

超幾何分布の分布関数の

Wise の近似式の延長

山 内 二 郎

まえがき

超幾何分布の分布関数を求めたいとき、標本の大きさ n が大きくなると、その計算はかきりわけらわしくなるので、その近似式をつくっておきたい。

筆者^{(1),(3)} (1953) はロットの大きさ N としたときの $N-1$ の級数展開によって、検取検査の設計のため^{9/}計算方法を求めておいたが、
M. E. Wise^{(2),(3)} (1954) は $\overbrace{M = N - (n-1)/2}^{(\text{標本の大きさ } n \text{ とし})}$ として M^{-2} の ~~級数~~ 級数展開によって近似式を求めることを試みた。Wise は n 項までを示しているが、 n 項を用いるとさらによくなるであろうと思って、求めておいた。

Wise は巧み¹⁰変数変換後の境界積分によって求めている。筆者は $\ln(N!/(N-n)!) の M^{-1} の 級数漸近 を求め、それをつかって計算を進$

めた。

1. $\ln(N!/(N-n)!) の漸近展開$

$M = N - (n-1)/2$ とし, ガンマ関数の対数の Stirling の級数展開を用いて, 一般的形式の漸近展開が得られた。 B_{2t-1} は Bernoulli 数である。

$$\begin{aligned}\ln(N!/(N-n)!) &= \ln \Gamma(N+1) - \ln \Gamma(N+1-n) \\ &= \ln \Gamma(M + (n+1)/2) - \ln \Gamma(M - (n-1)/2) \\ &= (M + n/2) \ln(M + (n+1)/2) - (M - n/2) \ln(M - (n-1)/2) \\ &\quad - (M + (n+1)/2) + (M - (n-1)/2)\end{aligned}$$

$$+ \sum_{t=1}^{\infty} (-1)^{t-1} \frac{B_{2t-1}}{2t(2t-1)} \left[\left(M + \frac{n+1}{2}\right)^{-2t+1} - \left(M - \frac{n-1}{2}\right)^{-2t+1} \right]$$

$$= n \ln M$$

$$- \sum_{t=1}^{\infty} \frac{1}{(2M)^{2t}} \left[n^{2t+1} \frac{1}{2t(2t+1)} - \sum_{s=1}^t n^{2t+1-2s} \binom{2t}{2s-1} \left\{ \frac{1}{2s} + \sum_{\tau=1}^s (-1)^{\tau} \frac{B_{2\tau-1}}{2\tau(2\tau-1)} 2^{2\tau} \binom{2s-1}{2\tau-2} \right\} \right]$$

$$- \sum_{t=1}^{\infty} \frac{1}{(2M)^{2t+1}} \sum_{s=1}^t n^{2t+1-2s} \binom{2t}{2s-1} \left\{ \frac{1}{2s+1} + \sum_{\tau=1}^s (-1)^{\tau} \frac{B_{2\tau-1}}{2\tau(2\tau-1)} 2^{2\tau} \binom{2s-1}{2\tau-2} \right\}$$

... (1.1)

$$- \text{よ} \quad y \operatorname{cosech} y = [2y / (e^{2y} - 1)] \cdot e^y \quad \text{よ} \quad z$$

$$1 + \sum_{s=1}^{\infty} (-1)^s B_{2s-1} \frac{y^{2s} 2(2^{2s-1} - 1)}{(2s)!}$$

$$= \left\{ 1 - \frac{2y}{2} + \sum_{t=1}^{\infty} (-1)^{t-1} B_{2t-1} \frac{(2y)^{2t}}{(2t)!} \right\} \cdot \left\{ 1 + \sum_{t=1}^{\infty} \frac{y^t}{t!} \right\}$$

$$= 1 + \frac{y}{1!} (1-1) + \sum_{s=2}^{\infty} \frac{y^s}{s!} \left[1 - \binom{s}{1} + \sum_{t=1}^{\lfloor s/2 \rfloor} (-1)^{t-1} B_{2t-1} 2^{2t} \binom{s}{2t} \right]$$

であるから,

$$\frac{y^{2s+1}}{(2s+1)!} \text{ の係数} = 0 = 1 - \binom{2s+1}{1} + \sum_{t=1}^s (-1)^{t-1} B_{2t-1} 2^{2t} \binom{2s+1}{2t}$$

$$\therefore \frac{1}{2s+1} + \sum_{t=1}^s (-1)^t \frac{B_{2t-1}}{2t(2t-1)} 2^{2t} \binom{2s-1}{2t-2} = 0$$

$$\frac{y^{2s}}{(2s)!} \text{ の係数} = (-1)^s B_{2s-1} 2 \binom{2s-1}{2s-1}$$

$$= 1 - \binom{2s}{1} + \sum_{t=1}^s (-1)^{t-1} B_{2t-1} 2^{2t} \binom{2s}{2t}$$

$$= -2s(2s-1) \left\{ \frac{1}{2s} + \sum_{t=1}^s (-1)^t \frac{B_{2t-1}}{2t(2t-1)} 2^{2t} \binom{2s-2}{2t-2} \right\}$$

$$\therefore - \left\{ \frac{1}{2s} + \sum_{t=1}^s (-1)^t \frac{B_{2t-1}}{2t(2t-1)} 2^{2t} \binom{2s-2}{2t-2} \right\}$$

$$= (-1)^s \frac{B_{2s-1}}{2s(2s-1)} 2 \binom{2s-1}{2s-1}$$

よって (1.1) にしたがって

$$\ln(N!/(N-n)!) = n \ln M$$

$$= \sum_{\tau=1}^{\infty} \frac{1}{(2M)^{2\tau}} \left\{ \frac{M^{2\tau+1}}{2\tau(2\tau+1)} + \sum_{s=1}^{\tau} M^{2\tau+1-2s} \binom{2\tau-1}{2s-2} (-1)^s \frac{B_{2s-1}}{2s(2s-1)} 2 \binom{2s-1}{2s-1} \right.$$

... (1.2)

よって (2M)^{-2τ} ののぞむ項まで書き下
せる。τの小さい所を書けば、次の通り。

$$\tau=1: n(n^2-1)/(2 \cdot 3) = (n^{(3)} + 3n^{(2)})/6$$

$$\begin{aligned}\tau=2: (3n^5 - 10n^3 + 7n)/(2^2 \cdot 3 \cdot 5) \\ = (3n^{(5)} + 30n^{(4)} + 65n^{(3)} + 15n^{(2)})/60 \\ = n(n^2-1)(3n^2-7)/60\end{aligned}$$

$$\begin{aligned}\tau=3: (3n^7 - 21n^5 + 49n^3 - 31n)/(2 \cdot 3^2 \cdot 7) \\ = (3n^{(7)} + 63n^{(6)} + 399n^{(5)} + 840n^{(4)} + 427n^{(3)} + 21n^{(2)})/126 \\ = n(n^2-1)(3n^4 - 18n^2 + 31)/126\end{aligned}$$

$$\begin{aligned}\tau=4: (5n^9 - 60n^7 + 294n^5 - 620n^3 + 381n)/(2^3 \cdot 3^2 \cdot 5) \\ = (5n^{(9)} + 180n^{(8)} + 2250n^{(7)} + 11970n^{(6)} + 26649n^{(5)} \\ + 20790n^{(4)} + 3795n^{(3)} + 45n^{(2)})/360 \\ = n(n^2-1)(5n^6 - 55n^4 + 239n^2 - 381)/360\end{aligned}$$

$$\begin{aligned}\tau=5: (3n^{11} - 55n^9 + 462n^7 - 2046n^5 + 4191n^3 - 2555n)/(2 \cdot 3 \cdot 5 \cdot 11) \\ = (3n^{(11)} + 165n^{(10)} + 3410n^{(9)} + 33660n^{(8)} + 167013n^{(7)} + 402633n^{(6)} \\ + 420519n^{(5)} + 151140n^{(4)} + 11231n^{(3)} + 33n^{(2)})/330 \\ = n(n^2-1)(3n^8 - 52n^6 + 410n^4 - 1636n^2 + 2555)/330\end{aligned}$$

こゝで $n^{(m)} = n(n-1)(n-2)\cdots(n-m+1)$ である。

2. 計算の準備

ロットの大きさを N , 標本の大きさを n , 母集団不良率 p_0 , 標本のなかの不良品箇数が c と超えない確率 P はよく知られた通り

$$P = \sum_{d=0}^c \frac{n! (N-n)! (Np_0)! (N-Np_0)!}{d! (n-d)! N! (Np_0-d)! (N-Np_0-n+d)!} \dots (2.1)$$

$\ln(N!/(N-n)!)$ は $M = N - (n-1)/2$ を用いて, (1.2) によつて $(2M)^{-2\tau}$ の漸近級数に展開できることと示しておいたが, (2.1) の $(Np_0)!/(Np_0-d)!$, $(N-Np_0)!/(N-Np_0-n+d)!$ があるから, これも同じように (1.2) によつて漸近級数に展開できるとあらう. その上で P の初項が標本の大きさ n , 母集団不良率 p で, 標本のなかの不良品数が c を超えない二項分布の確率であらわされる, $(2M)^{-1}$ の項が 0 となるように p を決定する.

$$M_p = Np_0 - \frac{d-1}{2}, \quad M_q = Nq_0 - \frac{n-d-1}{2} \dots (2.2)$$

$$p_0 = p(1 + \alpha/(2M)), \quad 1-p_0=q_0 = q - p\alpha/(2M) \dots (2.3)$$

とおくと

$$\begin{aligned} M_p &= M \left(1 + \frac{n-1}{2M}\right) p \left(1 + \frac{\alpha}{2M}\right) - \frac{d-1}{2} \\ &= Mp \left\{ 1 + \frac{n-1-(d-1)p^{-1}}{2M} + \frac{\alpha}{2M} \left(1 + \frac{n-1}{2M}\right) \right\} \dots (2.4p) \end{aligned}$$

$$M_q = Mq \left\{ 1 + \frac{n-1-(n-d-1)q^{-1}}{2M} - \frac{p}{q} \frac{\alpha}{2M} \left(1 + \frac{n-1}{2M}\right) \right\} \dots (2.4q)$$

ここから

$$x_1 = x(1 + (n-1)/(2M)) \quad \dots (2.5)$$

とおくと,

$$M_p = Mp \left\{ 1 + [n-1 - (d-1)p^{-1} + x_1]/(2M) \right\} \quad \dots (2.6p)$$

$$M_q = Mq \left\{ 1 + [n-1 - (n-d-1)q^{-1} - x_1 p q^{-1}]/(2M) \right\} \quad \dots (2.6q)$$

(1.2) 式の右辺第 1 項に よって

$$\begin{aligned} & -n \ln M + d \ln M_p + (n-d) \ln M_q \\ &= d \ln p + (n-d) \ln q + d \ln \left\{ 1 + [n-1 - (d-1)p^{-1} + x_1]/(2M) \right\} \\ & \quad + (n-d) \ln \left\{ 1 + [n-1 - (n-d-1)q^{-1} - x_1 p q^{-1}]/(2M) \right\} \\ & \quad \dots (2.7) \end{aligned}$$

(1.2) 式の右辺第 2 項は $(2M)^{-2}$ 以下はじまるから, (2.1) で $(2M)^{-1}$ の係数は $\frac{1}{2M}$ に対しては (2.7) からの x である. (2.7) に よって

$$\begin{aligned} \frac{1}{2M} \text{ の係数} &= \sum_{d=0}^c \binom{n}{d} p^d q^{n-d} \left\{ d^{(1)} [n-1 - (d-1)p^{-1} + x_1] \right. \\ & \quad \left. + (n-d)^{(1)} [n-1 - (n-d-1)q^{-1} - x_1 p q^{-1}] \right\} \\ &= n^{(1)} \left\{ (n-1+x_1) p B(c-1, n-1, p) + (n-1 - \frac{x_1 p}{q}) q B(c, n-1, p) \right\} \\ & \quad - n^{(2)} \left\{ p B(c-2, n-2, p) + q B(c, n-2, p) \right\} \quad \dots (2.8) \end{aligned}$$

ここから $B(c, n, p) = \sum_{d=0}^c \binom{n}{d} p^d q^{n-d}$ である. なお

$$b(d, n, p) = \binom{n}{d} p^d q^{n-d}$$

とし、特にまぎらわしくないとき $B(c, n), b(d, n)$ のように記す。

(2.8) を整理すると

$$\begin{aligned} \frac{1}{2M} \text{の係数} &= n^{(1)} \{ [(n-1)q - x, p] b(c, n-1, p) \\ &\quad + ((n-1+x), p + [(n-1)q - x, p]) B(c-1, n-1, p) \} \\ &\quad - n^{(2)} \{ p B(c-2, n-2, p) + q B(c, n-2, p) \} \\ &= n b(c, n-1, p) [(n-1)q - x, p] \\ &\quad + n^{(2)} \{ B(c-1, n-1, p) - p B(c-2, n-2, p) - q B(c, n-2, p) \} \\ &= n b(c, n-1, p) [(n-1)q - x, p] - n^{(2)} q b(c, n-2, p) \\ &= n b(c, n-1, p) [(n-1)q - x, p - (n-1-c)] = 0 \end{aligned}$$

$$\therefore x, p = (n-1)q - (n-1-c) = cq - (n-1-c)p \quad \dots (2.9)$$

$$\left. \begin{aligned} n-1+x, p &= n-1 + \frac{cq - (n-1-c)p}{p} = \frac{c}{p} \\ n-1 - \frac{x, p}{q} &= n-1 - \frac{cq - (n-1-c)p}{q} = \frac{n-1-c}{q} \end{aligned} \right\} \dots (2.10)$$

なお次の関係をつかっている。

$$\begin{aligned} B(c, n, p) &= \sum_{d=0}^c \binom{n-1}{d} p^d q^{n-d} + \sum_{d=0}^{c-1} \binom{n-1}{d} p^{d+1} q^{n-1-d} \\ &= B(c, n-1, p) \cdot q + B(c-1, n-1, p) \cdot p \\ &= \sum_{i=0}^{\wedge} B(c-i, n-1, p) \binom{\wedge}{i} p^i q^{\wedge-i} \quad \dots (2.11) \end{aligned}$$

(2.10) に よ っ て, (2.6) は

$$\left. \begin{aligned} M_p &= Mp \left(1 - \frac{d-1-c}{2Mp}\right) \\ M_q &= Mq \left(1 - \frac{n-d-1-(n-1-c)}{2Mq}\right) \end{aligned} \right\} \dots (2.12)$$

(2.3), (2.5) に よ っ て

$$\begin{aligned} p_0 &= p \left\{ 1 + \frac{x_1}{2M} \left(1 + \frac{n-1}{2M}\right)^{-1} \right\} \\ &= p + \frac{c q - (n-1-c)p}{2N} = \left(pM + \frac{c}{2}\right) N^{-1} \dots (2.13) \end{aligned}$$

$$Mp = Np_0 - \frac{c}{2}, \quad Mq = Np_0 - \frac{n-1-c}{2} \dots (2.14)$$

Wise o' $h, 1-x, p$ と し て い る の $\overbrace{h_2}^{(x_1, x_2)}$ に

て の $q, q_0, c+1$ と あ る。

(2.7) に よ っ て

$$\begin{aligned} &\left(1 - \frac{d-1-c}{2Mp}\right)^d \left(1 - \frac{n-d-1-(n-1-c)}{2Mq}\right)^{n-d} \\ &= \left\{ 1 + \sum_{\lambda=1}^d (-1)^\lambda \frac{d^{(\lambda)}}{\lambda!} \left(\frac{d-1-c}{2Mp}\right)^\lambda \right\} \cdot \left\{ 1 + \sum_{\lambda=1}^{n-d} (-1)^\lambda \frac{(n-d)^{(\lambda)}}{\lambda!} \left(\frac{n-d-1-(n-1-c)}{2Mq}\right)^\lambda \right\} \\ &= 1 + \sum_{\lambda=1}^n (-1)^\lambda \frac{1}{\lambda! (2M)^\lambda} G_\lambda \dots (2.15.1) \end{aligned}$$

$$\begin{aligned} G_\lambda &= \sum_{i=0}^{\lambda} \binom{\lambda}{i} d^{(\lambda-i)} (n-d)^{(i)} \left(\frac{d-1-c}{p}\right)^{\lambda-i} \left(\frac{n-d-1-(n-1-c)}{q}\right)^i \\ (G_0 &= 1 \text{ と おく}) \dots (2.15.2) \end{aligned}$$

(1.2) の 右 辺 の 2 項 E 書 き 直 し

と

$$= \sum_{\tau=1}^{\infty} \frac{1}{(2\tau)!(2M)^{2\tau}} f_{2\tau}(n)$$

と書くと, (2.1) は次のようになる. (ただし $f_{2\tau}(n)$ は附録1のF)に等しい)

$$P = \sum_{d=0}^c \binom{n}{d} p^d q^{n-d} \left\{ 1 + \sum_{A=1}^n (-1)^A \frac{1}{A!(2M)^A} G_A \right\}.$$

$$\cdot \exp \left[\sum_{\tau=1}^{\infty} \frac{1}{(2\tau)!(2M)^{2\tau}} \left\{ f_{2\tau}(n) - \frac{1}{p^{2\tau}} f_{2\tau}(d) \left(1 - \frac{d-1-c}{2Mp}\right)^{-2\tau} \right. \right. \\ \left. \left. - \frac{1}{q^{2\tau}} f_{2\tau}(n-d) \left(1 - \frac{n-d-1-(n-1-c)}{2Mq}\right)^{-2\tau} \right\} \right]$$

... (2.16)

(2.16) を整理するにため

$$\phi_m = \sum_{\tau=1}^{\left[\frac{m}{2}\right]} m^{(m-2\tau)} \binom{m-1}{m-2\tau} \left\{ \frac{1}{p^m} f_{2\tau}(d) (d-1-c)^{m-2\tau} \right. \\ \left. + \frac{1}{q^m} f_{2\tau}(n-d) \cdot (n-d-1-(n-1-c))^{m-2\tau} \right\}$$

... (2.17)

$$F_{2m} = \sum_{\alpha=1}^m \sum_{2m}^{\alpha} \frac{(2m)!}{\alpha_2! (2!)^{\alpha_2} \alpha_4! (4!)^{\alpha_4} \dots} (f_2(n))^{\alpha_2} (f_4(n))^{\alpha_4} \dots$$

... (2.18)

$$\sum_{2m}^{\alpha} 1 \text{ は } \left\{ \begin{array}{l} \alpha = \alpha_2 + \alpha_4 + \dots \\ 2m = 2\alpha_2 + 4\alpha_4 + \dots \end{array} \right. ; \alpha_2, \alpha_4, \dots \geq 0$$

すなわち $\alpha_2, \alpha_4, \dots$ のあらゆる組合せ

の和の意味

$$\Phi_m = \sum_{\alpha=1}^{\left[\frac{m}{2}\right]} (-1)^{\alpha} \sum_m^{\alpha} \frac{m!}{\alpha_2! (2!)^{\alpha_2} \alpha_3! (3!)^{\alpha_3} \alpha_4! (4!)^{\alpha_4} \dots}$$

$$\cdot \phi_2^{\alpha_2} \phi_3^{\alpha_3} \phi_4^{\alpha_4} \dots \quad \dots (2.19)$$

$$\sum_{m \geq 0} 12 \left\{ \begin{array}{l} \alpha = \alpha_2 + \alpha_3 + \alpha_4 + \dots \quad ; \quad \alpha_2, \alpha_3, \alpha_4, \dots \geq 0 \\ m = 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + \dots \end{array} \right\}$$

で ある $\alpha_2, \alpha_3, \alpha_4, \dots$ の あり 方 の 総 合

の 総 和 .

F_{2m} の 値 は 付 録 2 の $\{ \}$ に 依 る .

(2.17), (2.18), (2.19) を 用 い る と (2.16) は

$$\begin{aligned} P &= \sum_{d=0}^{\infty} \binom{n}{d} p^d q^{n-d} \left\{ 1 + \sum_{\lambda=1}^n (-1)^{\lambda} \frac{1}{\lambda! (2M)^{\lambda}} G_{\lambda} \right\} \cdot \\ &\quad \cdot \left\{ 1 + \sum_{m=1}^{\infty} \left[\frac{1}{(2m)! (2M)^{2m}} \sum_{s=0}^m \frac{(2m)!}{(2s)! (2m-2s)!} F_{2m-2s} \Phi_{2s} \right. \right. \\ &\quad \left. \left. + \frac{1}{(2m+1)! (2M)^{2m+1}} \sum_{s=1}^m \frac{(2m+1)!}{(2s+1)! (2m-2s)!} F_{2m-2s} \Phi_{2s+1} \right] \right\} \\ &= \sum_{d=0}^{\infty} \binom{n}{d} p^d q^{n-d} \left\{ 1 + \sum_{m=1}^{\infty} \frac{1}{(2m)! (2M)^{2m}} \left[F_{2m} \right. \right. \\ &\quad \left. \left. + \sum_{t=1}^m F_{2m-2t} \binom{2m}{2t} \left(\sum_{s=0}^t \binom{2t}{2s} \Phi_{2s} G_{2t-2s} \right. \right. \right. \\ &\quad \left. \left. \left. - \sum_{s=1}^{t-1} \binom{2t}{2s+1} \Phi_{2s+1} G_{2t-1-2s} \right) \right] \right. \\ &\quad \left. + \sum_{m=1}^{\infty} \frac{1}{(2m+1)! (2M)^{2m+1}} \left[- \binom{2m+1}{1} F_{2m} G_1 \right. \right. \\ &\quad \left. \left. + \sum_{t=1}^m F_{2m-2t} \binom{2m+1}{2t+1} \left(\sum_{s=1}^t \binom{2t+1}{2s+1} \Phi_{2s+1} G_{2t-2s} \right. \right. \right. \\ &\quad \left. \left. \left. - \sum_{s=0}^t \binom{2t+1}{2s} \Phi_{2s} G_{2t+1-2s} \right) \right] \right\} \end{aligned}$$

$\{ \}$ の $\dots$ (2.10)

と なる . (2.10) の $\{ \}$ の 示 す と は $(p, q), (d, n-d), (c, \bar{c})$ に ついて 変 換 を もつ と して ある .

3. $(2M)^{-(2m+1)}$ の 項 に ついて

(2.10) の $(2M)^{-(2m+1)}$ の 項 に ついて , こ の 項 の

と なる と して , m の 小 さ い と き に 計 算 し た .

右係数の第1項は

$$\begin{aligned}
 & \sum_{d=0}^c \binom{n}{d} p^d q^{n-d} G_1 \\
 &= \sum_{d=0}^c \binom{n}{d} p^d q^{n-d} \left[\frac{d(d-1-c)}{p} + \frac{(n-d)(n-d-1-(n-1-c))}{q} \right] \\
 &= n^{(2)} \sum_{d=0}^{c-2} \binom{n-2}{d} p^{d+1} q^{n-2-d} - n^{(1)} \sum_{d=0}^{c-1} \binom{n-1}{d} p^d q^{n-1-d-c} \\
 &\quad + n^{(2)} \sum_{d=0}^c \binom{n-2}{d} p^d q^{n-1-d} - n^{(1)} \sum_{d=0}^c \binom{n-1}{d} p^d q^{n-1-d-(n-1-c)} \\
 &= n^{(2)} B(c-2, n-2) p + n^{(2)} B(c, n-2) q \\
 &\quad - n B(c-1, n-1) c - n B(c, n-1) (n-1-c) \\
 &= -n b(c, n-1) \cdot (n-1-c) + n^{(2)} [B(c-2, n-2) p + B(c, n-2) q \\
 &\quad - B(c-1, n-1)] \\
 &= -n b(c, n-1) \cdot (n-1-c) + n^{(2)} b(c, n-2) q \\
 &= n b(c, n-1) \{ -(n-1-c) + (n-1-c) \} = 0 \quad \dots (3.1)
 \end{aligned}$$

次に $F_{2m-2c} \left(\begin{smallmatrix} 2m+1 \\ 2c+1 \end{smallmatrix} \right)$ の係数 について 12 は

$$\begin{aligned}
 A_{2c+1} &= \sum_{d=0}^c \binom{n}{d} p^d q^{n-d} \left[\sum_{s=1}^c \binom{2c+1}{2s+1} \Phi_{2s+1} G_{2c-2s} \right. \\
 &\quad \left. - \sum_{s=0}^c \binom{2c+1}{2s} \Phi_{2s} G_{2c+1-2s} \right] \\
 &\quad \dots (3.2)
 \end{aligned}$$

と おく と

$$A_3 = \sum_{d=0}^c \binom{n}{d} p^d q^{n-d} [\Phi_3 - \binom{3}{2} \Phi_2 G_1 - G_3] \quad \dots (3.3)$$

今便宜上 $\bar{c} = (n-1-c)$ とおき, $(p, q), (d, n-d),$

$(c, \bar{c})$ を同時に取リ換えたものを加え、意味をあらわして $+ \sim$ と記しておく

$$\begin{aligned}
 \Phi_3 &= 3\Phi_2 G_1 - G_3 \\
 &= -6 \left\{ \frac{1}{p^3} f_2(d) \cdot (d-1-c) + \sim \right\} + 3 \left\{ \frac{1}{p^2} f_2(d) + \sim \right\} \left\{ \frac{d(d-1-c)}{p} + \sim \right\} \\
 &\quad - \left\{ \frac{d^{(3)}(d-1-c)^3}{p^3} + 3 \frac{d^{(2)}(d-1-c)^2(n-d)(n-d-1-\bar{c})}{p^2 q} + \sim \right\} \\
 &= \frac{1}{p^3} [-d^{(6)} + d^{(5)}(c3-8) - d^{(4)}(c^{(2)}3 - c11+12) + d^{(3)}(c^{(3)} - c^{(2)}3 + c3)] \\
 &\quad + \frac{1}{p^2 q} [-d^{(4)}3 + d^{(3)}(c6-8) - d^{(2)}(c^{(2)}3 - c3)] \cdot [(n-d)^{(2)} - (n-d)\bar{c}] \\
 &\quad + \sim
 \end{aligned}$$

と q 3 次から, $(3-3) = \lambda$ 4 次

$$\begin{aligned}
 A_3 &= -n^{(6)} B(c, n-6) q^3 + n^{(5)} B(c, n-5) q^2 [\bar{c}3-8] \\
 &\quad - n^{(4)} B(c, n-4) q [\bar{c}^{(2)}3 - \bar{c}11+12] + n^{(3)} B(c, n-3) [\bar{c}^{(3)} - \bar{c}^{(2)}3 + \bar{c}3] \\
 &\quad - n^{(6)} B(c-2, n-6) p q^2 3 + n^{(5)} B(c-2, n-5) p q [\bar{c}6-8] - n^{(4)} B(c-2, n-4) p \cdot \\
 &\quad \quad \cdot [\bar{c}^{(2)}3 - \bar{c}3] \\
 &\quad + c \left\{ n^{(5)} B(c-1, n-5) q^2 3 - n^{(4)} B(c-1, n-4) q [\bar{c}6-8] + n^{(3)} B(c-1, n-3) [\bar{c}^{(2)}3 - \bar{c}3] \right\} \\
 &\quad - n^{(6)} B(c-4, n-6) p^2 q 3 + n^{(5)} B(c-3, n-5) p q [c6-8] - n^{(4)} B(c-2, n-4) q [c^{(2)}3 - c3] \\
 &\quad + \bar{c} \left\{ n^{(5)} B(c-4, n-5) p^2 3 - n^{(4)} B(c-3, n-4) p [c6-8] + n^{(3)} B(c-2, n-3) [c^{(2)}3 - c3] \right\} \\
 &\quad - n^{(6)} B(c-6, n-6) p^3 + n^{(5)} B(c-5, n-5) p^2 [c3-8] \\
 &\quad - n^{(4)} B(c-4, n-4) p [c^{(2)}3 - c11+12] + n^{(3)} B(c-3, n-3) [c^{(3)} - c^{(2)}3 + c3] \\
 &\quad \dots (3.4)
 \end{aligned}$$

こゝに $B(c, n-6)$ 等 $\bar{c}$ の $I\bar{Q}$ は 2 次

$$- n^{(6)} b(c-1, n-6) q^3 + n^{(5)} b(c-1, n-5) q^2 [\bar{c}3-8]$$

$$\begin{aligned}
& -n^{(4)}b(c-1, n-4)q[\bar{c}^{(2)}3 - \bar{c}^{(1)}1 + 12] + n^{(3)}b(c-1, n-3)[\bar{c}^{(3)} - \bar{c}^{(2)}3 + \bar{c}3] \\
& = n b(c, n-1) \frac{c^{(1)}}{p^\lambda q^{2-\lambda}} \left\{ -\bar{c}^{(5-\lambda)} + \bar{c}^{(4-\lambda)}[\bar{c}3 - 8] - \bar{c}^{(3-\lambda)}[\bar{c}^{(2)}3 - \bar{c}^{(1)}1 + 12] \right. \\
& \quad \left. + \bar{c}^{(2-\lambda)}[\bar{c}^{(2)} - \bar{c}^{(2)}3 + \bar{c}3] \right\}
\end{aligned}$$

$$= n b(c, n-1) \frac{c^{(1)}}{p^\lambda q^{2-\lambda}} \left\{ \bar{c}^{(3-\lambda)} \binom{1}{1} - \bar{c}^{(2-\lambda)} \left[\binom{1}{3}6 + \binom{1}{2}6 - \binom{1}{1}3 \right] \right\}$$

($0 \leq \lambda \leq 2$)

... (3.5)

$\lambda=0$ のときは $0 \leq \lambda \leq 3$.

次に

$$-n^{(6)}b(c-1-\lambda, n-6)pq^23 + n^{(5)}b(c-1-\lambda, n-5)pq[\bar{c}6-8]$$

$$-n^{(4)}b(c-1-\lambda, n-4)p[\bar{c}^{(2)}3 - \bar{c}3]$$

$$+ c \left\{ n^{(5)}b(c-1, n-5)q^23 - n^{(4)}b(c-1, n-4)q[\bar{c}6-8] + n^{(3)}b(c-1, n-3)[\bar{c}^{(2)}3 - \bar{c}3] \right\}$$

$$= n b(c, n-1) \frac{-c^{(1+\lambda)} + c \cdot c^{(1)}}{p^\lambda q^{2-\lambda}} \left\{ \bar{c}^{(4-\lambda)}3 - \bar{c}^{(3-\lambda)}[\bar{c}6-8] + \bar{c}^{(2-\lambda)}[\bar{c}^{(2)}3 - \bar{c}3] \right\}$$

$$= n b(c, n-1) \frac{c^{(1)}\lambda}{p^\lambda q^{2-\lambda}} \left\{ -\bar{c}^{(3-\lambda)} + \bar{c}^{(2-\lambda)} \left[\binom{1}{2}6 + \binom{1}{1}3 - 3 \right] \right\}$$

($1 \leq \lambda \leq 2$)

$$= n b(c, n-1) \frac{c^{(1)}}{p^\lambda q^{2-\lambda}} \left\{ -\bar{c}^{(3-\lambda)} \binom{1}{1} + \bar{c}^{(2-\lambda)} \left[\binom{1}{3}18 + \binom{1}{2}6 - \binom{1}{1}3 \right] \right\}$$

... (3.6)

以上の二式と対称式 T_j に関する c は (p, q, λ) (3.6) を用い

$$n b(c, n-1) \frac{\bar{c}^{(1)}}{p^{2-\lambda} q^\lambda} \left\{ -c^{(3-\lambda)} \binom{1}{1} + c^{(2-\lambda)} \left[\binom{1}{3}18 + \binom{1}{2}6 - \binom{1}{1}3 \right] \right\}$$

で $\lambda=0$ のときは 0 .

(3.5) と (3.6) とを $n \rightarrow 3$ と

$$n b(c, n-1) \frac{c^{(1)}}{p^\lambda q^{2-\lambda}} \bar{c}^{(2-\lambda)} \binom{1}{3}12 = 0 \quad (1 \leq \lambda \leq 2)$$

と等しい, (3.4) は λ_1 の λ_2 に等しい.

$$\begin{aligned}
 A_3 &= -n^{(6)}B(c-3, n-6)q^3 + n^{(5)}B(c-3, n-5)q^2[\bar{c}3-8] - n^{(4)}B(c-3, n-4)q[\bar{c}^{(2)}_3 - \bar{c}^{(1)}_{+12}] \\
 &\quad + n^{(3)}B(c-3, n-3)[\bar{c}^{(3)} - \bar{c}^{(2)}_3 + \bar{c}3] \\
 &\quad - n^{(6)}B(c-4, n-6)pq^3 + n^{(5)}B(c-4, n-5)pq[\bar{c}6-8] - n^{(4)}B(c-4, n-4)p[\bar{c}^{(2)}_3 - \bar{c}3] \\
 &\quad + c\{n^{(5)}B(c-3, n-5)q^23 - n^{(4)}B(c-3, n-4)q[\bar{c}6-8] + n^{(3)}B(c-3, n-3)[\bar{c}^{(2)}_3 - \bar{c}3]\} \\
 &\quad - n^{(6)}B(c-5, n-6)p^2q^3 + n^{(5)}B(c-4, n-5)pq[c6-8] - n^{(4)}B(c-3, n-4)[c^{(2)}_3 - c3] \\
 &\quad + \bar{c}\{n^{(5)}B(c-5, n-5)p^23 - n^{(4)}B(c-4, n-4)p[c6-8] + n^{(3)}B(c-3, n-3)[c^{(2)}_3 - c3]\} \\
 &\quad - n^{(6)}B(c-6, n-6)p^3 + n^{(5)}B(c-5, n-5)p^2[c3-8] - n^{(4)}B(c-4, n-4)p[c^{(2)}_3 - c^{(1)}_{+12}] \\
 &\quad + n^{(3)}B(c-3, n-3)[c^{(3)} - c^{(2)}_3 + c3] \\
 &= -n^{(6)}B(c-3, n-3) + n^{(5)}B(c-3, n-3)[(n-1)3-8] - n^{(4)}B(c-3, n-3)[(n-1)^{(2)}_3 - (n-1)^{(1)}_{+12}] \\
 &\quad + n^{(3)}B(c-3, n-3)[(n-1)^{(3)} - (n-1)^{(2)}_3 + (n-1)3] \\
 &= B(c-3, n-3)\{-n^{(6)} + n^{(5)}[(n-5)3+4] - n^{(4)}[(n-4)^{(2)}_3 + (n-4)^{(1)}_{+12}] \\
 &\quad + n^{(3)}[(n-3)^{(3)} + (n-3)^{(2)}_3 - (n-3)3]\} \\
 &= 0
 \end{aligned}$$

このようにして $A_5=0$, $A_7=0$ のように $\frac{1}{2}n$ まで成り立つ.

4. $(2M)^{-2m}$ の $I\mathbb{Q}$ に ついて.

(2.10) の $1/((2m)!(2M)^{2m})$ の $I\mathbb{Q}$ を K_{2m} とおくと

$$\begin{aligned}
 K_{2m} &= \sum_{d=0}^c \binom{n}{d} p^d q^{n-d} [F_{2m} + \sum_{t=1}^m F_{2m-2t} \binom{2m}{2t} A_{2t}] \\
 A_{2t} &= \left\{ \sum_{s=0}^t \binom{2t}{2s} \Phi_{2s} G_{2t-2s} - \sum_{s=1}^{t-1} \binom{2t}{2s+1} \Phi_{2s+1} G_{2t-1-2s} \right\} \dots (4.1) \\
 &\quad \left\{ \sum_{d=0}^c \binom{n}{d} p^d q^{n-d} \right\} \dots (4.2)
 \end{aligned}$$

(4.1) の F_{2m} の係数は (2.11) に よる

$$\sum_{d=0}^c \binom{n}{d} p^d q^{n-d} = B(c, n)$$

$$= B(c, n-2) q^2 + B(c-1, n-2) p q^2 \\ + B(c-2, n-2) p^2$$

$$= b(c, n-2) q^2 - b(c-1, n-2) p^2 + B(c-1, n-2)$$

$$= \frac{nb(c, n-1)}{n^{(2)}} A_{0,0} + B(c-1, n-2) \quad \dots (4.3)$$

$$A_{0,0} = \bar{c} q - c p \quad \dots (4.4)$$

同様に $\lambda \geq 1$ として

$$B(c-\lambda, n-2\lambda) = B(c-\lambda, n-2\lambda-2) q^2 \\ + B(c-\lambda-1, n-2\lambda-2) p q^2 + B(c-\lambda-2, n-2\lambda-2) p^2$$

$$= b(c-\lambda, n-2\lambda-2) q^2 - b(c-\lambda-1, n-2\lambda-2) p^2 \\ + B(c-\lambda-1, n-2\lambda-2)$$

$$= \frac{nb(c, n-1)}{n^{(2\lambda+2)}} A_{0,\lambda} + B(c-\lambda-1, n-2\lambda-2) \quad \dots (4.5)$$

$$A_{0,\lambda} = \frac{c^{(\lambda)} \bar{c}^{(\lambda+1)}}{p^\lambda q^{\lambda+1}} - \frac{c^{(\lambda+1)} \bar{c}^{(\lambda)}}{p^{\lambda+1} q^\lambda} \quad \dots (4.6)$$

次に A_2 について計算する

$$A_2 = \cancel{\Phi_2 + G_2} \sum_{d=0}^c \binom{n}{d} p^d q^{n-d} \{ \Phi_2 + G_2 \}$$

$$\begin{aligned}
&= \sum_{d=0}^c \binom{n}{d} p^d q^{n-d} \left\{ - \frac{d^{(3)} + d^{(2)} 3}{3p^2} - \sim \right. \\
&\quad + \frac{d^{(4)} - d^{(3)}(c-3) + d^{(2)}(c^{(2)} - c + 1)}{p^2} + \sim \\
&\quad \left. + \frac{2}{pq} (d^{(2)} - dc) ((n-d)^{(2)} - (n-d)\bar{c}) \right\}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{d=0}^c \binom{n}{d} p^d q^{n-d} \left\{ \frac{1}{p^2} [d^{(4)} - d^{(3)}(c-3) + d^{(2)}(c^{(2)} - c)] + \sim \right. \\
&\quad \left. + \frac{2}{pq} (d^{(2)} - dc) ((n-d)^{(2)} - (n-d)\bar{c}) \right\}
\end{aligned}$$

$$\begin{aligned}
&= n^{(4)} B(c, n-4) q^2 - n^{(3)} B(c, n-3) q (\bar{c} - \frac{8}{3}) + n^{(2)} B(c, n-2) (\bar{c}^{(2)} - \bar{c}) \\
&\quad + n^{(4)} B(c-4, n-4) p^2 - n^{(3)} B(c-3, n-3) p (c - \frac{8}{3}) + n^{(2)} B(c-2, n-2) (c^{(2)} - c) \\
&\quad + n^{(4)} B(c-2, n-4) pq^2 - n^{(3)} B(c-1, n-3) q c^2 \\
&\quad - n^{(3)} B(c-2, n-3) p \bar{c}^2 + n^{(2)} B(c-1, n-2) c \bar{c}^2
\end{aligned}$$

$$\begin{aligned}
&= n^{(4)} [b(c, n-4) q^2 - b(c-3, n-4) p^2] + n^{(4)} [B(c-1, n-4) q^2 \\
&\quad + B(c-2, n-4) pq^2 + B(c-3, n-4) p^2] \\
&\quad - n^{(3)} [b(c, n-3) q (\bar{c} - \frac{8}{3}) - b(c-2, n-3) p (c - \frac{8}{3})] \\
&\quad - n^{(3)} [B(c-1, n-3) q ((n-1)^2 - \frac{8}{3}) + B(c-2, n-3) p ((n-1)^2 - \frac{8}{3})] \\
&\quad + n^{(2)} [b(c, n-2) (\bar{c}^{(2)} - \bar{c}) - b(c-1, n-2) (c^{(2)} - c)] \\
&\quad + n^{(2)} B(c-1, n-2) ((n-1)^{(2)} - (n-1))
\end{aligned}$$

$$\begin{aligned}
&= nb(c, n-1) A_{2,0} + B(c-1, n-2) \left\{ n^{(4)} - n^{(3)} ((n-3)^2 + \frac{4}{3}) \right. \\
&\quad \left. + n^{(2)} ((n-2)^{(2)} + (n-2) - 1) \right\}
\end{aligned}$$

$$= nb(c, n-1) A_{2,0} - B(c-1, n-2) f_2(n) \quad \dots \quad (4.7)$$

$$\begin{aligned}
A_{2,0} &= \frac{\bar{c}^{(3)}}{q} - \frac{c^{(3)}}{p} - \frac{\bar{c}^{(2)}}{q} \left(\bar{c}^2 - \frac{8}{3} \right) + \frac{c^{(2)}}{p} \left(c^2 - \frac{8}{3} \right) \\
&\quad + \frac{\bar{c}}{q} (\bar{c}^{(2)} - \bar{c}) - \frac{c}{p} (c^{(2)} - c) \\
&= -\frac{\bar{c}(\bar{c}+2)}{3q} + \frac{c(c+2)}{3p} \quad \dots \dots (4.8)
\end{aligned}$$

よって (4.1) に よって

$$\begin{aligned}
K_{2m} &= nb(c, n-1) \left\{ \frac{F_{2m}}{n^{(2)}} A_{0,0} + \frac{F_{2m} - \binom{2m}{2} F_{2m-2} f_2(n)}{n^{(4)}} A_{0,1} \right. \\
&\quad \left. + \binom{2m}{2} F_{2m-2} A_{2,0} \right\}
\end{aligned}$$

$$+ [F_{2m} - \binom{2m}{2} F_{2m-2} f_2(n)] B(c-2, n-4)$$

$$+ \sum_{t=2}^m F_{2m-2t} \binom{2m}{2t} A_{2t} \quad \dots \dots (4.9)$$

したがって

$$K_2 = nb(c, n-1) \left\{ \frac{n+1}{3} A_{0,0} + A_{2,0} \right\} \quad \dots (4.10)$$

となる。

次に A_4 について計算する。

$$\begin{aligned}
A_4 &= \sum_{d=0}^c \binom{n}{d} p^d q^{n-d} \left\{ \sum_{s=0}^2 \binom{4}{2s} \Phi_{2s} G_{4-2s} - \binom{4}{1} \Phi_3 G_1 \right\} \\
&\quad \dots (4.11)
\end{aligned}$$

まず (4.11) の $\{ \}$ 内の d の x の項, $(n-d)$ の x の項を 集め

$$\begin{aligned}
& \sum_{d=0}^c \binom{n}{d} p^d q^{n-d} \left\{ \frac{1}{p^4} \left[d^{(8)} - d^{(7)}(c4-16) + d^{(6)}(c^{(2)}6 - c38 + \frac{208}{3}) \right. \right. \\
& \quad \left. \left. - d^{(5)}(c^{(3)}4 - c^{(2)}28 + c76 - \frac{384}{5}) \right. \right. \\
& \quad \left. \left. + d^{(4)}(c^{(4)} - c^{(3)}6 + c^{(2)}15 - c15) \right] + \sim \right\} \\
& = n^{(8)} B(c-8, n-8) p^4 - n^{(7)} B(c-7, n-7) p^3 (c4-16) \\
& \quad + n^{(6)} B(c-6, n-6) p^2 (c^{(2)}6 - c38 + \frac{208}{3}) - n^{(5)} B(c-5, n-5) p (c^{(3)}4 - c^{(2)}28 + c76 - \frac{384}{5}) \\
& \quad + n^{(4)} B(c-4, n-4) (c^{(4)} - c^{(3)}6 + c^{(2)}15 - c15) \\
& + n^{(8)} B(c, n-8) q^4 - n^{(7)} B(c, n-7) q^3 (\bar{c}4-16) \\
& \quad + n^{(6)} B(c, n-6) q^2 (\bar{c}^{(2)}6 - \bar{c}38 + \frac{208}{3}) - n^{(5)} B(c, n-5) q (\bar{c}^{(3)}4 - \bar{c}^{(2)}28 + \bar{c}76 - \frac{384}{5}) \\
& \quad + n^{(4)} B(c, n-4) (\bar{c}^{(4)} - \bar{c}^{(3)}6 + \bar{c}^{(2)}15 - \bar{c}15) \\
& = \sum_{\lambda=0}^i \left\{ n^{(8)} b(c-\lambda, n-8) q^4 - n^{(7)} b(c-\lambda, n-7) q^3 (\bar{c}4-16) + n^{(6)} b(c-\lambda, n-6) q^2 (\bar{c}^{(2)}6 - \bar{c}38 + \frac{208}{3}) \right. \\
& \quad \left. - n^{(5)} b(c-\lambda, n-5) q (\bar{c}^{(3)}4 - \bar{c}^{(2)}28 + \bar{c}76 - \frac{384}{5}) + n^{(4)} b(c-\lambda, n-4) (\bar{c}^{(4)} - \bar{c}^{(3)}6 + \bar{c}^{(2)}15 - \bar{c}15) \right. \\
& \quad \left. - n^{(8)} b(c-7+\lambda, n-8) p^4 + n^{(7)} b(c-6+\lambda, n-7) p^3 (c4-16) + n^{(6)} b(c-5+\lambda, n-6) p^2 (c^{(2)}6 - c38 + \frac{208}{3}) \right. \\
& \quad \left. + n^{(5)} b(c-4+\lambda, n-5) p (c^{(3)}4 - c^{(2)}28 + c76 - \frac{384}{5}) - n^{(4)} b(c-3+\lambda, n-4) (c^{(4)} - c^{(3)}6 + c^{(2)}15 - c15) \right\} \\
& + n^{(8)} B(c-2, n-8) q^4 - n^{(7)} B(c-2, n-7) q^3 (\bar{c}4-16) + n^{(6)} B(c-2, n-6) q^2 (\bar{c}^{(2)}6 - \bar{c}38 + \frac{208}{3}) \\
& \quad - n^{(5)} B(c-2, n-5) q (\bar{c}^{(3)}4 - \bar{c}^{(2)}28 + \bar{c}76 - \frac{384}{5}) + n^{(4)} B(c-2, n-4) (\bar{c}^{(4)} - \bar{c}^{(3)}6 + \bar{c}^{(2)}15 - \bar{c}15) \\
& + n^{(8)} B(c-6, n-8) p^4 - n^{(7)} B(c-5, n-7) p^3 (c4-16) + n^{(6)} B(c-4, n-6) p^2 (c^{(2)}6 - c38 + \frac{208}{3}) \\
& \quad - n^{(5)} B(c-3, n-5) p (c^{(3)}4 - c^{(2)}28 + c76 - \frac{384}{5}) + n^{(4)} B(c-2, n-4) (c^{(4)} - c^{(3)}6 + c^{(2)}15 - c15) \\
& = n b(c, n-1) \sum_{\lambda=0}^i \left\{ \frac{c^{(\lambda)}}{p^4 q^{3-\lambda}} \left[\bar{c}^{(7-\lambda)} - \bar{c}^{(6-\lambda)} (\bar{c}4-16) + \bar{c}^{(5-\lambda)} (\bar{c}^{(2)}6 - \bar{c}38 + \frac{208}{3}) \right. \right. \\
& \quad \left. \left. - \bar{c}^{(4-\lambda)} (\bar{c}^{(3)}4 - \bar{c}^{(2)}28 + \bar{c}76 - \frac{384}{5}) \right. \right. \\
& \quad \left. \left. + \bar{c}^{(3-\lambda)} (\bar{c}^{(4)} - \bar{c}^{(3)}6 - \bar{c}^{(2)}15 + \bar{c}15) \right] \right. \\
& \quad \left. - \bullet \sim \right\} + n^{(8)} B(c-2, n-8) q^4 - \dots
\end{aligned}$$

$$\begin{aligned}
&= n b(c, n-1) \sum_{\lambda=0}^1 \left\{ \frac{c^{(\lambda)}}{p^\lambda q^{3-\lambda}} \left[\bar{c}^{(5-\lambda)} \frac{1}{3} + \bar{c}^{(4-\lambda)} \left[-\binom{\lambda}{2} 4 - \binom{\lambda}{1} 2 + \frac{19}{5} \right] \right. \right. \\
&\quad \left. \left. + \bar{c}^{(3-\lambda)} \left[\binom{\lambda}{4} 24 + \binom{\lambda}{3} 36 - \binom{\lambda}{2} 6 - \binom{\lambda}{1} 9 + 9 \right] \right] \right. \\
&\quad \left. - \sim \right\} \\
&+ n^{(8)} B(c-2, n-8) q^4 - n^{(9)} B(c-2, n-7) q^3 (\bar{c} 4 - 16) + n^{(6)} B(c-2, n-6) q^2 (\bar{c}^{(2)} 6 - \bar{c} 38 + \frac{208}{3}) \\
&\quad - n^{(5)} B(c-2, n-5) q (\bar{c}^{(3)} 4 - \bar{c}^{(2)} 28 + \bar{c} 76 - \frac{384}{5}) + n^{(4)} B(c-2, n-4) (\bar{c}^{(4)} - \bar{c}^{(3)} 6 + \bar{c}^{(2)} 15 - \bar{c} 15) \\
&+ n^{(8)} B(c-6, n-8) p^4 - n^{(9)} B(c-5, n-7) p^3 (c 4 - 16) + n^{(6)} B(c-4, n-6) p^2 (c^{(2)} 6 - c 38 + \frac{208}{3}) \\
&\quad - n^{(5)} B(c-3, n-5) p (c^{(3)} 4 - c^{(2)} 28 + c 76 - \frac{384}{5}) + n^{(4)} B(c-2, n-4) (c^{(4)} - c^{(3)} 6 + c^{(2)} 15 - c 15) \\
&\quad \dots (4.12)
\end{aligned}$$

次に (4.11) の $\{ \}$ 内の $[(n-d)^{(2)} - (n-d)\bar{c}]$ を因子として
 (4.12) の d の 2 の項と $\bar{c}$ の 2 項の項とを $\bar{c}$ と $\bar{c}$ と

$$\begin{aligned}
&\sum_{d=0}^5 \binom{n}{d} p^d q^{n-d} \left\{ \frac{4}{p^3 q} \left[d^{(6)} - d^{(5)} (c 3 - 8) + d^{(4)} (c^{(2)} 3 - c 11 + 12) - d^{(3)} (c^{(3)} - c^{(2)} 3 + c 3) \right] \right. \\
&\quad \left. \cdot [(n-d)^{(2)} - (n-d)\bar{c}] + \sim \right\} \\
&= 4 \left\{ n^{(8)} B(c-2, n-8) p q^3 - n^{(9)} B(c-2, n-7) p q^2 (\bar{c} 3 - 8) + n^{(6)} B(c-2, n-6) p q (\bar{c}^{(2)} 3 - \bar{c} 11 + 12) \right. \\
&\quad \left. - n^{(5)} B(c-2, n-5) p (\bar{c}^{(3)} - \bar{c}^{(2)} 3 + \bar{c} 3) \right. \\
&\quad \left. + n^{(8)} B(c-6, n-8) p^3 q - n^{(9)} B(c-5, n-7) p^2 q (c 3 - 8) + n^{(6)} B(c-4, n-6) p q (c^{(2)} 3 - c 11 + 12) \right. \\
&\quad \left. - n^{(5)} B(c-3, n-5) q (c^{(3)} - c^{(2)} 3 + c 3) \right. \\
&\quad \left. - c \left[n^{(9)} B(c-1, n-7) q^3 - n^{(6)} B(c-1, n-6) q^2 (\bar{c} 3 - 8) + n^{(5)} B(c-1, n-5) q (\bar{c}^{(2)} 3 - \bar{c} 11 + 12) \right. \right. \\
&\quad \left. \left. - n^{(4)} B(c-1, n-4) (\bar{c}^{(3)} - \bar{c}^{(2)} 3 + \bar{c} 3) \right] \right. \\
&\quad \left. - \bar{c} \left[n^{(9)} B(c-6, n-7) p^3 - n^{(6)} B(c-5, n-6) p^2 (c 3 - 8) + n^{(5)} B(c-4, n-5) p (c^{(2)} 3 - c 11 + 12) \right. \right. \\
&\quad \left. \left. - n^{(4)} B(c-3, n-4) (c^{(3)} - c^{(2)} 3 + c 3) \right] \right\} \dots (4.13)
\end{aligned}$$

右項の $c-\lambda$ と $-\bar{c}$ だけの中の方に寄せると

$$\begin{aligned}
& n^{(8)} b(c-2, n-8) p q^3 - n^{(7)} b(c-2, n-7) p q^2 (\bar{c}3-8) + n^{(6)} b(c-2, n-6) p q (\bar{c}^{(2)}3 - \bar{c}11+12) \\
& - n^{(5)} b(c-2, n-5) p (\bar{c}^{(3)}2 - \bar{c}^{(2)}3 + \bar{c}3) + \sim \\
& - c [n^{(7)} b(c-1, n-7) q^3 - n^{(6)} b(c-1, n-6) q^2 (\bar{c}3-8) + n^{(5)} b(c-1, n-5) q (\bar{c}^{(2)}3 - \bar{c}11+12) \\
& - n^{(4)} b(c-1, n-4) (\bar{c}^{(3)}2 - \bar{c}^{(2)}3 + \bar{c}3)] - \sim \\
& = n b(c, n-1) \left\{ \frac{c^{(2)} - c \cdot c}{p q^2} [\bar{c}^{(5)} - \bar{c}^{(4)} (\bar{c}3-8) + \bar{c}^{(3)} (\bar{c}^{(2)}3 - \bar{c}11+12) - \bar{c}^{(2)} (\bar{c}^{(3)}2 - \bar{c}^{(2)}3 + \bar{c}3)] \right. \\
& \quad \left. + \sim \right\} \\
& = 0
\end{aligned}$$

と なる、(4.13) は 27 の よう に なる。

$$\begin{aligned}
(4.13) = & 4 \left\{ n^{(8)} B(c-3, n-8) p q^3 - n^{(7)} B(c-3, n-7) p q^2 (\bar{c}3-8) \right. \\
& + n^{(6)} B(c-3, n-6) p q (\bar{c}^{(2)}3 - \bar{c}11+12) - n^{(5)} B(c-3, n-5) p (\bar{c}^{(3)}2 - \bar{c}^{(2)}3 + \bar{c}3) \\
& + n^{(6)} B(c-5, n-8) p^3 q - n^{(7)} B(c-4, n-7) p^2 q (\bar{c}3-8) \\
& + n^{(6)} B(c-3, n-6) p q (c^{(2)}3 - c11+12) - n^{(5)} B(c-2, n-5) q (c^{(2)} - c^{(2)}3 + c3) \\
& - c [n^{(7)} B(c-2, n-7) q^3 - n^{(6)} B(c-2, n-6) q^2 (\bar{c}3-8) + n^{(5)} B(c-2, n-5) q (\bar{c}^{(2)}3 - \bar{c}11+12) \\
& - n^{(4)} B(c-2, n-4) (\bar{c}^{(3)}2 - \bar{c}^{(2)}3 + \bar{c}3)] \\
& - \bar{c} [n^{(7)} B(c-5, n-7) p^3 - n^{(6)} B(c-4, n-6) p^2 (\bar{c}3-8) + n^{(5)} B(c-3, n-5) p (c^{(2)}3 - c11+12) \\
& \left. - n^{(4)} B(c-2, n-4) (c^{(3)}2 - c^{(2)}3 + c3) \right] \} \quad \dots (4.14)
\end{aligned}$$

(4.11) の 34.1) の $I\bar{K}$ に つ い て 述 べ る と

$$\begin{aligned}
\sum_{d=0}^c \binom{n}{d} p^d q^{n-d} \left\{ - \frac{2}{p^2 q^2} [d^{(3)} + d^{(2)}3] [(n-d)^{(4)} - (n-d)^{(3)} (\bar{c}2-3) + (n-d)^{(2)} (\bar{c}^{(2)}2 - \bar{c}+1)] \right. \\
- \sim \\
+ \frac{6}{p^2 q^2} [d^{(4)} - d^{(3)} (\bar{c}2-3) + d^{(2)} (c^{(2)} - c+1)] [(n-d)^{(4)} - (n-d)^{(3)} (\bar{c}2-3) + (n-d)^{(2)} (\bar{c}^{(2)}2 - \bar{c}+1)] \\
\left. + \frac{2}{3} \frac{1}{p^2 q^2} [d^{(3)} + d^{(2)}3] [(n-d)^{(3)} + (n-d)^{(2)}3] \right\}
\end{aligned}$$

$$\begin{aligned}
&= -2 \left\{ n^{(7)} B(c-3, n-7) p q^2 - n^{(6)} B(c-3, n-6) p q (\bar{c}-3) + n^{(5)} B(c-3, n-5) p (\bar{c}^{(2)} - \bar{c} + 1) \right. \\
&\quad \left. + n^{(7)} B(c-4, n-7) p^2 q - n^{(6)} B(c-3, n-6) p q (c-3) + n^{(5)} B(c-2, n-5) q (c^{(2)} - c + 1) \right\} \\
&- 6 \left\{ n^{(6)} B(c-2, n-6) q^2 - n^{(5)} B(c-2, n-5) q (\bar{c}-3) + n^{(4)} B(c-2, n-4) (\bar{c}^{(2)} - \bar{c} + 1) \right. \\
&\quad \left. + n^{(6)} B(c-4, n-6) p^2 - n^{(5)} B(c-3, n-5) p (c-3) + n^{(4)} B(c-2, n-4) (c^{(2)} - c + 1) \right\} \\
&+ 6 \left\{ n^{(8)} B(c-4, n-8) p^2 q^2 - n^{(7)} B(c-3, n-7) p q^2 (c-3) - n^{(7)} B(c-4, n-7) p^2 q (\bar{c}-3) \right. \\
&\quad + n^{(6)} B(c-2, n-6) q^2 (c^{(2)} - c + 1) + n^{(6)} B(c-3, n-6) p q (c-3) (\bar{c}-3) + n^{(6)} B(c-4, n-6) p^2 (\bar{c}^{(2)} - \bar{c} + 1) \\
&\quad - n^{(5)} B(c-2, n-5) q (c^{(2)} - c + 1) (\bar{c}-3) - n^{(5)} B(c-3, n-5) p (\bar{c}^{(2)} - \bar{c} + 1) (c-3) \\
&\quad \left. + n^{(4)} B(c-2, n-4) (c^{(2)} - c + 1) (\bar{c}^{(2)} - \bar{c} + 1) \right\} \\
&+ n^{(6)} B(c-3, n-6) p q \frac{2}{3} + n^{(5)} B(c-2, n-5) q + n^{(5)} B(c-3, n-5) p^2 \\
&+ n^{(4)} B(c-2, n-4) 6 \quad \dots (4.15)
\end{aligned}$$

(4.12), (4.14), (4.15) $\in \mathcal{O}_2 \bar{z}$

$$\begin{aligned}
A_4 &= m b(c, n-1) A_{4,0} + B(c-2, n-4) \left\{ n^{(8)} - n^{(7)} [(n-1)^4 - 16] \right. \\
&\quad + n^{(6)} [(n-1)^4 6 - (n-1) 38 + \frac{208}{3}] - n^{(5)} [(n-1)^4 4 - (n-1)^2 28 + (n-1) 76 - \frac{384}{5}] \\
&\quad \left. + n^{(4)} [(n-1)^4 - (n-1)^3 6 + (n-1)^2 15 - (n-1) 15] \right\}
\end{aligned}$$

$$B(c-2, n-4) \text{ on } \left\{ \frac{2}{q^3} \right\} \text{ is } n^{(6)} \frac{1}{3} + n^{(5)} \frac{19}{5} + n^{(4)} 9 = -f_4(n) + \frac{4!}{2!(2!)^2} (f_2(n))^2$$

$\in \mathcal{O}_2$

$\dots (4.16)$

$$A_4 = m b(c, n-1) A_{4,0} - B(c-2, n-4) \left\{ f_4(n) - \frac{4!}{2!(2!)^2} (f_2(n))^2 \right\}$$

$\dots (4.17)$

$$\begin{aligned}
A_{4,0} &= \frac{1}{q^3} [\bar{c}^{(5)} \frac{1}{3} + \bar{c}^{(4)} \frac{19}{5} + \bar{c}^{(3)} 9] - \frac{1}{p^3} [c^{(5)} \frac{1}{3} + c^{(4)} \frac{19}{5} + c^{(3)} 9] \\
&\quad + \frac{c}{p q^2} [\bar{c}^{(4)} \frac{1}{3} + \bar{c}^{(3)} \frac{9}{5}] - \frac{\bar{c}}{p^2 q} [c^{(4)} \frac{1}{3} + c^{(3)} \frac{9}{5}] \dots (4.18) \\
&= ((\bar{c}^2 + \bar{c} - 2)(\bar{c} - \frac{13}{5}) + \frac{24}{5}) \frac{1}{q^3} [\frac{\bar{c}^3 + \bar{c}^2}{3q} + \frac{c\bar{c}}{3p}] - \sim
\end{aligned}$$

よって (4.9) に よって

$$\begin{aligned}
 K_{2m} = mb(c, n-1) & \left\{ \frac{F_{2m}}{n^{(2)}} A_{0,0} + \frac{F_{2m} - \binom{2m}{2} F_{2m-2} f_2(n)}{n^{(4)}} A_{0,1} \right. \\
 & + \frac{F_{2m} - \binom{2m}{2} F_{2m-2} f_2(n) - \binom{2m}{4} F_{2m-4} \left[f_4(n) - \frac{4!}{2!(2!)} (f_2(n))^2 \right]}{n^{(6)}} A_{0,2} \\
 & + \left. \left(\binom{2m}{2} F_{2m-2} A_{2,0} + \binom{2m}{4} F_{2m-4} A_{4,0} \right) \right\} \\
 & + \left\{ F_{2m} - \binom{2m}{2} F_{2m-2} f_2(n) - \binom{2m}{4} F_{2m-4} \left[f_4(n) - \frac{4!}{2!(2!)} (f_2(n))^2 \right] \right\} \cdot \\
 & \quad \cdot B(c-3, n-6) \\
 & + \sum_{t=3}^m F_{2m-2t} \binom{2m}{2t} A_{2t} \quad \dots (4.19)
 \end{aligned}$$

よって K_4 は次のようになる。

$$\begin{aligned}
 K_4 = mb(c, n-1) & \left\{ \frac{n+1}{3} \left[n(n^2-1) + (n^2-1) \frac{18}{5} - \frac{24}{5} \right] A_{0,0} \right. \\
 & - \frac{n+1}{3} \left[n+1 + \frac{2}{5} \right] A_{0,1} \\
 & + \left. 2n(n^2-1) A_{2,0} + A_{4,0} \right\} \quad \dots (4.20)
 \end{aligned}$$

(4.20) は筆者が別⁽⁴⁾に求めたもの⁽⁴⁾と一致して

いる。

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むすび

できるだけ一般的に計算を進めたいと思っ

て工夫もしてみたが、今はこの程度にとまった。なお筆者が前に書いたもの<sup>(4)</sup>とは記号も計算の進め方もちがっているので、それと混同されるときは注意してほしい。

(計算機による取値例については、戸田英雄電子がやっている。

### 参考文献

(1) 山内：統計的品質管理（電気学会編，昭和28年5月）第6章抜取検査。p. 291.

(2) M.E. Wise: "A Quickly Convergent Expansion for Cumulative Hypergeometric Probabilities, Direct and Inverse," *Biometrika*, vol. 41 (1954) pp. 317~329

(3) G.J. Lieberman and ~~David~~ B. Owen: *Tables of the Hypergeometric Probability Distribution* (1961) (Stanford Univ. Press) pp. 16~22

(4) 山内：本文の同じ題目で近刊「応用統計学」に投稿中。

附録 1  $f_{2n}(n)$  の値

$$f_2 = (n+1)^{(3)} / 3$$

$$f_4 = \frac{3!}{5} \left\{ (n+2)^{(5)} + (n+1)^{(3)} \frac{5}{3} \right\}$$

$$f_6 = \frac{5!}{7} \left\{ (n+3)^{(7)} + (n+2)^{(5)} 7 + (n+1)^{(3)} \frac{7}{3} \right\}$$

$$f_8 = \frac{7!}{9} \left\{ (n+4)^{(9)} + (n+3)^{(7)} 9 \cdot 2 + (n+2)^{(5)} \frac{9 \cdot 21}{5} + (n+1)^{(3)} \frac{9}{3} \right\}$$

$$f_{10} = \frac{9!}{11} \left\{ (n+5)^{(11)} + (n+4)^{(9)} \frac{11 \cdot 10}{3} + (n+3)^{(7)} 11 \cdot 21 + (n+2)^{(5)} 11 \cdot 17 + (n+1)^{(3)} \frac{11}{3} \right\}$$

$$f_{12} = \frac{11!}{13} \left\{ (n+6)^{(13)} + (n+5)^{(11)} 13 \cdot 5 + (n+4)^{(9)} \frac{13 \cdot 11 \cdot 19}{3} + (n+3)^{(7)} \frac{13 \cdot 11 \cdot 128}{7} \right. \\ \left. + (n+2)^{(5)} \frac{13 \cdot 11 \cdot 31}{5} + (n+1)^{(3)} \frac{13}{3} \right\}$$

附録 2  $F_{2m}$  の値

$$F_2 = (n+1)^{(3)} / 3$$

$$F_4 = (n+3)^{(6)} / 3 + (n+3)^{(5)} / 5$$

$$F_6 = (n+5)^{(9)} 5 / 3^2 + (n+5)^{(8)} + (n+5)^{(7)} / 7$$

$$F_8 = (n+7)^{(12)} 35 / 3^3 + (n+7)^{(11)} 14 / 3 + (n+7)^{(10)} 41 / 15 \\ + (n+7)^{(9)} / 9$$

$$F_{10} = (n+9)^{(15)} 35 / 3^2 + (n+9)^{(14)} 70 / 3 + (n+9)^{(13)} 31 \\ + (n+9)^{(12)} 23 / 3 + (n+9)^{(11)} / 11$$

$$F_{12} = (n+11)^{(18)} 11 \cdot 7 \cdot 5 / 3^3 + (n+11)^{(17)} 11 \cdot 7 \cdot 5 / 3 + (n+11)^{(16)} 11 \cdot 83 / 3 \\ + (n+11)^{(15)} 11 \cdot 268 / 15 + (n+11)^{(14)} 157 / 7 \\ + (n+11)^{(13)} / 13$$



超幾何分布に於ける 山内の式

$$[記号] \quad p_d = \frac{\binom{Np_0}{d} \binom{N-Np_0}{n-d}}{\binom{N}{n}}, \quad (d=0, 1, \dots, n) \quad (1)$$

$$0 < Np_0 < N$$

$d=0, 1, \dots, n$  なる整数  $d$  に対して確率  $p_0, p_1, \dots, p_n$  が出現する.

$$P = \sum_{d=0}^c p_d \equiv H(N, n, Np_0, c) \quad (2)$$

$$p_d \sim \binom{n}{d} p_0^d (1-p_0)^{n-d} \equiv b(d, n) \quad (3)$$

$$P \sim \sum_{d=0}^c \binom{n}{d} p_0^d (1-p_0)^{n-d} \equiv B(c, n, p_0) \quad (4)$$

[山内の近似式 (1953)]

$$\left. \begin{aligned} P &= B(c, n, p_0) - n \binom{n-1}{c} p_0^c (1-p_0)^{n-c-1} \cdot \left[ \frac{A_1}{N} + \frac{A_2}{N^2} - \frac{A_1^3}{N^3 p_0 (1-p_0)} \right] \\ A_1 &= - (c - (n-1)p_0) / 2 \\ A_2 &= A_1^2 \left[ \frac{1}{6 p_0 (1-p_0)} + \frac{7}{6 p_0} \right] + A_1 \left[ \frac{3c}{4 p_0} + \frac{1}{6(1-p_0)} + \frac{1}{6} \right] \\ &\quad + \frac{c}{6} - \frac{c}{12 p_0} \end{aligned} \right\} \quad (5)$$

実用的に  $H(N, n, Np_0, c)$  を  $B(c, n, p_0)$  で近似するときの目安として  $\frac{1}{N}$  の

次の項:

$$\frac{n}{N} \frac{c - (n-1)p_0}{2} \binom{n-1}{c} p_0^c (1-p_0)^{n-1-c}$$

が近似の尺度となる

[Wick - 山田の式 (1971)]

$$p = \frac{Np_0 - \frac{c}{2}}{N - \frac{n-1}{2}}, \quad \left( N - \frac{n-1}{2} = M \text{ とおく} \right)$$

なる  $p$  を代入して  $M^{-2}$  の級数展開が得られた。

$$P = B(c, n, p)$$

$$\begin{aligned}
 & + \frac{n \cdot b(c, n-1, p)}{2! (2M)^2} \left\{ \frac{n+1}{3} [(n-1-c)g - cp] - \frac{(n-1-c)^2 + 2(n-1-c)}{3g} + \frac{c^2 + 2c}{3p} \right\} \\
 & \downarrow \text{山田} \\
 & + \frac{n \cdot b(c, n-1, p)}{4! (2M)^4} \left\{ \frac{n+1}{3} [(n-1-c)g - cp] \cdot \left[ n^2 - n + \frac{(n^2-1)18-24}{5} \right] \right. \\
 & \quad \left. + \left[ -\frac{(n-1-c)^2 + 2(n-1-c)}{3g} + \frac{c^2 + 2c}{3p} \right] 2(n^3 - n) \right. \\
 & \quad \left. + (n+1 + \frac{2}{5})(n+1) \cdot \left[ \frac{c(n-1-c)}{3g} \cdot (c-1) - \frac{c(n-1-c)}{3p} \cdot (n-1-c-1) \right] \right. \\
 & \quad \left. + \left( \left[ (n-1-c)^2 + 2(n-1-c) \right] (n-1-c - \frac{13}{5}) + \frac{24}{5} \right) \frac{1}{6^2} \left[ \frac{(n-1-c)^3 + 2(n-1-c)}{3g} + \frac{c(n-1-c)}{3p} \right] \right. \\
 & \quad \left. - \left( [c^2 + 2c] (c - \frac{13}{5}) + \frac{24}{5} \right) \frac{1}{p^2} \left[ \frac{c^2 + 2c}{3p} + \frac{c(n-1-c)}{3g} \right] \right\} \\
 & + \dots
 \end{aligned}$$

| N  | 項数 | $\exp(\ln \Gamma(N+1))$ | $N!$                   | $\frac{\Gamma(N+1)-N!}{N!}$ |
|----|----|-------------------------|------------------------|-----------------------------|
| 2  | 25 | 2000001123195979D+01    | 2000000000000000000+01 | 5615976740978831D-06        |
| 3  | 23 | 6000000016053141D+01    | 6000000000000000000+01 | 2675523423541415D-08        |
| 4  | 15 | 2399999999999564D+02    | 2400000000000000000+02 | 187834077111860D-12         |
| 5  | 13 | 1200000000000065D+03    | 1200000000000000000+03 | 5391057970408699D-13        |
| 6  | 11 | 7199999999999560D+03    | 7200000000000000000+03 | 6112394541131097D-13        |
| 7  | 11 | 503999999999949D+04     | 5040000000000000000+04 | 103289162507302D-13         |
| 8  | 11 | 403199999999930D+05     | 4032000000000000000+05 | 1741111664015328D-14        |
| 9  | 9  | 3628800000000099D+06    | 3628800000000000000+06 | 1895281893410150D-13        |
| 10 | 9  | 362880000000024D+07     | 3628800000000000000+07 | 6611681611764134D-14        |
| 11 | 9  | 399168000000009D+08     | 3991680000000000000+08 | 2235458357893051D-14        |
| 12 | 9  | 4790016000000002D+09    | 4790016000000000000+09 | 382540927953019D-15         |
| 13 | 9  | 622702079999996D+10     | 6227020300000000000+10 | 5928617880601521D-15        |
| 14 | 9  | 871782911999998D+11     | 8717829120000000000+11 | 1258709588373926D-14        |
| 15 | 9  | 130767436800002D+13     | 1307674368000000000+13 | 189801399363573D-14         |
| 16 | 9  | 209227898800003D+14     | 2092278988000000000+14 | 1669345644200852D-14        |
| 17 | 7  | 355687428095990D+15     | 3556874280960000000+15 | 2837540109732523D-14        |
| 18 | 7  | 6402373705727990D+16    | 6402373705728000000+16 | 1502128420182006D-14        |
| 19 | 7  | 1216451004088319D+18    | 121645100408332000+18  | 831825290178001D-15         |
| 20 | 7  | 2432902008176639D+19    | 243290200817664000+19  | 542561926272061D-15         |
| 21 | 7  | 5109094217170941D+20    | 510909421717094000+20  | 53383996924242D-15          |
| 22 | 7  | 112400072777607D+22     | 11240007277760800+22   | 6472878371160204D-15        |
| 23 | 7  | 258520167388495D+23     | 25852016738849800+23   | 8663512880336322D-15        |
| 24 | 7  | 6204484017332387D+24    | 620448401733239000+24  | 1170345314729657D-14        |
| 25 | 7  | 155112100433096D+26     | 15511210043309900+26   | 15588712635254D-14          |
| 26 | 7  | 403291461126604D+27     | 40329146112660500+27   | 184640644972072D-14         |
| 27 | 7  | 108888945041833D+29     | 10888894504183500+29   | 221909615191703D-14         |
| 28 | 7  | 304883446117130D+30     | 30488344611713900+30   | 267315234738445D-14         |
| 29 | 7  | 88417619937970D+31      | 88417619937970200+31   | 304220499071046D-14         |
| 30 | 7  | 2652528598121901D+33    | 26525285981219100+33   | 3461494721226618D-14        |
| 31 | 7  | 822283865417795D+34     | 822283865417792300+34  | 3643663155018681D-14        |
| 32 | 7  | 263130836933694D+36     | 26313083693369300+36   | 3432663155018681D-14        |
| 33 | 7  | 8683317618811915D+37    | 868331761881188700+37  | 322535353644924D-14         |
| 34 | 7  | 295232799039605D+39     | 29523279903960400+39   | 308634621650236D-14         |
| 35 | 7  | 1033314796638617D+41    | 103331479663861400+41  | 2735941532359694D-14        |
| 36 | 7  | 319933267899021D+42     | 31993326789901200+42   | 2486142049874794D-14        |
| 37 | 7  | 1376373509122638D+44    | 13763735091226300+44   | 219689140070408D-14         |
| 38 | 7  | 5230226174666021D+45    | 523022617466601100+45  | 1938309742910059D-14        |
| 39 | 7  | 2039788208119748D+47    | 203978820811974400+47  | 1619194489273398D-14        |
| 40 | 7  | 815915283247898D+48     | 81591528324789700+48   | 1273998839536251D-14        |
| 41 | 7  | 3345252661316384D+50    | 33452526613163100+50   | 1030110948422173D-14        |
| 42 | 7  | 1405006117752881D+52    | 14050061177528800+52   | 619469730352833D-15         |
| 43 | 7  | 6041526306337385D+53    | 604152630633738400+53  | 2867073888580512D-15        |
| 44 | 7  | 2658271574788449D+55    | 265827157478844900+55  | 5850405841969392D-16        |
| 45 | 7  | 119622208654801D+57     | 11962220865480200+57   | 397538883234512D-15         |
| 46 | 7  | 5502622159812085D+58    | 550262215981208900+58  | 8088677052610630D-15        |
| 47 | 7  | 2586232415111679D+60    | 258623241511168200+60  | 1134970925726716D-14        |
| 48 | 7  | 124139155923605D+62     | 124139155923607D+62    | 148969056680679D-14         |
| 49 | 7  | 6082818640342664D+63    | 608281864034876D+63    | 182890300432227D-14         |
| 50 | 7  | 3041409320171331D+65    | 3041409320172338D+65   | 2319329052372524D-14        |
| 51 | 7  | 155111875287378D+67     | 155111875287382D+67    | 265810102148830D-14         |
| 52 | 7  | 8065817517094363D+68    | 8065817517094388D+68   | 30396609130860D-14          |
| 53 | 7  | 42748324060010D+70      | 42748324060026D+70     | 3575780523895119D-14        |
| 54 | 7  | 2308436973392405D+72    | 2308436973392414D+72   | 406354439165213D-14         |
| 55 | 7  | 1269640335365822D+74    | 1269640335365828D+74   | 433275931794272D-14         |

□-1  $\exp(\ln \Gamma(N+1))$  と  $N!$  の比較

(2)

(1)

N 項数

 $\ln \Gamma(N+1)$ 

秒

 $\ln N + \ln(N-1) + \dots + \ln 2$ 

秒

 $(N) - (2)$  $\ln(N!)$ 

秒

171

|    |    |                       |    |                       |    |                        |                       |    |
|----|----|-----------------------|----|-----------------------|----|------------------------|-----------------------|----|
| 2  | 25 | 69314774215777710+00  | 10 | 69314718055994540+00  | 02 | 25615978316765730-06   | 69314718055994540+00  | 01 |
| 3  | 23 | 17917594719035780+01  | 08 | 17917594692280350+01  | 02 | 2675233297867580-08    | 17917594692280350+01  | 02 |
| 4  | 15 | 31780538303477640+01  | 07 | 31780538303477640+01  | 04 | -1819985134210670-12   | 31780538303477640+01  | 02 |
| 5  | 13 | 478749174278201000+01 | 07 | 478749174278201000+01 | 05 | 153651527665010690-13  | 478749174278201000+01 | 02 |
| 6  | 11 | 65792512120100400+01  | 06 | 65792512120100400+01  | 07 | -61360638792251620-13  | 65792512120100400+01  | 02 |
| 7  | 11 | 85251613610654040+01  | 05 | 85251613610654040+01  | 08 | -101933314483970-13    | 85251613610654040+01  | 02 |
| 8  | 11 | 10604602902745250+02  | 06 | 10604602902745250+02  | 09 | -20192381260370030-14  | 10604602902745250+02  | 03 |
| 9  | 9  | 12801827480081490+02  | 05 | 12801827480081490+02  | 10 | 1185962356247130-13    | 12801827480081490+02  | 03 |
| 10 | 9  | 1510441257307520+02   | 05 | 1510441257307520+02   | 12 | 6210310043986960-14    | 1510441257307520+02   | 03 |
| 11 | 9  | 17502307845873890+02  | 06 | 17502307845873890+02  | 13 | 18318679906315080-14   | 17502307845873890+02  | 03 |
| 12 | 9  | 19987214495661890+02  | 06 | 19987214495661890+02  | 14 | 7000000000000000+00    | 19987214495661890+02  | 03 |
| 13 | 9  | 2252163853123420+02   | 05 | 2252163853123420+02   | 15 | -9159339953157540-15   | 2252163853123420+02   | 04 |
| 14 | 9  | 25191221182738680+02  | 05 | 25191221182738680+02  | 16 | -1998801083243960-14   | 25191221182738680+02  | 04 |
| 15 | 9  | 27899271383840890+02  | 05 | 27899271383840890+02  | 18 | 11748601263784220-14   | 27899271383840890+02  | 04 |
| 16 | 9  | 30671860106080670+02  | 06 | 30671860106080670+02  | 19 | 148016565685892600-14  | 30671860106080670+02  | 04 |
| 17 | 7  | 33505073450136890+02  | 05 | 33505073450136890+02  | 20 | -31918911957973250-14  | 33505073450136890+02  | 04 |
| 18 | 7  | 3639544520803050+02   | 05 | 3639544520803050+02   | 21 | -19151347174783950-14  | 3639544520803050+02   | 05 |
| 19 | 7  | 39339884187199490+02  | 05 | 39339884187199490+02  | 23 | -11304512053934550-14  | 39339884187199490+02  | 04 |
| 20 | 7  | 42335616460753480+02  | 04 | 42335616460753480+02  | 24 | -10547118733938990-14  | 42335616460753480+02  | 04 |
| 21 | 7  | 4538013898476910+02   | 05 | 4538013898476910+02   | 25 | -10547118733938990-14  | 4538013898476910+02   | 04 |
| 22 | 7  | 48471181351835220+02  | 05 | 48471181351835220+02  | 26 | -11657341758564140-14  | 48471181351835220+02  | 05 |
| 23 | 7  | 5160667567764370+02   | 05 | 5160667567764370+02   | 28 | -13600232051658170-14  | 5160667567764370+02   | 05 |
| 24 | 7  | 54784729398112320+02  | 05 | 54784729398112320+02  | 29 | -16098233857064770-14  | 54784729398112320+02  | 05 |
| 25 | 7  | 58003605222980320+02  | 05 | 58003605222980320+02  | 30 | -19151347174783950-14  | 58003605222980320+02  | 06 |
| 26 | 7  | 61261701764100200+02  | 05 | 61261701764100200+02  | 31 | -212926904736346840-14 | 61261701764100200+02  | 05 |
| 27 | 7  | 64557538627006330+02  | 05 | 64557538627006330+02  | 32 | -244249654705340-14    | 64557538627006330+02  | 06 |
| 28 | 7  | 67889743137181530+02  | 05 | 67889743137181530+02  | 34 | -2773557561562890-14   | 67889743137181530+02  | 06 |
| 29 | 7  | 7125703897168010+02   | 05 | 7125703897168010+02   | 35 | -30531133177191800-14  | 7125703897168010+02   | 06 |
| 30 | 7  | 7465823634830160+02   | 05 | 7465823634830160+02   | 36 | -3330690738754080-14   | 7465823634830160+02   | 06 |
| 31 | 7  | 780922353315310+02    | 05 | 780922353315310+02    | 37 | -39412917374193060-14  | 780922353315310+02    | 06 |
| 32 | 7  | 8155795956115040+02   | 05 | 8155795956115040+02   | 39 | -36082248300317590-14  | 8155795956115040+02   | 06 |
| 33 | 7  | 85054467017581520+02  | 05 | 85054467017581520+02  | 40 | -327515792264120-14    | 85054467017581520+02  | 07 |
| 34 | 7  | 885082752197680+02    | 05 | 885082752197680+02    | 42 | -29976021664879230-14  | 885082752197680+02    | 06 |
| 35 | 7  | 92136175603687400+02  | 05 | 92136175603687400+02  | 44 | -27200464103316340-14  | 92136175603687400+02  | 08 |
| 36 | 7  | 95719694542143210+02  | 05 | 95719694542143210+02  | 46 | -2386979502944080-14   | 95719694542143210+02  | 07 |
| 37 | 7  | 99330612454787430+02  | 05 | 99330612454787430+02  | 48 | -19984014443252820-14  | 99330612454787430+02  | 07 |
| 38 | 7  | 10298819861451280+03  | 05 | 10298819861451280+03  | 49 | -16653345369377350-14  | 10298819861451280+03  | 07 |
| 39 | 7  | 10663176026064350+03  | 05 | 10663176026064350+03  | 51 | -13322676295501880-14  | 10663176026064350+03  | 08 |
| 40 | 7  | 11032063971475740+03  | 05 | 11032063971475740+03  | 52 | -99920072216264080-15  | 11032063971475740+03  | 08 |
| 41 | 7  | 11403421178143470+03  | 05 | 11403421178143470+03  | 53 | -72164496600635180-15  | 11403421178143470+03  | 08 |
| 42 | 7  | 117718139974310+03    | 05 | 117718139974310+03    | 54 | -38857805861380480-15  | 117718139974310+03    | 09 |
| 43 | 7  | 1215330815154360+03   | 05 | 1215330815154360+03   | 55 | -9000000000000000+00   | 1215330815154360+03   | 08 |
| 44 | 7  | 1251272114033550+03   | 05 | 1251272114033550+03   | 56 | -1277557561582490-15   | 1251272114033550+03   | 09 |
| 45 | 7  | 12912393363912720+03  | 05 | 12912393363912720+03  | 57 | -36613381477509380-15  | 12912393363912720+03  | 09 |
| 46 | 7  | 132525753516130+03    | 05 | 132525753516130+03    | 58 | -199820072216264080-15 | 132525753516130+03    | 09 |
| 47 | 7  | 13680272263732460+03  | 05 | 13680272263732460+03  | 59 | -135322676295501880-14 | 13680272263732460+03  | 10 |
| 48 | 7  | 14067392364232460+03  | 05 | 14067392364232460+03  | 60 | -15343123447452190-14  | 14067392364232460+03  | 09 |
| 49 | 7  | 1446574394634490+03   | 05 | 1446574394634490+03   | 61 | -1887379141822760-14   | 1446574394634490+03   | 10 |
| 50 | 7  | 1484776695177300+03   | 05 | 1484776695177300+03   | 62 | -22204404602490310-14  | 1484776695177300+03   | 10 |
| 51 | 7  | 15240959258449740+03  | 05 | 15240959258449740+03  | 63 | -2535129566378600-14   | 15240959258449740+03  | 09 |
| 52 | 7  | 15636083630307080+03  | 05 | 15636083630307080+03  | 64 | -2175557561528910-14   | 15636083630307080+03  | 10 |
| 53 | 7  | 1603511221663990+03   | 05 | 1603511221663990+03   | 65 | -3419646714129540-14   | 1603511221663990+03   | 10 |
| 54 | 7  | 16432011226315020+03  | 05 | 16432011226315020+03  | 66 | -3527136780005010-14   | 16432011226315020+03  | 11 |
| 55 | 7  | 1683274451442490+03   | 05 | 1683274451442490+03   | 67 | -43174750263755320-14  | 1683274451442490+03   | 10 |
| 56 | 7  | 1722276713511200+03   | 05 | 1722276713511200+03   | 68 | -44078219111507000-14  | 1722276713511200+03   | 10 |

|     |   |                      |     |                      |      |                         |                      |    |
|-----|---|----------------------|-----|----------------------|------|-------------------------|----------------------|----|
| 57  | 7 | 1763958494069973D+03 | .05 | 1763958494069973D+03 | .70  | -743229869798038111D-14 | 1683274454484277D+03 | 10 |
| 58  | 7 | 1804562914175438D+03 | .05 | 1804562914175438D+03 | .71  | -77735959003888173D-14  | 1683274454484277D+03 | 10 |
| 59  | 7 | 184338288614950D+03  | .05 | 184338288614950D+03  | .72  | -4884981308350689D-14   | 1683274454484277D+03 | 10 |
| 60  | 5 | 1886281734236716D+03 | .04 | 1886281734236716D+03 | .74  | -4996003610813204D-14   | 1683274454484277D+03 | 10 |
| 61  | 5 | 1927390472878449D+03 | .05 | 1927390472878449D+03 | .75  | -5440092820663287D-14   | 1683274454484277D+03 | 10 |
| 62  | 5 | 196866181672890D+03  | .05 | 196866181672890D+03  | .76  | -5662137425588298D-14   | 1683274454484277D+03 | 10 |
| 63  | 5 | 2010093163992815D+03 | .04 | 2010093163992815D+03 | .77  | -9325873406851315D-14   | 1683274454484277D+03 | 10 |
| 64  | 5 | 2051681994826412D+03 | .05 | 2051681994826412D+03 | .78  | -9214851108388799D-14   | 1683274454484277D+03 | 10 |
| 65  | 5 | 2093425867523368D+03 | .04 | 2093425867523368D+03 | .80  | -8770761894538737D-14   | 1683274454484277D+03 | 10 |
| 66  | 5 | 2135322414945633D+03 | .04 | 2135322414945633D+03 | .81  | -8437694987151190D-14   | 1683274454484277D+03 | 10 |
| 67  | 5 | 217369341139542D+03  | .04 | 217369341139542D+03  | .83  | -8215650382226158D-14   | 1683274454484277D+03 | 10 |
| 68  | 5 | 2219564418191303D+03 | .05 | 2219564418191303D+03 | .83  | -799360577301127D-14    | 1683274454484277D+03 | 10 |
| 69  | 5 | 2261905483237276D+03 | .04 | 2261905483237276D+03 | .84  | -766053869913580D-14    | 1683274454484277D+03 | 10 |
| 70  | 5 | 2304390435657770D+03 | .04 | 2304390435657770D+03 | .86  | -743849426498859D-14    | 1683274454484277D+03 | 10 |
| 71  | 5 | 2347017234428183D+03 | .05 | 2347017234428183D+03 | .87  | -721649660063518D-14    | 1683274454484277D+03 | 10 |
| 72  | 5 | 2389783895618343D+03 | .05 | 2389783895618343D+03 | .88  | -68838275267591D-14     | 1683274454484277D+03 | 10 |
| 73  | 5 | 243268849002927D+03  | .05 | 243268849002927D+03  | .89  | -6661338147750959D-14   | 1683274454484277D+03 | 10 |
| 74  | 5 | 2475729140961869D+03 | .04 | 2475729140961869D+03 | .91  | -621724893790087D-14    | 1683274454484277D+03 | 10 |
| 75  | 5 | 2518904022097232D+03 | .04 | 2518904022097232D+03 | .93  | -5995204332975845D-14   | 1683274454484277D+03 | 10 |
| 76  | 5 | 256221135550095D+03  | .05 | 256221135550095D+03  | .93  | -555115121212783D-14    | 1683274454484277D+03 | 10 |
| 77  | 5 | 2605649409718632D+03 | .04 | 2605649409718632D+03 | .95  | -529070518200751D-14    | 1683274454484277D+03 | 10 |
| 78  | 5 | 2649216497985528D+03 | .04 | 2649216497985528D+03 | .96  | -4884981308350689D-14   | 1683274454484277D+03 | 10 |
| 79  | 5 | 269210976510198D+03  | .04 | 269210976510198D+03  | .97  | -4884981308350689D-14   | 1683274454484277D+03 | 10 |
| 80  | 5 | 273673124285637D+03  | .05 | 273673124285637D+03  | .98  | -421884793375532D-14    | 1683274454484277D+03 | 10 |
| 81  | 5 | 2780675734403661D+03 | .04 | 2780675734403661D+03 | 1.00 | -399680288850564D-14    | 1683274454484277D+03 | 10 |
| 82  | 5 | 2824742926876304D+03 | .04 | 2824742926876304D+03 | 1.01 | -399680288850564D-14    | 1683274454484277D+03 | 10 |
| 83  | 5 | 2868931332954270D+03 | .04 | 2868931332954270D+03 | 1.02 | -377475828372532D-14    | 1683274454484277D+03 | 10 |
| 84  | 5 | 2913239500942703D+03 | .05 | 2913239500942703D+03 | 1.03 | -3108624468950438D-14   | 1683274454484277D+03 | 10 |
| 85  | 5 | 2957666013507606D+03 | .04 | 2957666013507606D+03 | 1.05 | -3108624468950438D-14   | 1683274454484277D+03 | 10 |
| 86  | 5 | 300220948670141D+03  | .04 | 300220948670141D+03  | 1.06 | -2664532525100316D-14   | 1683274454484277D+03 | 10 |
| 87  | 5 | 3046868567656687D+03 | .04 | 3046868567656687D+03 | 1.07 | -2220446049250313D-14   | 1683274454484277D+03 | 10 |
| 88  | 5 | 309164195801469D+03  | .05 | 309164195801469D+03  | 1.08 | -2220446049250313D-14   | 1683274454484277D+03 | 10 |
| 89  | 5 | 3136528299498791D+03 | .04 | 3136528299498791D+03 | 1.10 | -1776356839400250D-14   | 1683274454484277D+03 | 10 |
| 90  | 5 | 3181526396202093D+03 | .04 | 3181526396202093D+03 | 1.11 | -1554312234475219D-14   | 1683274454484277D+03 | 10 |
| 91  | 5 | 3226634991267262D+03 | .04 | 3226634991267262D+03 | 1.12 | -1332267629550186D-14   | 1683274454484277D+03 | 10 |
| 92  | 5 | 3271852877037752D+03 | .04 | 3271852877037752D+03 | 1.13 | -1110223024625157D-14   | 1683274454484277D+03 | 10 |
| 93  | 5 | 3317178871969285D+03 | .04 | 3317178871969285D+03 | 1.15 | -6661338147750939D-15   | 1683274454484277D+03 | 10 |
| 94  | 5 | 3362611819791985D+03 | .05 | 3362611819791985D+03 | 1.16 | -4440892098500626D-15   | 1683274454484277D+03 | 10 |
| 95  | 5 | 3408150588707990D+03 | .04 | 3408150588707990D+03 | 1.17 | -2220446049250313D-15   | 1683274454484277D+03 | 10 |
| 96  | 5 | 3453794070622669D+03 | .04 | 3453794070622669D+03 | 1.19 | -2220446049250313D-15   | 1683274454484277D+03 | 10 |
| 97  | 5 | 3499541180407702D+03 | .05 | 3499541180407702D+03 | 1.19 | -2220446049250313D-15   | 1683274454484277D+03 | 10 |
| 98  | 5 | 3545390855194408D+03 | .05 | 3545390855194408D+03 | 1.20 | -4440892098500626D-15   | 1683274454484277D+03 | 10 |
| 99  | 5 | 3591342053695754D+03 | .05 | 3591342053695754D+03 | 1.22 | -1881784197001252D-15   | 1683274454484277D+03 | 10 |
| 100 | 5 | 3637393755555635D+03 | .04 | 3637393755555635D+03 | 1.23 | -1110223024625157D-14   | 1683274454484277D+03 | 10 |

图-2  $\ln(\Gamma(Nn))$  と  $\ln N!$  の比較

10-7 N! 5  
overline

N= 100 SN= 40 C= 16

| NP0  | P1       | P2       | P3       |
|------|----------|----------|----------|
| 10.0 | .999996  | .999996  | .999995  |
| 15.0 | .999990  | .999990  | .999990  |
| 20.0 | .999977  | 1.000023 | .999918  |
| 25.0 | .9981362 | .9989400 | .9988530 |
| 30.0 | .9723940 | .9770684 | .9770343 |
| 35.0 | .8490185 | .8572767 | .8575598 |
| 40.0 | .5808146 | .5833520 | .5834712 |
| 45.0 | .2761124 | .2698345 | .2695708 |
| 50.0 | .0832474 | .0764569 | .0763262 |

TIME= .7000

N= 200 SN= 80 C= 32

| NP0   | P1       | P2       | P3       |
|-------|----------|----------|----------|
| 20.0  | .9999997 | .9999997 | .9999996 |
| 30.0  | .9999986 | .9999986 | .9999986 |
| 40.0  | .9999979 | .9999979 | .9999978 |
| 50.0  | .9999593 | .9999890 | .9999810 |
| 60.0  | .9947863 | .9962339 | .9961442 |
| 70.0  | .9054632 | .9129043 | .9130719 |
| 80.0  | .5574017 | .5591999 | .5592844 |
| 90.0  | .1627297 | .1551904 | .1549481 |
| 100.0 | .0179172 | .0149796 | .0150384 |

TIME= .8400

173

NP0 PE ER

|      |          |           |
|------|----------|-----------|
| 10.0 | .9999980 | .0000015  |
| 15.0 | .9999995 | -.0000005 |
| 20.0 | .9999944 | -.0000025 |
| 25.0 | .9988522 | .0000008  |
| 30.0 | .9770260 | .0000083  |
| 35.0 | .8575714 | -.0000116 |
| 40.0 | .5834769 | -.0000057 |
| 45.0 | .2695580 | .0000128  |
| 50.0 | .0763256 | .0000006  |

TIME= 1.0700

NP0 PE ER

|       |          |           |
|-------|----------|-----------|
| 20.0  | .9999987 | .0000010  |
| 30.0  | .9999955 | .0000031  |
| 40.0  | .9999948 | .0000031  |
| 50.0  | .9999845 | -.0000035 |
| 60.0  | .9961402 | .0000040  |
| 70.0  | .9130753 | -.0000034 |
| 80.0  | .5592884 | -.0000040 |
| 90.0  | .1549392 | .0000089  |
| 100.0 | .0150432 | -.0000048 |

TIME= 1.1400