^{-\alpha x}}{e^{\alpha x} + e^{-\alpha x}}\right), \end{aligned} \quad (\text{B.111})$$

where the coefficients T_{11} , T_{12} , T_{21} , and T_{22} are defined as

$$T_{11} = \frac{\Gamma(1 + \frac{i\omega}{\alpha}) \Gamma(\frac{i\omega}{\alpha})}{\Gamma(1 - \lambda + \frac{i\omega}{\alpha}) \Gamma(\lambda + \frac{i\omega}{\alpha})}, \quad (\text{B.112})$$

$$T_{21} = \frac{\Gamma(1 + \frac{i\omega}{\alpha}) \Gamma(-\frac{i\omega}{\alpha})}{\Gamma(1 - \lambda) \Gamma(\lambda)}, \quad (\text{B.113})$$

$$T_{12} = \frac{\Gamma(1 - \frac{i\omega}{\alpha}) \Gamma(\frac{i\omega}{\alpha})}{\Gamma(1 - \lambda) \Gamma(\lambda)}, \quad (\text{B.114})$$

$$T_{22} = \frac{\Gamma(1 - \frac{i\omega}{\alpha}) \Gamma(-\frac{i\omega}{\alpha})}{\Gamma(1 - \lambda - \frac{i\omega}{\alpha}) \Gamma(\lambda - \frac{i\omega}{\alpha})}. \quad (\text{B.115})$$

In the region $x \rightarrow +\infty$, the wave function $\psi(x)$ behaves as

$$f_+(x) \simeq T_{11}(\omega) e^{i\omega x} + T_{21}(\omega) e^{-i\omega x}, \quad (\text{B.116})$$

$$f_-(x) \simeq T_{12}(\omega) e^{i\omega x} + T_{22}(\omega) e^{-i\omega x}, \quad (\text{B.117})$$

$$\psi(x) \simeq A' e^{i\omega x} + B' e^{-i\omega x}, \quad (\text{B.118})$$

where A' and B' are related to A and B by the equation

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}. \quad (\text{B.119})$$

B.2. Solvable models and symmetry

When we take the point $x = 0$ as a reference point of the scattering, the waves with coefficients A and B' are ingoing modes, while B and A' are outgoing modes. As boundary conditions, we put $A = B' = 0$ and obtain a relation

$$T_{22}(\omega) = \frac{\Gamma(1 - \frac{i\omega}{\alpha})\Gamma(-\frac{i\omega}{\alpha})}{\Gamma(1 - \lambda - \frac{i\omega}{\alpha})\Gamma(\lambda - \frac{i\omega}{\alpha})} = 0. \quad (\text{B.120})$$

The gamma functions in the denominator have poles in the complex ω -plane, and they determine the quasinormal modes $\omega_n^{(\pm)}$ as

$$\omega_n^{(\pm)} = -i\alpha \left(n + \lambda^{(\pm)} \right) \quad (n = 0, 1, \dots), \quad (\text{B.121})$$

$$\lambda^{(+)} := \lambda, \quad \lambda^{(-)} := 1 - \lambda. \quad (\text{B.122})$$

In this case, the wave function $\psi(x) = B \psi_n^{(\pm)}(x)$ is written by the hypergeometric function as

$$\psi_n^{(\pm)}(x) = (e^{\alpha x} + e^{-\alpha x})^{i\omega_n^{(\pm)}/\alpha} F \left(\begin{matrix} \lambda - \frac{i\omega_n^{(\pm)}}{\alpha}, & 1 - \lambda - \frac{i\omega_n^{(\pm)}}{\alpha} \\ 1 - \frac{i\omega_n^{(\pm)}}{\alpha} \end{matrix} ; \frac{e^{\alpha x}}{e^{\alpha x} + e^{-\alpha x}} \right) \quad (\text{B.123})$$

$$= (e^{\alpha x} + e^{-\alpha x})^{n+\lambda^{(\pm)}} F \left(\begin{matrix} -n, & 1 - 2\lambda^{(\pm)} - n \\ 1 - \lambda^{(\pm)} - n \end{matrix} ; \frac{e^{\alpha x}}{e^{\alpha x} + e^{-\alpha x}} \right). \quad (\text{B.124})$$

Now we set $\lambda = \frac{1}{2} + i\nu$ ($\nu \in \mathbb{R}$) to relate this model to the system of quasinormal modes in the black hole. It corresponds to the high barrier case and is related to the continuous representation of the Lie algebra $\mathfrak{sl}(2; \mathbb{R})$. In that case, the height of $V(x)$ is $V_0 = \alpha^2(\frac{1}{4} + \nu^2)$, and the frequencies $\omega_n^{(\pm)}$ are

$$\omega_n^{(\pm)} = -i\alpha \left(n + \frac{1}{2} \right) \pm \alpha\nu. \quad (\text{B.125})$$

The wave propagates along the x -direction according to $e^{-i\omega t}$. Its amplitude decreases exponentially as $e^{-i\omega_n^{(\pm)}t} = e^{-\alpha(n+\frac{1}{2})t} e^{\mp i\alpha\nu t}$.

The associated wave functions are expressed as

$$\psi_n^{(\pm)}(x) = (2 \cosh \alpha x)^{n+\frac{1}{2} \pm i\nu} F \left(\begin{matrix} -n, & -n \mp 2i\nu \\ -n + \frac{1}{2} \mp i\nu \end{matrix} ; \frac{1+y}{2} \right). \quad (\text{B.126})$$

Here $y := \tanh \alpha x$ and

$$F \left(\begin{matrix} -n, & -n \mp 2i\nu \\ -n + \frac{1}{2} \mp i\nu \end{matrix} ; \frac{1+y}{2} \right) = \left[\frac{(-1)^n}{n!} \frac{\Gamma(\mp i\nu + \frac{1}{2})}{\Gamma(\mp i\nu + \frac{1}{2} - n)} \right]^{-1} P_n^{(\mp i\nu - \frac{1}{2} - n, \mp i\nu - \frac{1}{2} - n)}(y), \quad (\text{B.127})$$

$$(\text{B.128})$$

where

$$P_n^{(\alpha, \beta)}(z) = \frac{(-1)^n}{2^n n!} (1-z)^{-\alpha} (1+z)^{-\beta} \frac{d^n}{dz^n} (1-z)^\alpha (1+z)^\beta (1-z^2)^n \quad (\text{B.129})$$

is the Jacobi polynomial. It can also be written as

$$P_n^{(\alpha, \beta)}(z) = \frac{(-1)^n}{n!} \frac{\Gamma(n+1+\beta)}{\Gamma(1+\beta)} F \left(\begin{matrix} -n, & n + \alpha + \beta + 1 \\ 1 + \beta \end{matrix} ; \frac{1+x}{2} \right)$$

B.2. Solvable models and symmetry

$$= \frac{1}{n!} \frac{\Gamma(n+1+\alpha)}{\Gamma(1+\alpha)} F\left(\begin{matrix} -n, & n+\alpha+\beta+1 \\ & 1+\alpha \end{matrix}; \frac{1-x}{2}\right). \quad (\text{B.130})$$

We also collect several properties of the hypergeometric functions here. The hypergeometric equation is defined as

$$z(1-z) \frac{d^2 f}{dz^2} + [c - (a+b+1)z] \frac{df}{dz} - abf = 0. \quad (\text{B.131})$$

There is a set of solutions that is analytic around a point $z = 0$

$$f = F\left(\begin{matrix} a, & b \\ & c \end{matrix}; z\right) \quad (\text{B.132})$$

$$f = z^{1-c} F\left(\begin{matrix} a-c+1, & b-c+1 \\ & 2-c \end{matrix}; z\right). \quad (\text{B.133})$$

They are suitable linearly independent solutions that converge under conditions $|z| < 1$, $\text{Re}(a+b-c) < 0$.

One can reexpress the hypergeometric function defined around $z = 0$ in terms of analytic functions around $z = 1$ through an analytic continuation,

$$\begin{aligned} F\left(\begin{matrix} a, & b \\ & c \end{matrix}; z\right) &= \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} F\left(\begin{matrix} c-a, & c-b \\ & c-a-b+1 \end{matrix}; 1-z\right) \\ &+ \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F\left(\begin{matrix} a, & b \\ & a+b-c+1 \end{matrix}; 1-z\right), \quad (\text{for } |\arg(1-z)| < \pi). \end{aligned} \quad (\text{B.134})$$

SL(2; $\mathbb{R}$) symmetry and spectra

The Pöschl-Teller model can be solved by the Jacobi polynomials in the high barrier case. We will discuss the symmetry of this model proposed in [152].

By introducing a variable $\xi := \alpha x$, we rewrite the Schrödinger equation as

$$\left(\frac{d^2}{d\xi^2} + \frac{\omega^2}{\alpha^2} + \frac{\lambda(\lambda-1)}{\cosh^2 \xi}\right) \psi = 0 \quad (\text{B.135})$$

and define a new parameter

$$K_n = \frac{i}{\alpha} \omega_n^{(\pm)} = \begin{cases} n+\lambda & (+) \\ n+1-\lambda & (-) \end{cases} \quad (n = 0, 1, \dots). \quad (\text{B.136})$$

From now on, we abbreviate the superscripts $(\pm)$ and introduce a set of operators $J^\pm$ and J^3 that act on the solutions $\{\psi_n\}$ ($n = 0, 1, \dots$) as

$$J_n^- = i(-\cosh \xi \partial_\xi + K_n \sinh \xi) : \psi_n \mapsto \psi_{n-1}, \quad (\text{B.137})$$

$$J_{n+1}^+ = i(\cosh \xi \partial_\xi + K_n \sinh \xi) : \psi_n \mapsto \psi_{n+1}, \quad (\text{B.138})$$

$$J_n^3 \psi_n = K_n \psi_n : \psi_n \mapsto \psi_n. \quad (\text{B.139})$$

When the operators act on the set of solutions $\{\psi_n\}$, one can show that the operators satisfy the commutation relations of $\mathfrak{su}(1, 1)$ algebra

$$[J^3, J^\pm] = \pm J^\pm \quad [J^+, J^-] = -2J^3. \quad (\text{B.140})$$

B.2. Solvable models and symmetry

These are obtained from the following calculation

$$(J_n^3)^2 \psi_n = K_n^2 \psi_n, \quad (\text{B.141})$$

$$J_n^+ J_n^- \psi_n = \cosh^2 \xi \partial_\xi^2 \psi_n - K_n (1 + K_n \sinh^2 \xi) \psi_n, \quad (\text{B.142})$$

$$J_{n+1}^- J_{n+1}^+ \psi_n = \cosh^2 \xi \partial_\xi^2 \psi_n - K_n (-1 + K_n \sinh^2 \xi) \psi_n, \quad (\text{B.143})$$

$$J_{n-1}^3 J_n^- \psi_n = K_{n-1} J_n^- \psi_n, \quad (\text{B.144})$$

$$J_n^- J_n^3 \psi_n = K_n J_n^- \psi_n, \quad (\text{B.145})$$

$$J_{n+1}^3 J_{n+1}^+ \psi_n = K_{n+1} J_{n+1}^+ \psi_n, \quad (\text{B.146})$$

$$J_{n+1}^+ J_n^3 \psi_n = K_n J_{n+1}^+ \psi_n. \quad (\text{B.147})$$

Next, we rewrite the Schrödinger equation (B.135) into the form

$$\cosh^2 \xi \partial_\xi^2 \psi_n = K_n^2 \cosh^2 \xi \psi_n - \lambda(\lambda - 1) \psi_n, \quad (\text{B.148})$$

and evaluate a bilinear of the generators

$$-J_n^+ J_n^- \psi_n - J_{n+1}^- J_{n+1}^+ \psi_n = -2 \cosh^2 \xi \partial_\xi^2 \psi_n + 2K_n^2 \sinh^2 \xi \psi_n \quad (\text{B.149})$$

$$= -2(K_n^2 \cosh^2 \xi \psi_n - \lambda(\lambda - 1) \psi_n) + 2K_n^2 \sinh^2 \xi \psi_n \quad (\text{B.150})$$

$$= -2K_n^2 \psi_n + 2\lambda(\lambda - 1) \psi_n, \quad (\text{B.151})$$

to obtain

$$\left[-J^+ J^- - J^- J^+ + 2(J^3)^2 \right] \psi_n = 2\lambda(\lambda - 1) \psi_n. \quad (\text{B.152})$$

Then, one can write down the Casimir invariant J^2 in this model by using the generators as

$$J^2 = -(J^3)^2 + \frac{1}{2}(J^+ J^- + J^- J^+) = -\lambda(\lambda - 1). \quad (\text{B.153})$$

The representation of $\mathfrak{su}(1, 1)$ is classified in terms of the eigenvalue of J^2 . It is related to the height V_0 of the peak of $V(x)$ and we can write down the V_0 in terms of the Casimir J^2 as

$$V_0 = -\alpha^2 \lambda(\lambda - 1) \cong \alpha^2 J^2. \quad (\text{B.154})$$

Note that, although the Casimir J^2 is the hermitian operator and its eigenvalue is real, the generator J^3 is not hermitian and its eigenvalue K_n can take complex value $n + \frac{1}{2} \pm i\nu$ ($\nu \in \mathbb{R}$), namely, the representation could be non-unitary.

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