

A REMARK ON CANNONE-KARCH SOLUTIONS TO THE HOMOGENEOUS BOLTZMANN EQUATION FOR MAXWELLIAN MOLECULES

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ABSTRACT. The purpose of this paper is to extend the result concerning the existence and the uniqueness of infinite energy solutions, given by Cannone-Karch, of the Cauchy problem for the spatially homogeneous Boltzmann equation of Maxwellian molecules without Grad's angular cutoff assumption in the mild singularity case, to the strong singularity case. This extension follows from a simple observation of the symmetry on the unit sphere for the Bobylev formula which is the Fourier transform of the Boltzmann collision term.

1. Introduction. We consider the Cauchy problem for the spatially homogeneous Boltzmann equation,

$$\partial_t f = Q(f, f), \quad f(0, v) = f_0(v), \quad (1)$$

where $f = f(t, v)$ is the density distribution function of particles with velocity $v \in \mathbb{R}^3$ at time t . The right hand side of (1) is given by the Boltzmann bilinear collision operator

$$Q(g, f)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma) \{g(v'_*)f(v') - g(v_*)f(v)\} d\sigma dv_*,$$

which is well-defined for suitable functions f and g (, even more probability measures,) specified later. Notice that the collision operator $Q(\cdot, \cdot)$ acts only on the velocity variable $v \in \mathbb{R}^3$. In the following discussion, we will use the σ -representation, that is, for $\sigma \in \mathbb{S}^2$,

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma,$$

which follow from the conservations of the moment and energy between pre- and post- collisional velocities. The nonnegative cross section $B(z, \sigma)$ depends only on $|z|$ and the scalar product $\frac{z}{|z|} \cdot \sigma$. For the physical model, it usually takes the form

$$B(|v - v_*|, \cos \theta) = \Phi(|v - v_*|)b(\cos \theta), \quad \cos \theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma, \quad 0 \leq \theta \leq \frac{\pi}{2},$$

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where

$$\Phi(|z|) = \Phi_\gamma(|z|) = |z|^\gamma, \text{ for some } \gamma > -3, \quad (2)$$

$$b(\cos \theta) \theta^{2+2s} \rightarrow K \text{ when } \theta \rightarrow 0+, \text{ for } 0 < s < 1 \text{ and } K > 0. \quad (3)$$

In fact, if the inter-molecule potential satisfies the inverse power law potential $U(\rho) = \rho^{-(q-1)}$, $q > 2$ (where ρ denotes the distance between two interacting molecules), then s and γ are given by

$$s = 1/(q-1) (< 1), \quad \gamma = 1 - 4s = 1 - 4/(q-1) (> -3).$$

In this physical model, we have $\gamma = 0$ if $s = 1/4$ (, which is called the Maxwell gas). Instead of this special case, in what follows we are interested in the case where

$$\gamma = 0, \quad 0 < s < 1,$$

which is called Maxwellian molecules type. Namely we assume $B = b(\cos \theta)$ throughout this paper.

The angle θ is the deviation angle, i.e., the angle between pre- and post- collisional velocities. The range of θ is a full interval $[0, \pi]$, but it is customary [11] to restrict it to $[0, \pi/2]$, replacing $b(\cos \theta)$ by its “symmetrized” version

$$[b(\cos \theta) + b(\cos(\pi - \theta))] \mathbf{1}_{0 \leq \theta \leq \pi/2},$$

which is possible due to the invariance of the product $f(v')f(v'_*)$ in the collision operator $Q(f, f)$ under the change of variables $\sigma \rightarrow -\sigma$. It should be noted that b is not integrable, that is,

$$\int_{\mathbb{S}^2} b\left(\frac{v - v_*}{|v - v_*|} \cdot \sigma\right) d\sigma = 2\pi \int_0^{\pi/2} b(\cos \theta) \sin \theta d\theta = \infty.$$

The case where $0 < s < 1/2$, that is, $\int_0^{\pi/2} \theta b(\cos \theta) \sin \theta d\theta < \infty$ is called the mild singularity, and another case $1/2 \leq s < 1$ is called the strong singularity.

In the study of the spatially homogeneous Boltzmann equation, it is natural to assume that the nonnegative initial datum satisfies

$$\int_{\mathbb{R}^3} f_0(v) dv = 1, \quad \int_{\mathbb{R}^3} v_j f_0(v) dv = 0, \quad j = 1, 2, 3, \quad (4)$$

$$\int_{\mathbb{R}^3} |v|^2 f_0(v) dv = 3, \quad (5)$$

because these relations are interpreted as the unit mass, the zero mean value and the unit temperature of the gas, respectively. The existence of a unique solution of the Cauchy problem (1) under the assumptions (4) and (5) is known (see [10, 9] and also [8]).

Recently Cannone-Karch [4] extended this result for the initial datum of a probability measure without (5) (, that is, which may have the infinite energy). Following [4] we introduce the set $\mathcal{K}$ of “characteristic functions” which are Fourier transforms of probability measures.

Definition 1.1. A function $\psi : \mathbb{R}^3 \rightarrow \mathbb{C}$ is called a characteristic function if there is a probability measure Ψ (i.e., a nonnegative Borel measure with $\int_{\mathbb{R}^3} d\Psi(v) = 1$) such that we have the identity $\psi(\xi) = \int_{\mathbb{R}^3} e^{-iv \cdot \xi} d\Psi(v)$. We denote the set of all characteristic functions by $\mathcal{K}$.

Inspired by [9], Cannone-Karch defined a subspace $\mathcal{K}^\alpha$ for $\alpha \geq 0$ as follows:

$$\mathcal{K}^\alpha = \{\varphi \in \mathcal{K}; \|\varphi - 1\|_\alpha < \infty\}, \quad (6)$$

where

$$\|\varphi - 1\|_\alpha = \sup_{\xi \in \mathbb{R}^3} \frac{|\varphi(\xi) - 1|}{|\xi|^\alpha}. \quad (7)$$

The space $\mathcal{K}^\alpha$ endowed with the distance

$$\|\varphi - \tilde{\varphi}\|_\alpha = \sup_{\xi \in \mathbb{R}^3} \frac{|\varphi(\xi) - \tilde{\varphi}(\xi)|}{|\xi|^\alpha} \quad (8)$$

is a complete metric space (see Proposition 3.10 of [4]). It follows that $\mathcal{K}^\alpha = \{1\}$ for all $\alpha > 2$ and the embeddings (Lemma 3.12 of [4])

$$\{1\} \subset \mathcal{K}^\alpha \subset \mathcal{K}^\beta \subset \mathcal{K}^0 = \mathcal{K} \quad \text{for all } 2 \geq \alpha \geq \beta \geq 0.$$

The space $\mathcal{K}^\alpha$ is natural because we have the following (Lemma 3.15 of [4]);

Lemma 1.2. *Let Ψ be a probability measure on $\mathbb{R}^3$ such that*

$$\begin{aligned} \exists \alpha \in (0, 2], \exists D > 0; \int |v|^\alpha d\Psi(v) \leq D, \\ \text{and moreover } \int v_j d\Psi(v) = 0, j = 1, 2, 3, \text{ when } \alpha > 1. \end{aligned} \quad (9)$$

Then the Fourier transform of Ψ , that is, $\psi(\xi) = \int e^{-iv \cdot \xi} d\Psi(v)$ belongs to $\mathcal{K}^\alpha$, more precisely

$$\sup_{\xi \in \mathbb{R}^3} \frac{|\psi(\xi) - 1|}{|\xi|^\alpha} \leq 3D. \quad (10)$$

The inverse of the lemma does not hold, in fact, the space $\mathcal{K}^\alpha$ is bigger than the set of the Fourier transforms of probability measures satisfying (10) (Remark 3.16 of [4]).

Before stating an extension of the main result given by [4], we introduce the condition

$$\exists \alpha_0 \in (0, 2] \quad \text{such that } (\sin \theta/2)^{\alpha_0} b(\cos \theta) \sin \theta \in L^1((0, \pi/2]), \quad (11)$$

which is fulfilled for b with (3) if $2s < \alpha_0$. If we consider the Cauchy problem (1) for the initial datum of the probability measure $\Psi_0(v)$ and if we set $\psi_0(\xi) = \int_{\mathbb{R}^3} e^{-iv \cdot \xi} d\Psi_0(v)$ and denote the Fourier transform of the probability measure solution by $\psi(t, v)$ then it follows from the Bobylev formula that the Cauchy problem (1) is reduced to

$$\begin{cases} \partial_t \psi(t, \xi) = \int_{\mathbb{S}^2} b\left(\frac{\xi \cdot \sigma}{|\xi|}\right) \left(\psi(t, \xi^+) \psi(t, \xi^-) - \psi(t, \xi) \psi(t, 0)\right) d\sigma, \\ \psi(0, \xi) = \psi_0(\xi), \text{ where } \xi^\pm = \frac{\xi}{2} \pm \frac{|\xi|}{2} \sigma. \end{cases} \quad (12)$$

Theorem 1.3. *Assume that b satisfies (11) for some $\alpha_0 \in (0, 2]$. Then for each $\alpha \in [\alpha_0, 2]$ and every $\psi_0 \in \mathcal{K}^\alpha$ there exists a classical solution $\psi \in C([0, \infty), \mathcal{K}^\alpha)$ of the Cauchy problem (12). The solution is unique in the space $C([0, \infty), \mathcal{K}^{\alpha_0})$. Furthermore, if $\alpha \in [\alpha_0, 2]$ and if $\psi(t, \xi), \varphi(t, \xi) \in C([0, \infty), \mathcal{K}^\alpha)$ are two solutions*

to the Cauchy problem (12) with initial data $\psi_0, \varphi_0 \in \mathcal{K}^\alpha$, respectively, then for any $t > 0$ we have

$$\|\psi(t) - \varphi(t)\|_\alpha \leq e^{\lambda_\alpha t} \|\psi_0 - \varphi_0\|_\alpha, \quad (13)$$

where

$$\lambda_\alpha = 2\pi \int_0^{\pi/2} b(\cos \theta) \left\{ \cos^\alpha \frac{\theta}{2} + \sin^\alpha \frac{\theta}{2} - 1 \right\} \sin \theta d\theta. \quad (14)$$

Remark 1. The assumption (11) with $\alpha = \alpha_0$ can be written as

$$(1 - \tau)^{\alpha_0/2} b(\tau) \in L^1([0, 1]), \quad (15)$$

by means of the change of variable $\tau = \cos \theta$. Theorem 1.3 ameliorates Theorem 2.2 of [4], where (15) is assumed with $\alpha_0/2$ replaced by $\alpha_0/4$, see (2.6) of [4] and note that b is now “symmetrized”. Hence Theorem 2.2 of [4] covers only the non-cutoff Boltzmann equation of the mild singularity case, and requires the space $\mathcal{K}^\alpha$ with $\alpha > 1$ to find a solution for the Maxwell gas ($s=1/4$).

At the end of this introduction we remark the smoothing effect for the homogeneous non-cutoff Boltzmann equation for Maxwellian molecules. We can not always expect the smoothing effect for solutions in the space of the probability measures whose Fourier transforms are in $\mathcal{K}^\alpha$, because $\mathcal{K}^\alpha$ contains, 1, which is the Fourier transform of the Dirac mass. However we have a result of the smoothing effect for a certain sum of Dirac masses as follows:

Example 1.4. Let $\mathbf{b}_k$, $k = 1, \dots, 4$, be four points in $\mathbb{R}^3$ which are not on the same plane. Put

$$F_0 = \sum_{k=1}^4 m_k \delta(v - \mathbf{b}_k),^1$$

where $m_k > 0$ with $\sum_{k=1}^4 m_k = 1$. If b satisfies (3) with $0 < s < 1$ and if $f(t, v)$ is a unique solution of (1) with the initial datum F_0 then $f(t, v) \in H^\infty(\mathbb{R}^3)$ for any $t > 0$.

To show this fact, we recall the following proposition concerning H^∞ -smoothing effect given in [7] (see also [2]).

Proposition 1.5. Assume that b satisfies (3) with $0 < s < 1$ and let $\psi(t, \xi) \in C([0, \infty), \mathcal{K}^\alpha)$ with $\alpha \in (2s, 2]$ be a unique solution of (12). If for any $T > 0$ there exists a constant $D_T > 0$ such that the solution $\psi(t, \xi)$ satisfies

$$\inf_{t \in [0, T]} \left(1 - |\psi(t, \xi)| \right) \geq D_T \min\{1, |\xi|^2\}, \quad (16)$$

then the inverse Fourier transform of $\psi(t, \xi)$ belongs to $H^\infty(\mathbb{R}^3)$ for any $t \in (0, T]$.

Remark 2. If $f(t, v) \in C([0, T], L^1)$ satisfies a certain moment condition and the entropy condition, that is,

$$\exists \delta > 0; \int_{\mathbb{R}^3} |v|^\delta f(t, v) dv < \infty, \quad \int_{\mathbb{R}^3} f(t, v) \log(1 + f(t, v)) dv < \infty, \quad (17)$$

uniformly in $[0, T]$ then the Fourier transform of f satisfies (16) (see Lemma 3 of [1]). The inequality (16) is a key for the coercive estimate (see (2.2) of [7] and (24) below). We refer the reader to [7, 5, 3] and references therein for further results about the smoothing effect for the spatially homogeneous non-cutoff Boltzmann equation.

¹suggested by C. Villani[12], whose more general conjecture has been proved recently in [13].

The plan of this paper is as follows: The proof of Theorem 1.3 will be given in the next section. In the last section 3 we will give a short proof of Proposition 1.5 for the convenience of the reader, and show H^∞ smoothing for Example 1.4.

2. Proof of Theorem 1.3. We first show the uniqueness part of the theorem since the construction of the solution can be done by the quite same way as in [4], by using the cut-off approximation, together with the uniform estimate (20) below.

Lemma 2.1. *For any characteristic function $\varphi \in \mathcal{K}$ we have*

$$|\varphi(\xi)\varphi(\eta) - \varphi(\xi + \eta)|^2 \leq (1 - |\varphi(\xi)|^2)(1 - |\varphi(\eta)|^2) \quad (18)$$

for all $\xi, \eta \in \mathbb{R}^3$ and moreover if $\varphi \in \mathcal{K}^\alpha$ then

$$|\varphi(\xi) - \varphi(\xi + \eta)| \leq \|\varphi - 1\|_\alpha \left(4|\xi|^{\alpha/2}|\eta|^{\alpha/2} + |\eta|^\alpha \right). \quad (19)$$

Proof. For the proof of (18) we refer to (3.5) of [4] and also Lemma 3.5.10 of [6]. Using (18) we show (19). For any $z \in \mathbb{C}$ with $|z| \leq 1$ we have

$$1 - |z|^2 = (1 + \bar{z})(1 - z) + 2i \operatorname{Im} z \leq 2|1 - z| + 2|1 - z| = 4|1 - z|.$$

Since $|\varphi| \leq 1$, it follows from (18) that

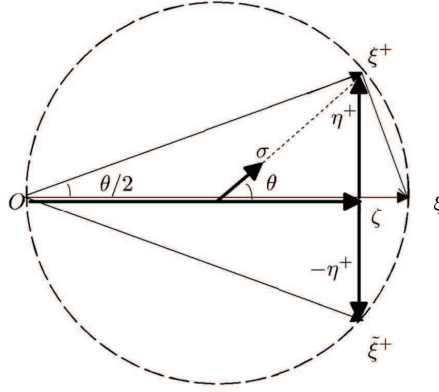
$$\begin{aligned} |\varphi(\xi) - \varphi(\xi + \eta)| &\leq |\varphi(\xi)\varphi(\eta) - \varphi(\xi + \eta)| + |\varphi(\xi)||\varphi(\eta) - 1| \\ &\leq 4|1 - \varphi(\xi)|^{1/2}|1 - \varphi(\eta)|^{1/2} + |\varphi(\eta) - 1| \\ &\leq 4\|1 - \varphi\|_\alpha^{1/2}|\xi|^{\alpha/2}\|1 - \varphi\|_\alpha^{1/2}|\eta|^{\alpha/2} + \|\varphi - 1\|_\alpha|\eta|^\alpha, \end{aligned}$$

which gives the desired estimate (19). $\square$

Lemma 2.2. *Assume that b satisfies (11) for some $\alpha_0 \in (0, 2]$. If $\varphi \in \mathcal{K}^\alpha$ for $\alpha \in [\alpha_0, 2]$ then*

$$\begin{aligned} &\left| \int_{\mathbb{S}^2} b\left(\frac{\xi \cdot \sigma}{|\xi|}\right) (\varphi(\xi^+) \varphi(\xi^-) - \varphi(\xi)) d\sigma \right| \\ &\leq 2^4 \pi \left(\int_0^1 (1 - \tau)^{\alpha/2} b(\tau) d\tau \right) \|1 - \varphi\|_\alpha |\xi|^\alpha. \end{aligned} \quad (20)$$

Proof. Put $\zeta = \left(\xi^+ \cdot \frac{\xi}{|\xi|} \right) \frac{\xi}{|\xi|}$ and consider $\tilde{\xi}^+ = \zeta - (\xi^+ - \zeta)$, which is symmetric to ξ^+ on $\mathbb{S}^2$, see Figure 1.

FIGURE 1. $\cos \theta = \frac{\xi \cdot \sigma}{|\xi|}$, $\eta^+ = \xi^+ - \zeta$

We divide the left hand side of (20) into

$$\begin{aligned}
 & \int_{\mathbb{S}^2} b \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \left(\varphi(\xi^+) \varphi(\xi^-) - \varphi(\xi) \right) d\sigma \\
 &= \int_{\mathbb{S}^2} b \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \left(\varphi(\xi^+) - \varphi(\xi) \right) d\sigma + \int_{\mathbb{S}^2} b \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \varphi(\xi^+) \left(\varphi(\xi^-) - 1 \right) d\sigma \\
 &= \frac{1}{2} \int_{\mathbb{S}^2} b \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \left(\varphi(\xi^+) + \varphi(\tilde{\xi}^+) - 2\varphi(\xi) \right) d\sigma \\
 &\quad + \int_{\mathbb{S}^2} b \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \varphi(\xi^+) \left(\varphi(\xi^-) - 1 \right) d\sigma \\
 &= \frac{1}{2} \int_{\mathbb{S}^2} b \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \left(\varphi(\xi^+) + \varphi(\tilde{\xi}^+) - 2\varphi(\zeta) \right) d\sigma \\
 &\quad + \int_{\mathbb{S}^2} b \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \left(\varphi(\zeta) - \varphi(\xi) \right) d\sigma + \int_{\mathbb{S}^2} b \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \varphi(\xi^+) \left(\varphi(\xi^-) - 1 \right) d\sigma \\
 &= I_1 + I_2 + I_3.
 \end{aligned}$$

Putting $\eta^+ = \xi^+ - \zeta$, we have

$$\begin{aligned}
 |\varphi(\xi^+) + \varphi(\tilde{\xi}^+) - 2\varphi(\zeta)| &= \left| \int_{\mathbb{R}^3} e^{-i\zeta \cdot v} \left(e^{-i\eta^+ \cdot v} + e^{i\eta^+ \cdot v} - 2 \right) d\Psi(v) \right| \\
 &\leq \int_{\mathbb{R}^3} |e^{-i\zeta \cdot v}| \left(2 - e^{-i\eta^+ \cdot v} - e^{i\eta^+ \cdot v} \right) d\Psi(v) \\
 &= 2 - \varphi(\eta^+) - \varphi(-\eta^+) \\
 &\leq 2\|1 - \varphi\|_\alpha |\eta^+|^\alpha \leq 2\|1 - \varphi\|_\alpha \left(|\xi| \sin(\theta/2) \right)^\alpha
 \end{aligned}$$

because $|\eta^+| = |\xi^+| \sin(\theta/2)$. Hence

$$|I_1| \leq 4\pi \|1 - \varphi\|_\alpha |\xi|^\alpha \int_0^{\pi/2} \sin^\alpha(\theta/2) b(\cos \theta) \sin \theta d\theta.$$

Thanks to (19) with $\eta = \zeta - \xi$ we have

$$|I_2| \leq 10\pi \|1 - \varphi\|_\alpha |\xi|^\alpha \int_0^{\pi/2} \sin^\alpha(\theta/2) b(\cos \theta) \sin \theta d\theta$$

because $|\zeta - \xi| = |\xi| \sin^2(\theta/2)$. Since the similar estimate for I_3 holds, we obtain the desired estimate (20). $\square$

From the proof of the above lemma, we have

Corollary 2.3. *Let b and φ be the same as in Lemma 2.2. If for any $\varepsilon > 0$ we put*

$$\Omega_\varepsilon = \Omega_\varepsilon(\xi) = \left\{ \sigma \in \mathbb{S}^2; 1 - \frac{\xi}{|\xi|} \cdot \sigma \leq 2 \left(\frac{\varepsilon}{\pi} \right)^2 \right\}$$

and

$$R_\varepsilon(\xi) = \int_{\mathbb{S}^2 \cap \Omega_\varepsilon} b \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \frac{(\varphi(\xi^+) \varphi(\xi^-) - \varphi(\xi))}{|\xi|^\alpha} d\sigma,$$

then it follows that

$$|R_\varepsilon(\xi)| \leq 16\pi \|1 - \varphi\|_\alpha |\xi|^\alpha \int_0^\varepsilon \sin^\alpha(\theta/2) b(\cos \theta) \sin \theta d\theta \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Proof of the uniqueness. Let $\psi(t, \xi), \varphi(t, \xi) \in C([0, \infty), \mathcal{K}^\alpha)$ with $\alpha \geq \alpha_0$ be solutions to the Cauchy problem (12) with the initial data $\psi_0(\xi), \varphi_0(\xi)$, respectively.

Putting $h(t, \xi) = \frac{\psi(t, \xi) - \varphi(t, \xi)}{|\xi|^\alpha}$, we have

$$\begin{aligned} \partial_t h(t, \xi) &= \int_{\mathbb{S}^2 \cap \Omega_\varepsilon^c} b \left(\frac{\xi}{|\xi|} \cdot \sigma \right) \frac{\{\psi(t, \xi^-) \psi(t, \xi^+) - \varphi(t, \xi^-) \varphi(t, \xi^+)\}}{|\xi|^\alpha} d\sigma \\ &\quad - \left(\int_{\mathbb{S}^2 \cap \Omega_\varepsilon} b \left(\frac{\xi}{|\xi|} \cdot \sigma \right) d\sigma \right) h(t, \xi) \\ &\quad + \int_{\mathbb{S}^2 \cap \Omega_\varepsilon} b \left(\frac{\xi}{|\xi|} \cdot \sigma \right) \frac{\{\psi(t, \xi^-) \psi(t, \xi^+) - \psi(t, 0) \psi(t, \xi)\}}{|\xi|^\alpha} d\sigma \\ &\quad - \int_{\mathbb{S}^2 \cap \Omega_\varepsilon} b \left(\frac{\xi}{|\xi|} \cdot \sigma \right) \frac{\{\varphi(t, \xi^-) \varphi(t, \xi^+) - \varphi(t, 0) \varphi(t, \xi)\}}{|\xi|^\alpha} d\sigma \\ &= I_\varepsilon(t, \xi) - S_\varepsilon h(t, \xi) + R_{\varepsilon, \psi}(t, \xi) - R_{\varepsilon, \varphi}(t, \xi), \end{aligned}$$

where we have put

$$S_\varepsilon = 2\pi \int_{[0, \pi/2] \cap \{\sin \theta/2 > \varepsilon/\pi\}} b(\cos \theta) \sin \theta d\theta,$$

and it may diverge as $\varepsilon \rightarrow +0$. Let $R > 0$ and note that for any ξ with $|\xi| \leq R$ we have

$$\begin{aligned} &\left| \frac{\{\psi(t, \xi^-) \psi(t, \xi^+) - \varphi(t, \xi^-) \varphi(t, \xi^+)\}}{|\xi|^\alpha} \right| \\ &= \left| \psi(t, \xi^-) h(t, \xi^+) \frac{|\xi^+|^\alpha}{|\xi|^\alpha} + \varphi(t, \xi^+) h(t, \xi^-) \frac{|\xi^-|^\alpha}{|\xi|^\alpha} \right| \\ &\leq \left(\sup_{|\xi| \leq R} |h(t, \xi)| \right) \frac{|\xi^+|^\alpha + |\xi^-|^\alpha}{|\xi|^\alpha} \end{aligned}$$

because $|\xi^\pm| \leq |\xi|$. If we put $H_R(t) = \sup_{|\xi| \leq R} |h(t, \xi)|$, then for any ξ with $|\xi| \leq R$ we get

$$|I_\varepsilon(t, \xi)| \leq \gamma_{\alpha, \varepsilon} H_R(t),$$

where

$$\gamma_{\alpha, \varepsilon} = 2\pi \int_{[0, \pi/2] \cap \{\sin \frac{\theta}{2} > \varepsilon/\pi\}} b(\cos \theta) \left\{ \cos^\alpha \frac{\theta}{2} + \sin^\alpha \frac{\theta}{2} \right\} \sin \theta d\theta.$$

Notice that

$$\gamma_{\alpha, \varepsilon} - S_\varepsilon \nearrow \lambda_\alpha = 2\pi \int_0^{\pi/2} b(\cos \theta) \left\{ \cos^\alpha \frac{\theta}{2} + \sin^\alpha \frac{\theta}{2} - 1 \right\} \sin \theta d\theta \quad (\varepsilon \rightarrow +0).$$

Since $\psi(t, \xi), \varphi(t, \xi) \in C([0, \infty), \mathcal{K}^\alpha)$, it follows from Corollary 2.3 that for any fixed $T > 0$

$$\sup_{t \in (0, T], \xi} \left(|R_{\varepsilon, \psi}(t, \xi)| + |R_{\varepsilon, \varphi}(t, \xi)| \right) = r_\varepsilon \rightarrow 0, \quad \varepsilon \rightarrow +0.$$

Finally, if $|\xi| \leq R$, then we have

$$|\partial_t h(t, \xi) + S_\varepsilon h(t, \xi)| \leq \gamma_{\alpha, \varepsilon} H_R(t) + r_\varepsilon.$$

and hence

$$H_R(t) \leq e^{(\gamma_{\alpha, \varepsilon} - S_\varepsilon)t} H_R(0) + \frac{r_\varepsilon}{\gamma_{\alpha, \varepsilon} - S_\varepsilon} \left(e^{(\gamma_{\alpha, \varepsilon} - S_\varepsilon)t} - 1 \right).$$

Taking the limit $\varepsilon \rightarrow 0$ we obtain the stability estimate

$$\sup_{|\xi| \leq R} \frac{|\psi(t, \xi) - \varphi(t, \xi)|}{|\xi|^\alpha} \leq e^{\lambda_\alpha t} \sup_{|\xi| \leq R} \frac{|\psi_0(\xi) - \varphi_0(\xi)|}{|\xi|^\alpha} \quad (21)$$

which corresponds to (4.21) of [4]. Letting $R \rightarrow \infty$ we get (13). Now the proof of the uniqueness is complete. $\square$

Proof of the existence. Consider the increasing sequence of bounded collision kernels

$$b_n(\cos \theta) = \min\{b(\cos), n\} \leq b(\cos \theta), \quad n \in \mathbb{N},$$

and let $\psi_n(t, \xi) \in C([0, \infty), \mathcal{K}^\alpha)$ be a solution of the Cauchy problem (12) for the initial datum $\psi_0(\xi) \in \mathcal{K}^\alpha$ with b replaced by the cutoff b_n , given in Section 4 of [4]. If $\alpha \in [\alpha_0, 2]$ then it follows from (13) that

$$\|1 - \psi_n(t)\|_\alpha \leq e^{\lambda_\alpha t} \|1 - \psi_0\|_\alpha. \quad (22)$$

The equi-continuity in time is a direct consequence of (20) because we have for $0 \leq t < t' \leq T$

$$\begin{aligned} & |\psi_n(t, \xi) - \psi_n(t', \xi)| \\ & \leq \int_t^{t'} \left| \int_{\mathbb{S}^2} b_n \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \left(\psi_n(\rho, \xi^+) \psi_n(\rho, \xi^-) - \psi_n(\rho, \xi) \right) d\sigma \right| d\rho \\ & \lesssim \left(\int_0^1 (1 - \tau)^{\alpha/2} b(\tau) d\tau \right) |\xi|^\alpha \left(\max_{\rho \in [0, T]} \|1 - \psi_n(\rho)\|_\alpha \right) |t - t'| \\ & \lesssim \left(\int_0^1 (1 - \tau)^{\alpha/2} b(\tau) d\tau \right) |\xi|^\alpha e^{\lambda_\alpha T} \|1 - \psi_0\|_\alpha |t - t'|. \end{aligned} \quad (23)$$

The equi-continuity in ξ variable follows from (19) and (22). Since $|\psi_n(t, \xi)| \leq 1$, the Ascoli-Arzelà theorem gives a convergent subsequence $\{\psi_{n_k}(t)\}_{k=1}^\infty$. By means of (20) and its proof, the Lebesgue convergence theorem shows the limit $\psi(t, \xi) = \lim_{k \rightarrow \infty} \psi_{n_k}(t, \xi)$ is a solution of (12). $\square$

3. Proof of Proposition 1.5. The proof of the proposition is quite the same as in [7]. We use the time dependent weight function

$$M_\delta(t, \xi) = \langle \xi \rangle^{Nt-4} \langle \delta \xi \rangle^{-2N_0}, \quad \langle \xi \rangle^2 = 1 + |\xi|^2,$$

where $N_0 = NT/2 + 2$, $N \in \mathbb{N}$ and $\delta > 0$. We multiply $M_\delta(t, \xi)^2 \overline{\psi(t, \xi)}$ by the first equation of (12) and integrate with respect to ξ over $\mathbb{R}^3$. Denote $\psi^\pm = \psi(t, \xi^\pm)$ and $M^+ = M_\delta(t, \xi^+)$ to simplify the notation and note that

$$\begin{aligned} -2\operatorname{Re} \left\{ (\psi^+ \psi^- - \psi) M^2 \overline{\psi} \right\} &= \left(|M\psi|^2 + |M^+ \psi^+|^2 - 2\operatorname{Re} \left\{ \psi^- (M^+ \psi^+) \overline{M\psi} \right\} \right) \\ &\quad + \left(|M\psi|^2 - |M^+ \psi^+|^2 \right) + 2\operatorname{Re} \left\{ \psi^- ((M - M^+) \psi^+) \overline{M\psi} \right\} \\ &= J_1 + J_2 + J_3. \end{aligned}$$

Using the Cauchy-Schwarz inequality for the third term of J_1 we have, by (16),

$$J_1 \geq (1 - |\psi^-|) \left(|M\psi|^2 + |M^+ \psi^+|^2 \right) \geq D_T \min\{1, |\xi^-|^2\} |M\psi|^2,$$

so that

$$\begin{aligned} \int_{\mathbb{R}^3 \times \mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) J_1 d\sigma d\xi &\gtrsim \int_{\mathbb{R}^3} |M\psi|^2 \left(\int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \min\{1, |\xi^-|^2\} d\sigma \right) d\xi \\ &\gtrsim \int_{\mathbb{R}^3 \setminus \{|\xi| \leq 2\}} |\xi|^{2s} |M\psi|^2 d\xi \end{aligned} \quad (24)$$

because if $|\xi| > 2$ then it follows from (3) that

$$\int_0^{|\xi|^{-1}} b(\cos \theta) \sin \theta \min\{1, |\xi|^2 \sin^2(\theta/2)\} d\theta \sim |\xi|^{2s}.$$

If we use the change of variable $\xi \rightarrow \xi^+$ for the term $M^+ \psi^+$ in J_2 , similar to the cancellation lemma (Lemma 1 of [1]), we have

$$\begin{aligned} &\left| \int_{\mathbb{R}^3 \times \mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) J_2 d\sigma d\xi \right| \\ &= 2\pi \left| \int_{\mathbb{R}^3} |M\psi|^2 \left(\int_0^{\pi/2} b(\cos \theta) \sin \theta \left(1 - \frac{1}{\cos^3(\theta/2)}\right) d\theta \right) d\xi \right| \\ &\lesssim \int_{\mathbb{R}^3} |M\psi|^2 d\xi. \end{aligned}$$

Since $|M - M^+| \lesssim \sin^2(\theta/2) M^+$ (see (3.4) of [7]), by the Cauchy-Schwarz inequality we also have the same upper bound estimate for J_3 by using again the change of variable $\xi \rightarrow \xi^+$ for the term including $M^+ \psi^+$. Since

$$2\operatorname{Re} \left(\frac{\partial \psi}{\partial t} M^2 \overline{\psi} \right) = \frac{\partial |M\psi|^2}{\partial t} - 2N \log \langle \xi \rangle |M\psi|^2,$$

and since $|\xi|^{2s} / \log \langle \xi \rangle \rightarrow \infty$ as $|\xi| \rightarrow \infty$, we have the Gronwall inequality

$$\frac{d}{dt} \int_{\mathbb{R}^3} |M_\delta(t, \xi) \psi(t, \xi)|^2 d\xi \lesssim \int_{\mathbb{R}^3} |M_\delta(t, \xi) \psi(t, \xi)|^2 d\xi,$$

which gives for $t \in (0, T]$

$$\int_{\mathbb{R}^3} |\langle \xi \rangle^{Nt-4} (1 + \delta |\xi|^2)^{-N_0} \psi(t, \xi)|^2 d\xi \lesssim \int_{\mathbb{R}^3} |\langle \xi \rangle^{-4} \psi_0(\xi)|^2 d\xi.$$

Letting $\delta \rightarrow 0$ we obtain the conclusion of the proposition since we can take an arbitrarily large N .

We show the H^∞ smoothness of the solution in Example 1.4. Let ψ_0 and $\psi(t)$ be the Fourier transforms of F_0 and $f(t)$, respectively. By the translation we may assume that $\int v_j dF_0(v) = 0$, $j = 1, 2, 3$, which yield $\sum_k m_k \mathbf{b}_k = 0$. Hence

$$\begin{aligned} 1 - \psi_0(\xi) &= 2 \sum_{k=1}^4 m_k \sin^2 \frac{\mathbf{b}_k \cdot \xi}{2} \\ &\quad + 2i \sum_{k=1}^4 m_k \left\{ \left(\frac{\mathbf{b}_k \cdot \xi}{2} - \sin \frac{\mathbf{b}_k \cdot \xi}{2} \right) + 2 \sin \frac{\mathbf{b}_k \cdot \xi}{2} \sin^2 \frac{\mathbf{b}_k \cdot \xi}{4} \right\}, \end{aligned}$$

so that $\psi_0 \in \mathcal{K}^2$ and $\psi(t) \in C([0, \infty); \mathcal{K}^2)$. On the other hand, if $\beta_k = \mathbf{b}_k \cdot \xi$ satisfy $|\beta_k| \leq \pi/4$, $k = 1, \dots, 4$ then the direct calculation gives

$$\begin{aligned} |\psi_0(\xi)|^2 &= 1 - 4 \sum_{k < \ell} m_k m_\ell (\sin \beta_k - \sin \beta_\ell)^2 \\ &\quad + 8 \sum_{k < \ell} m_k m_\ell \sin \beta_k \sin \beta_\ell (\sin \beta_k \sin \beta_\ell + \cos \beta_k \cos \beta_\ell - 1), \end{aligned}$$

and

$$\begin{aligned} 1 - |\psi_0(\xi)| &\geq 8 \sum_{k < \ell} m_k m_\ell \sin^2 \frac{\beta_k - \beta_\ell}{2} \cos^2 \frac{\beta_k + \beta_\ell}{2} + O(|\xi|^4) \\ &\geq \frac{4}{\pi^2} \sum_{k < \ell} m_k m_\ell |\beta_k - \beta_\ell|^2 + O(|\xi|^4) \gtrsim |\xi|^2, \quad (|\xi| \rightarrow 0), \end{aligned}$$

where the last inequality follows from the hypothesis for four points. Since we have $|\psi(t, \xi) - \psi_0(\xi)| \lesssim t|\xi|^2$ by the similar way as in (23), we get (16) for a sufficiently small $T > 0$. Consequently, for a $t_0 > 0$ $f(t_0, v) \in H^\infty$ and $\in L^\infty$, which yields $\int f(t_0, v) \log(1 + f(t_0, v)) dv < \infty$. Once we could show the energy conservation

$$\int |v|^2 f(t_0, v) dv = \int |v|^2 F_0(dv) = 1 \quad (25)$$

we would obtain (16) for any $T > 0$ by regarding $f(t, v)$ as a weak solution constructed in [10] starting from t_0 . In fact, as stated in Remark 2, (16) follows from (17), which is fulfilled for any t if the initial datum satisfies, according to [10]. To check (25), it suffices to take an approximation sequence of the initial datum F_0

$$F_0^n(v) = \sum_{k=1}^4 m_k n^3 \mu(n(v - \mathbf{b}_k)), \quad n = 1, 2, \dots, \quad \mu(v) = \frac{e^{-|v|^2/2}}{(2\pi)^{3/2}}.$$

Since it follows from (13) that $f^n(t, v) \rightarrow f(t, v)$ on $\mathcal{S}'(\mathbb{R}_v^3)$, by the Fatou lemma we get firstly

$$\int |v|^2 f(t_0, v) dv < \infty$$

from (25) for $f^n(t, v)$ and $F_0^n(v)$, and finally (25) by the same arguments as in pp-292-293 of [10].

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