

KHOVANOV-LAUDA-ROUQUIER ALGEBRAS AND CRYSTAL BASES FOR FINITE CLASSICAL TYPE

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ABSTRACT. We present a crystal basis theoretic construction of irreducible modules over Khovanov-Lauda-Rouquier algebras and their cyclotomic quotients of finite classical type.

INTRODUCTION

The *Khovanov-Lauda-Rouquier algebras* (or *Hecke quiver algebras*) were introduced independently Khovanov-Lauda [11, 12] and Rouquier [16] for providing a categorification of quantum groups associated with symmetrizable Cartan data. Let $U_q(\mathfrak{g})$ be a quantum group and let $R(\alpha)$ be the corresponding Khovanov-Lauda-Rouquier algebra of weight $\alpha \in Q^+$. For a dominant integral weight $\lambda \in P^+$, the algebra $R(\alpha)$ has a special quotient $R^\lambda(\alpha)$, which gives a categorification of the irreducible highest weight $U_q(\mathfrak{g})$ -module $V(\lambda)$ with highest weight λ [4, 18]. The crystal structures of $U_q^-(\mathfrak{g})$ and $V(\lambda)$ also were interpreted in [14] in terms of irreducible modules over $R(\alpha)$ and $R^\lambda(\alpha)$. The Khovanov-Lauda-Rouquier algebras were generalized to the quantum generalized Kac-Moody algebras [5, 7] and, when the Cartan datum is symmetric, geometric realizations of the Khovanov-Lauda-Rouquier algebras were given in [6, 17] via quiver varieties.

In this paper, we announce the main result of our previous work [1, 8], which is an explicit construction of irreducible modules over $R(\alpha)$ and $R^\lambda(\alpha)$ of finite classical type. This construction differs from the one given in [2, 13] and is based on the theory of crystal bases. Though this paper is rely on [1], the description of irreducible modules in this paper is different and more combinatorial than the description given in [1].

Let us explain more precisely. Let $\mathfrak{B}(\infty)$ (resp. $\mathfrak{B}(\lambda)$) be the set of all isomorphism classes of irreducible graded $R(\alpha)$ -modules (resp. $R^\lambda(\alpha)$ -modules) for $\alpha \in Q^+$. It is shown in [14] that there exists a crystal isomorphism $\mathfrak{B}(\infty) \simeq B(\infty)$ (resp. $\mathfrak{B}(\lambda) \simeq B(\lambda)$). We first define *segments* $\mathfrak{s}$ to be unordered pairs of comparable elements in the basic crystals $\mathbb{B}_X$ ($X = A, B, C, D$) given in Section 3 and give a partial order $\preceq$ to them. Then we set *multisegments* $\underline{\mathfrak{m}}$ of $\mathbb{B}_X$ to be multisets of segments satisfying the conditions (3.1). Proposition

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3.1 says that the set $\mathcal{M}_X$ of multisegments is in a 1-1 correspondence Υ to the crystal $B(\infty)$. For each segment $\mathfrak{s}$, we define 1 or 2-dimensional module $\nabla(\mathfrak{s})$ out of the crystal $\mathbb{B}_X$. Then a multisegment $\underline{\mathfrak{m}} = \{\mathfrak{s}_1 \preceq \mathfrak{s}_2 \preceq \mathfrak{s}_3 \preceq \cdots\}$ of $\mathbb{B}_X$ gives the outer tensor product

$$\nabla(\underline{\mathfrak{m}}) = \nabla(\mathfrak{s}_1) \boxtimes \nabla(\mathfrak{s}_2) \boxtimes \nabla(\mathfrak{s}_3) \boxtimes \cdots.$$

It follows from [1] that

$$\text{hdInd} \nabla(\underline{\mathfrak{m}}) = \tilde{f}_{\nu_X}^{\mathbf{e}_1(\underline{\mathfrak{m}})} \cdots \tilde{f}_{\nu_X}^{\mathbf{e}_n(\underline{\mathfrak{m}})} \mathbf{1}$$

for $\underline{\mathfrak{m}} \in \mathcal{M}_X$ and the map $\Psi : \mathcal{M}_X \rightarrow \mathfrak{B}(\infty)$ defined by

$$\Psi(\underline{\mathfrak{m}}) = \text{hdInd} \nabla(\underline{\mathfrak{m}}) \quad \text{for } \underline{\mathfrak{m}} \in \mathcal{M}_X$$

is bijective (Theorem 3.2). Hence it can be deduced from Proposition 2.3 that the composition $\Psi \circ \Upsilon^{-1} : B(\infty) \rightarrow \mathfrak{B}(\infty)$ is a crystal isomorphism. In the cyclotomic cases, using the crystal embedding $B(\lambda) \hookrightarrow B(\infty) \otimes T_\lambda \otimes C$, we obtain the same results; i.e., the composition $\Psi^\lambda \circ \Upsilon_\lambda^{-1} : B(\lambda) \rightarrow \mathfrak{B}(\lambda)$ is a crystal isomorphism (Corollary 3.4).

This paper is organized as follows. Section 1 contains a brief review of the crystal basis theory for quantum generalized Kac-Moody algebras. In Section 2, we give the definition of Khovanov-Lauda-Rouquier algebras which is the most general version given in [6] associated with a Borchers-Cartan datum (or a quiver positively with loops), and introduce some results of Khovanov-Lauda-Rouquier algebras on crystal bases. In Section 3, we restrict to the case of finite classical types and present a crystal basis theoretic construction of irreducible modules over Khovanov-Lauda-Rouquier algebras and their cyclotomic quotients in terms of segments.

1. QUANTUM GENERALIZED KAC-MOODY ALGEBRAS

Let I be an index set. A square matrix $A = (a_{ij})_{i,j \in I}$ is called a *symmetrizable Borchers-Cartan matrix* if it satisfies (i) $a_{ii} = 2$ or $a_{ii} \in 2\mathbb{Z}_{\leq 0}$ for $i \in I$, (ii) $a_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$, (iii) $a_{ij} = 0$ if $a_{ji} = 0$ for $i, j \in I$, (vi) there is a diagonal matrix $D = \text{diag}(\mathbf{d}_i \in \mathbb{Z}_{>0} \mid i \in I)$ such that DA is symmetric. Let $I^{\text{re}} = \{i \in I \mid a_{ii} = 2\}$ and $I^{\text{im}} = I \setminus I^{\text{re}}$.

A *Borchers-Cartan datum* $(A, P, \Pi, \Pi^\vee)$ consists of

- (1) a symmetrizable Borchers-Cartan matrix A ,
- (2) a free abelian group P , called the *weight lattice*,
- (3) the set $\Pi = \{\alpha_i \mid i \in I\} \subset P$ of *simple roots*,
- (4) the set $\Pi^\vee = \{h_i \mid i \in I\} \subset P^\vee := \text{Hom}(P, \mathbb{Z})$ of *simple coroots*,

which satisfy the following properties:

- (i) $\langle h_i, \alpha_j \rangle := \alpha_j(h_i) = a_{ij}$ for all $i, j \in I$,
- (ii) $\Pi \subset \mathfrak{h}^*$ is linearly independent, where $\mathfrak{h} := \mathbb{C} \otimes_{\mathbb{Z}} P^\vee$,

(iii) for each $i \in I$, there exists $\Lambda_i \in P$ such that $\langle h_j, \Lambda_i \rangle = \delta_{ij}$ for all $j \in I$.

We denote by $P^+ = \{\lambda \in P \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0}, i \in I\}$ the set of *dominant integral weights*. The free abelian group $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ is the *root lattice*, and $Q^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$ is the *positive root lattice*. For $\alpha = \sum_{i \in I} k_i \alpha_i \in Q^+$, the *height* of α is $|\alpha| := \sum_{i \in I} k_i$. There is a symmetric bilinear form $(\mid)$ on $\mathfrak{h}^*$ such that

$$(\alpha_i \mid \alpha_j) = d_i a_{ij} \text{ for } i, j \in I, \quad \langle h_i, \lambda \rangle = \frac{2(\alpha_i \mid \lambda)}{(\alpha_i \mid \alpha_i)} \text{ for } \lambda \in \mathfrak{h}^* \text{ and } i \in I.$$

Let q be an indeterminate and $m, n \in \mathbb{Z}_{\geq 0}$. For $i \in I^{\text{re}}$, let $q_i = q^{d_i}$ and

$$[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! = \prod_{k=1}^n [k]_i, \quad \begin{bmatrix} m \\ n \end{bmatrix}_i = \frac{[m]_i!}{[m-n]_i! [n]_i!}.$$

Definition 1.1. The *quantum generalized Kac-Moody algebra* $U_q(\mathfrak{g})$ associated with a Borcherds-Cartan datum $(A, P, \Pi, \Pi^\vee)$ is the associative algebra over $\mathbb{Q}(q)$ with 1 generated by e_i, f_i ($i \in I$) and q^h ($h \in P^\vee$) satisfying following relations:

- (1) $q^0 = 1, q^h q^{h'} = q^{h+h'}$ for $h, h' \in P^\vee$,
- (2) $q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i, q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i$ for $h \in P^\vee, i \in I$,
- (3) $e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$, where $K_i = q_i^{h_i}$,
- (4) $\sum_{k=0}^{1-a_{ij}} \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_i e_i^{1-a_{ij}-k} e_j e_i^k = 0$ if $i \in I^{\text{re}}$ and $i \neq j$,
- (5) $\sum_{k=0}^{1-a_{ij}} \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_i f_i^{1-a_{ij}-k} f_j f_i^k = 0$ if $i \in I^{\text{re}}$ and $i \neq j$,
- (6) $e_i e_j - e_j e_i = 0, f_i f_j - f_j f_i = 0$ if $a_{ij} = 0$.

Note that, if all diagonal entries of A are 2, then A is a generalized Cartan matrix and $U_q(\mathfrak{g})$ is the usual quantum group associated with A .

Let $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) be the subalgebra of $U_q(\mathfrak{g})$ generated by the elements e_i (resp. f_i) for $i \in I$. For $n \in \mathbb{Z}_{>0}$, set

$$e_i^{(n)} = \begin{cases} \frac{e_i^n}{[n]_i!} & \text{if } i \in I^{\text{re}}, \\ e_i^n & \text{if } i \in I^{\text{im}}, \end{cases} \quad f_i^{(n)} = \begin{cases} \frac{f_i^n}{[n]_i!} & \text{if } i \in I^{\text{re}}, \\ f_i^n & \text{if } i \in I^{\text{im}}. \end{cases}$$

For an element $u \in U_q^-(\mathfrak{g})$, u can be expressed uniquely as

$$(1.1) \quad u = \sum_{k \geq 0} f_i^{(k)} u_k,$$

where $u_k \in \ker e'_i$ and $u_k = 0$ for $k \gg 0$. Here, e'_i is the endomorphism $U_q^-(\mathfrak{g}) \rightarrow U_q^-(\mathfrak{g})$ given in [3, Section 6] and [9, Section 3.3]. The *Kashiwara operators* $\tilde{e}_i, \tilde{f}_i$ ($i \in I$) of $U_q^-(\mathfrak{g})$ are defined by

$$\tilde{e}_i u = \sum_{k \geq 1} f_i^{(k-1)} u_k, \quad \tilde{f}_i u = \sum_{k \geq 0} f_i^{(k+1)} u_k.$$

Let $\mathbb{A}_0 = \{f/g \in \mathbb{Q}(q) \mid f, g \in \mathbb{Q}[q], g(0) \neq 0\}$.

Definition 1.2. A *crystal basis* of $U_q^-(\mathfrak{g})$ is a pair (L, B) satisfying the following conditions:

- (1) L is a free $\mathbb{A}_0$ -module of $U_q^-(\mathfrak{g})$ such that $U_q^-(\mathfrak{g}) = \mathbb{Q}(q) \otimes_{\mathbb{A}_0} L$ and $L = \bigoplus_{\alpha \in Q^+} L_{-\alpha}$, where $L_{-\alpha} := L \cap U_q^-(\mathfrak{g})_{-\alpha}$,
- (2) B is a $\mathbb{Q}$ -basis of L/qL such that $B = \bigsqcup_{\alpha \in Q^+} B_{-\alpha}$, where $B_{-\alpha} := B \cap (L_{-\alpha}/qL_{-\alpha})$,
- (3) $\tilde{e}_i B \subset B \sqcup \{0\}$, $\tilde{f}_i B \subset B$ for all $i \in I$,
- (4) For $b, b' \in B$ and $i \in I$, $b' = \tilde{f}_i b$ if and only if $b = \tilde{e}_i b'$.

Let $L(\infty)$ be the free $\mathbb{A}_0$ -module of $U_q^-(\mathfrak{g})$ generated by $\{\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} 1 \mid r \geq 0, i_k \in I\}$ and let

$$B(\infty) = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} 1 + qL(\infty) \mid r \geq 0, i_k \in I\} \setminus \{0\}.$$

Then, it is proved in [3, 9] that the pair $(L(\infty), B(\infty))$ is a unique crystal basis of $U_q^-(\mathfrak{g})$.

Let M be a $U_q(\mathfrak{g})$ -module in the category $\mathcal{O}_{int}$ defined in [3, Definition 3.1]. For any $i \in I$, any element $u \in M_\mu$ can be expressed uniquely as

$$u = \sum_{k \geq 0} f_i^{(k)} u_k,$$

where $u_k \in M_{\mu+k\alpha_i} \cap \ker e_i$. The *Kashiwara operators* $\tilde{e}_i, \tilde{f}_i$ ($i \in I$) are defined by

$$\tilde{e}_i u = \sum_{k \geq 1} f_i^{(k-1)} u_k, \quad \tilde{f}_i u = \sum_{k \geq 0} f_i^{(k+1)} u_k.$$

Definition 1.3. A *crystal basis* of $U_q(\mathfrak{g})$ -module M is a pair (L, B) satisfying the following conditions:

- (1) L is a free $\mathbb{A}_0$ -module of M such that $M = \mathbb{Q}(q) \otimes_{\mathbb{A}_0} L$ and $L = \bigoplus_{\lambda \in P} L_\lambda$, where $L_\lambda := L \cap M_\lambda$,
- (2) B is $\mathbb{Q}$ -basis of L/qL such that $B = \bigsqcup_{\lambda \in P} B_\lambda$, where $B_\lambda := B \cap L_\lambda/qL_\lambda$,
- (3) $\tilde{e}_i B \subset B \sqcup \{0\}$, $\tilde{f}_i B \subset B \sqcup \{0\}$ for all $i \in I$,
- (4) For $b, b' \in B$ and $i \in I$, $b' = \tilde{f}_i b$ if and only if $b = \tilde{e}_i b'$.

For a dominant integral weight $\lambda \in P^+$, we denote by $V(\lambda)$ the irreducible highest weight $U_q(\mathfrak{g})$ -module with highest weight λ . Note that $V(\lambda)$ is contained in $\mathcal{O}_{int}$. Let $L(\lambda)$ be the free $\mathbb{A}_0$ -module of $V(\lambda)$ generated by $\{\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} v_\lambda \mid r \geq 0, i_k \in I\}$ and let

$$B(\lambda) = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} v_\lambda + qL(\lambda) \mid r \geq 0, i_k \in I\} \setminus \{0\}.$$

Then the pair $(L(\lambda), B(\lambda))$ is a unique crystal basis of $V(\lambda)$ [3, 9].

2. KHOVANOV-LAUDA-ROUQUIER ALGEBRAS

Let $\mathbf{k} = \bigoplus_{n \in \mathbb{Z}} \mathbf{k}_n$ be a commutative graded ring such that $\mathbf{k}_0$ is a field and $\mathbf{k}_n = 0$ for $n < 0$. For $\alpha \in \mathbb{Q}^+$ with $|\alpha| = m$, let

$$I^\alpha = \{\nu = (\nu_1, \dots, \nu_m) \in I^m \mid \alpha_{\nu_1} + \dots + \alpha_{\nu_m} = \alpha\}.$$

Note that the symmetric group $S_m = \langle s_k \mid k = 1, \dots, m-1 \rangle$ acts naturally on I^α . For $t = 1, \dots, m-1$, we define the operator ∂_t on $\mathbf{k}[x_1, \dots, x_m]$ by

$$\partial_t(f) = \frac{s_t f - f}{x_t - x_{t+1}},$$

where $wf(x_1, \dots, x_m) = f(x_{w(1)}, \dots, x_{w(m)})$ for $w \in S_m$ and $f(x_1, \dots, x_m) \in \mathbf{k}[x_1, \dots, x_m]$.

For each $i \in I$, we choose a polynomial $\mathcal{P}_i(u, v) \in \mathbf{k}[u, v]$ of the form

$$\mathcal{P}_i(u, v) = \sum_{k, l \geq 0} p_{i; k, l} u^k v^l,$$

where $p_{i; k, l} \in \mathbf{k}_{d_i(2-a_{ii})-2d_i k-2d_i l}$ and $p_{i; 1-\frac{a_{ii}}{2}, 0}, p_{i; 0, 1-\frac{a_{ii}}{2}} \in \mathbf{k}_0^\times$. We also take a matrix $(\mathcal{Q}_{ij}(u, v))_{i, j \in I}$ in $\mathbf{k}[u, v]$ such that $\mathcal{Q}_{ij}(u, v) = \mathcal{Q}_{ji}(v, u)$ and $\mathcal{Q}_{ij}(u, v)$ has the form

$$\mathcal{Q}_{ij}(u, v) = \begin{cases} 0 & \text{if } i = j, \\ \sum_{k, l \geq 0} q_{i; j; k, l} u^k v^l & \text{if } i \neq j, \end{cases}$$

where $q_{i; j; k, l} \in \mathbf{k}_{-2(\alpha_i|\alpha_j)-2d_i k-2d_j l}$ and $q_{i; j; -a_{ij}, 0} \in \mathbf{k}_0^\times$.

Definition 2.1. Let $(\mathbf{A}, \mathbf{P}, \Pi, \Pi^\vee)$ be a Borcherds-Cartan datum and $\alpha \in \mathbb{Q}^+$ with height m . The *Khovanov-Lauda-Rouquier algebra* $R(\alpha)$ of weight α associated with the data $(\mathbf{A}, \mathbf{P}, \Pi, \Pi^\vee)$, $(\mathcal{P}_i)_{i \in I}$ and $(\mathcal{Q}_{ij})_{i, j \in I}$ is the associative graded $\mathbf{k}$ -algebra generated by $e(\nu)$ ($\nu = (\nu_1, \dots, \nu_m) \in I^\alpha$), x_k ($1 \leq k \leq m$), τ_t ($1 \leq t \leq m-1$) satisfying the following defining relations:

$$e(\nu)e(\nu') = \delta_{\nu, \nu'} e(\nu), \quad \sum_{\nu \in I^\alpha} e(\nu) = 1, \quad x_k e(\nu) = e(\nu) x_k, \quad x_k x_l = x_l x_k,$$

$$\tau_t e(\nu) = e(s_t(\nu)) \tau_t, \quad \tau_t \tau_s = \tau_s \tau_t \text{ if } |t-s| > 1,$$

$$\tau_t^2 e(\nu) = \begin{cases} \partial_t \mathcal{P}_{\nu_t}(x_t, x_{t+1}) \tau_t e(\nu) & \text{if } \nu_t = \nu_{t+1}, \\ \mathcal{Q}_{\nu_t \nu_{t+1}}(x_t, x_{t+1}) e(\nu) & \text{if } \nu_t \neq \nu_{t+1}, \end{cases}$$

$$(\tau_t x_k - x_{s_t(k)} \tau_t) e(\nu) = \begin{cases} -\mathcal{P}_{\nu_t}(x_t, x_{t+1}) e(\nu) & \text{if } k = t \text{ and } \nu_t = \nu_{t+1}, \\ \mathcal{P}_{\nu_t}(x_t, x_{t+1}) e(\nu) & \text{if } k = t+1 \text{ and } \nu_t = \nu_{t+1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\begin{aligned}
& (\tau_{t+1}\tau_t\tau_{t+1} - \tau_t\tau_{t+1}\tau_t)e(\nu) \\
&= \begin{cases} \mathcal{P}_{\nu_t}(x_t, x_{t+2})\overline{\mathcal{Q}}_{\nu_t, \nu_{t+1}}(x_t, x_{t+1}, x_{t+2})e(\nu) & \text{if } \nu_t = \nu_{t+2} \neq \nu_{t+1}, \\ \overline{\mathcal{P}}'_{\nu_t}(x_t, x_{t+1}, x_{t+2})\tau_te(\nu) + \overline{\mathcal{P}}''_{\nu_t}(x_t, x_{t+1}, x_{t+2})\tau_{t+1}e(\nu) & \text{if } \nu_t = \nu_{t+1} = \nu_{t+2}, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

where

$$\begin{aligned}
\overline{\mathcal{P}}'_i(u, v, w) &:= \frac{\mathcal{P}_i(v, u)\mathcal{P}_i(u, w)}{(u-v)(u-w)} + \frac{\mathcal{P}_i(u, w)\mathcal{P}_i(v, w)}{(u-w)(v-w)} - \frac{\mathcal{P}_i(u, v)\mathcal{P}_i(v, w)}{(u-v)(v-w)}, \\
\overline{\mathcal{P}}''_i(u, v, w) &:= -\frac{\mathcal{P}_i(u, v)\mathcal{P}_i(u, w)}{(u-v)(u-w)} - \frac{\mathcal{P}_i(u, w)\mathcal{P}_i(w, v)}{(u-w)(v-w)} + \frac{\mathcal{P}_i(u, v)\mathcal{P}_i(v, w)}{(u-v)(v-w)}, \\
\overline{\mathcal{Q}}_{i,j}(u, v, w) &:= \frac{\mathcal{Q}_{i,j}(u, v) - \mathcal{Q}_{i,j}(w, v)}{u-w}.
\end{aligned}$$

The algebra $R(\alpha)$ has the $\mathbb{Z}$ -grading given by

$$\deg(e(\nu)) = 0, \quad \deg(x_k(\nu)) = 2\mathbf{d}_{\nu_k}, \quad \deg(\tau_t(\nu)) = -(\alpha_{\nu_t} | \alpha_{\nu_{t+1}}),$$

where $x_k(\nu) = x_k e(\nu)$ and $\tau_t(\nu) = \tau_t e(\nu)$ for $\nu \in I^\alpha$. A diagrammatic presentation of $R(\alpha)$ using planar diagrams with dots and strands is given in [7, 11, 12].

For $\lambda \in \mathbf{P}^+$ and $i \in I$, let us fix a polynomial $a_i^\lambda(u)$ of the form

$$a_i^\lambda(u) = \sum_{k=0}^{\lambda(h_i)} c_{i,k}^\lambda u^{\lambda(h_i)-k},$$

where $c_{i,k}^\lambda \in \mathbf{k}_{2\mathbf{d}_i, k}$ and $c_{i,0}^\lambda = 1$. Set $a^\lambda(x) = \sum_{\nu \in I} a_{\nu_1}^\lambda(x) e(\nu)$. Then the *cyclotomic Khovanov-Lauda-Rouquier algebra* $R^\lambda(\alpha)$ is defined to be the quotient algebra

$$R^\lambda(\alpha) = R(\alpha) / R(\alpha) a^\lambda(x) R(\alpha).$$

Let $R(\alpha)\text{-mod}$ (resp. $R^\lambda(\alpha)\text{-mod}$) be the category of finite-dimensional graded left $R(\alpha)$ -modules (resp. finite-dimensional graded left $R^\lambda(\alpha)$ -modules). For a $\mathbb{Z}$ -graded module $M = \bigoplus_{k \in \mathbb{Z}} M_k$ and $t \in \mathbb{Z}$, let $M\langle t \rangle = \bigoplus_{k \in \mathbb{Z}} M\langle t \rangle_k$ be the $\mathbb{Z}$ -graded module obtained from M by setting $M\langle t \rangle_k := M_{t+k}$. The q -character $\text{ch}_q(M)$ and character $\text{ch}(M)$ of M are defined by

$$\text{ch}_q(M) = \sum_{\nu \in I^\alpha} \dim_q(e(\nu)M) \nu, \quad \text{ch}(M) = \sum_{\nu \in I^\alpha} \dim(e(\nu)M) \nu,$$

where $\dim_q(N) := \sum_{i \in \mathbb{Z}} (\dim N_i) q^i$ for any graded module $N = \bigoplus_{i \in \mathbb{Z}} N_i$.

For $M, N \in R(\alpha)\text{-mod}$, let $\text{Hom}(M, N)$ be the set of homogeneous homomorphisms of degree 0, and let $\text{HOM}(M, N) = \bigoplus_{k \in \mathbb{Z}} \text{Hom}(M, N\langle k \rangle)$. For $\beta_1, \dots, \beta_m \in \mathbf{Q}^+$, we define

$$e(\beta_1, \dots, \beta_m) = \sum_{\nu_j \in I^{\beta_j}} e(\nu_1 * \dots * \nu_m),$$

where $\nu_1 * \cdots * \nu_m$ is the concatenation of ν_k 's. The natural embedding $R(\beta_1) \otimes \cdots \otimes R(\beta_m) \hookrightarrow R(\beta_1 + \cdots + \beta_m)$ gives the functors

$$\text{Ind}_{\beta_1, \dots, \beta_m} : R(\beta_1) \otimes \cdots \otimes R(\beta_m)\text{-mod} \rightarrow R(\beta_1 + \cdots + \beta_m)\text{-mod},$$

$$\text{Res}_{\beta_1, \dots, \beta_m} : R(\beta_1 + \cdots + \beta_m)\text{-mod} \rightarrow R(\beta_1) \otimes \cdots \otimes R(\beta_m)\text{-mod}$$

defined by $\text{Ind}_{\beta_1, \dots, \beta_m} L := R(\beta_1 + \cdots + \beta_m) \otimes_{R(\beta_1) \otimes \cdots \otimes R(\beta_m)} L$ and $\text{Res}_{\beta_1, \dots, \beta_m} M := e(\beta_1, \dots, \beta_m)M$ for $L \in R(\beta_1) \otimes \cdots \otimes R(\beta_m)\text{-mod}$ and $M \in R(\beta_1 + \cdots + \beta_m)\text{-mod}$.

Let $\mathfrak{B}(\infty)$ (resp. $\mathfrak{B}(\lambda)$) be the set of all isomorphism classes of irreducible graded $R(\beta)$ -modules (resp. $R^\lambda(\beta)$ -modules) for all $\beta \in \mathbb{Q}^+$. Set $\mathbf{1}$ to be the 1-dimensional trivial $R(0)$ -module. For $M \in R(\beta)\text{-mod}$, we define

$$\tilde{e}_i(M) = \text{soc}(\text{Res}_{\alpha_i, \beta - \alpha_i} M) \in R(\beta - \alpha_i)\text{-mod}$$

$$\tilde{f}_i(M) = \text{hd Ind}_{\alpha_i, \beta}(L(i) \boxtimes M) \in R(\beta + \alpha_i)\text{-mod},$$

$$\text{wt}(M) = \begin{cases} -\beta & \text{if } M \in R(\beta)\text{-mod}, \\ \lambda - \beta & \text{if } M \in R^\lambda(\beta)\text{-mod}, \end{cases}$$

$$\varepsilon_i(M) = \max\{k \geq 0 \mid \tilde{e}_i^k M \neq 0\}, \quad \varphi_i(M) = \varepsilon_i(M) + \text{wt}(M)(h_i),$$

where $L(i)$ is the 1-dimensional $R(\alpha_i)$ -module with $\dim_q L(i) = 1$. Here, for an $R(\alpha)$ -module N , $\text{soc}(N)$ (resp. $\text{hd}(N)$) is the maximal completely reducible submodule (resp. the maximal completely reducible quotient) of N . Then, when $a_{ii} \neq 0$ for all $i \in I$, the sets $\mathfrak{B}(\infty)$ and $\mathfrak{B}(\lambda)$ with the above maps have crystal structures [7, 14].

Theorem 2.2 ([7, 14]). *If $a_{ii} \neq 0$ for all $i \in I$, then the crystal $(\mathfrak{B}(\infty), \text{wt}, \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i)$ (resp. $(\mathfrak{B}(\lambda), \text{wt}, \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i)$) is isomorphic to the crystal $B(\infty)$ (resp. $B(\lambda)$).*

We now assume that $a_{ii} \neq 0$ for all $i \in I$. Let $n = |I|$ be the rank of $U_q(\mathfrak{g})$, and let $I_{(k)} \subset I$ ($k = 1, \dots, n$) be subsets of $I = I_{(n+1)}$ such that $I_{(k)} \subset I_{(k+1)}$ and $|I_{(k)}| = k$ for all k . Let U_k denote the subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i ($i \in I_{(k)}$) and q^h ($h \in \mathbb{P}^\vee$) and let $\mathcal{B}_k$ be the crystal obtained from $B(\infty)$ by forgetting the i -arrows for $i \notin I_{(k)}$. Then $\mathcal{B}_k$ can be understood as a U_k -crystal and every connected component of $\mathcal{B}_k$ has a unique highest weight vector [1, Lemma 1.9].

Take an element $v \in B(\infty)$. Let $u_0 = v$ and let u_k be the highest weight vector of the connected component $\mathcal{C}_k$ of $\mathcal{B}_k$ containing v for $k = 1, \dots, n$. By construction, there is a chain of injective maps

$$\mathcal{C}_1 \hookrightarrow \mathcal{C}_2 \hookrightarrow \cdots \hookrightarrow \mathcal{C}_{n-1} \hookrightarrow B(\infty).$$

For $k = 1, \dots, n$, let

$$(2.1) \quad \mathcal{N}_k(v) = \tilde{f}_{\nu_{k,1}} \cdots \tilde{f}_{\nu_{k,t_k}} \mathbf{1} \in \mathfrak{B}(\infty),$$

where $\nu_k = (\nu_{k,1}, \dots, \nu_{k,t_k})$ is a sequence of I such that $u_{k-1} = \tilde{f}_{\nu_{k,1}} \cdots \tilde{f}_{\nu_{k,t_k}} u_k$.

Hence, for each $v \in B(\infty)$, we obtain the corresponding n -tuple $(\mathcal{N}_1(v), \mathcal{N}_2(v), \dots, \mathcal{N}_n(v))$ of irreducible modules in $\mathfrak{B}(\infty)$. Then, using the same argument as in [1, Proposition 1.10], one can obtain the following proposition.

Proposition 2.3. [1, Proposition 1.10]

- (1) For $v \in B(\infty)$, $\text{hd Ind}(\boxtimes_{k=1}^n \mathcal{N}_k(v))$ is irreducible.
- (2) The map $\Phi : B(\infty) \longrightarrow \mathfrak{B}(\infty)$ defined by

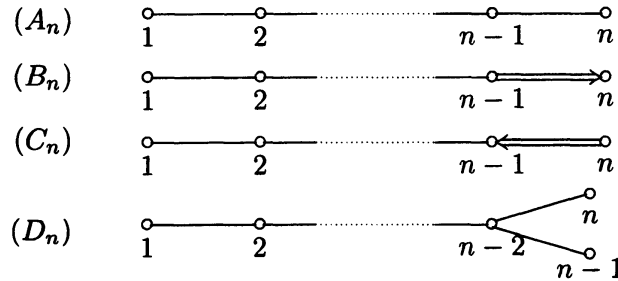
$$\Phi(v) = \text{hd Ind}(\boxtimes_{k=1}^n \mathcal{N}_k(v)) \quad \text{for } v \in B(\infty)$$

is a crystal isomorphism.

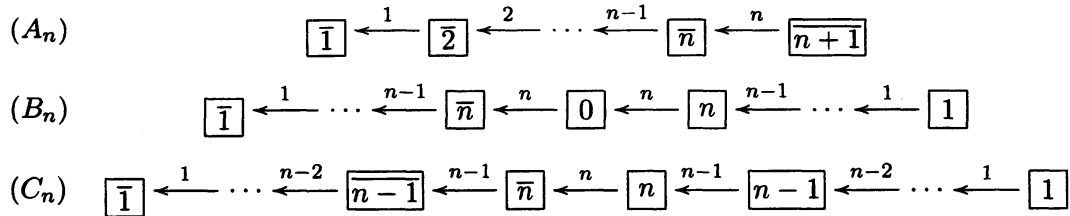
3. CRYSTAL BASES AND IRREDUCIBLE REPRESENTATIONS

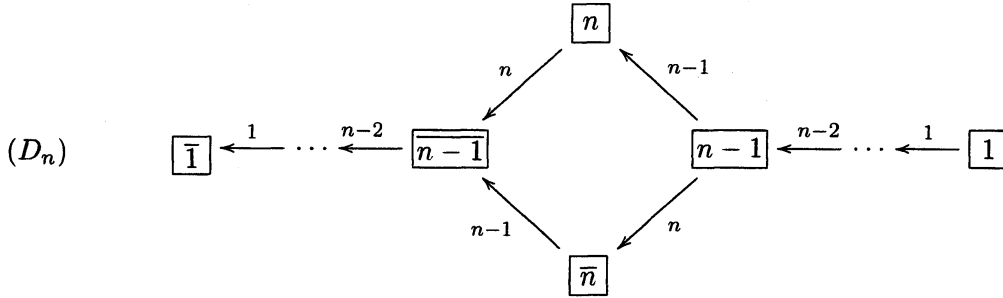
In this section, we give a crystal basis theoretic construction of irreducible modules over Khovanov-Lauda-Rouquier algebras and their cyclotomic quotients for finite classical types. Though this section is based on our previous work [1], the description of irreducible modules in this section is different and more combinatorial than the description given in [1]. Throughout this section, we assume that $\mathbf{k} = \mathbb{C}$ and A is a generalized Cartan matrix of finite classical type A_n , B_n , C_n and D_n .

Let $I = \{1, 2, \dots, n\}$ and let Γ be the following Dynkin diagram:



We set $\mathbb{B}_X$ ($X = A, B, C, D$) to be the crystal defined by





with the entries ordered by

$$\begin{aligned}
 (A_n) \quad & \bar{1} \succ \bar{2} \succ \dots \succ \overline{n+1}, \\
 (B_n) \quad & \bar{1} \succ \bar{2} \succ \dots \succ \bar{n} \succ 0 \succ n \succ \dots \succ 2 \succ 1, \\
 (C_n) \quad & \bar{1} \succ \bar{2} \succ \dots \succ \overline{n-1} \succ \bar{n} \succ n \succ n-1 \succ \dots \succ 1, \\
 (D_n) \quad & \bar{1} \succ \bar{2} \succ \dots \succ \overline{n-1} \succ \bar{n}, n \succ n-1 \succ \dots \succ 1.
 \end{aligned}$$

For $\nu = (\nu_1, \dots, \nu_m) \in I^m$ and $\mathbf{k} = (k_1, \dots, k_m) \in (\mathbb{Z}_{\geq 0})^m$, let $\tilde{f}_\nu^{\mathbf{k}} = \tilde{f}_{\nu_1}^{k_1} \dots \tilde{f}_{\nu_m}^{k_m}$ (resp. $\tilde{e}_\nu^{\mathbf{k}} = \tilde{e}_{\nu_1}^{k_1} \dots \tilde{e}_{\nu_m}^{k_m}$). If $\mathbf{k} = (1, \dots, 1)$, then we write $\tilde{f}_\nu$ (resp. $\tilde{e}_\nu$) for $\tilde{f}_\nu^{\mathbf{k}}$ (resp. $\tilde{e}_\nu^{\mathbf{k}}$).

A *segment* $\mathfrak{s}$ of the crystal $\mathbb{B}_X$ is a subset $\mathfrak{s} = \{a, b\} \subset \mathbb{B}_X$ such that the two elements a and b of $\mathfrak{s}$ are distinct and comparable. Note that any distinct two elements $\{a, b\}$ of $\mathbb{B}_X$ except the case $\{a, b\} = \{n, \bar{n}\}$ ($X = D$) can be viewed as a segment of $\mathbb{B}_X$. For a segment $\mathfrak{s} = \{a, b\}$ with $a \succ b$, set $h(\mathfrak{s}) = a$ and $t(\mathfrak{s}) = b$, respectively. We define a partial order $\succeq$ on the set of segments of $\mathbb{B}_X$ as follows: for two segments $\mathfrak{s}$ and $\mathfrak{s}'$ of $\mathbb{B}_X$,

$$\mathfrak{s} \succeq \mathfrak{s}' \quad \text{if and only if} \quad (h(\mathfrak{s}) \succ h(\mathfrak{s}')) \text{ or } (h(\mathfrak{s}) = h(\mathfrak{s}') \text{ and } (t(\mathfrak{s}') \succ t(\mathfrak{s})))$$

For $b \in \mathbb{B}_X$ and $X = B, C, D$, we set $b^\vee$ to be a unique element in $\mathbb{B}_X$ such that $\text{wt}(b^\vee) = -\text{wt}(b)$. Let $b_X = \bar{n}$ ($X = A, B, C$), $b_X = \overline{n-1}$ ($X = D$). A *multisegment* $\underline{\mathfrak{m}}$ of $\mathbb{B}_X$ is a multiset of segments of the crystal $\mathbb{B}_X$ such that

$$\begin{aligned}
 (3.1) \quad & \text{(i) } h(\mathfrak{s}) \succeq b_X \text{ for } \mathfrak{s} \in \underline{\mathfrak{m}}, \\
 & \text{(ii) } t(\mathfrak{s}) \succeq h(\mathfrak{s})^\vee \text{ if } X = B, C, D, \\
 & \text{(iii) any two segments in } \underline{\mathfrak{m}} \text{ are comparable.}
 \end{aligned}$$

When no confusion can arise, we write $\underline{\mathfrak{m}} = \{\mathfrak{s}_1 \preceq \mathfrak{s}_2 \preceq \mathfrak{s}_3 \preceq \dots\}$. For $i = 1, \dots, n$, we set

$$\underline{\mathfrak{m}}(i) = \{\mathfrak{s} \in \underline{\mathfrak{m}} \mid \tilde{e}_i(h(\mathfrak{s})) \neq 0\}.$$

Note that $\underline{\mathfrak{m}} = \bigcup_{i \in I} \underline{\mathfrak{m}}(i)$.

Let $\mathcal{M}_X$ be the set of all multisegments of $\mathbb{B}_X$. We will show that $\mathcal{M}_X$ parameterizes the crystal $B(\infty)$. Let $\ell_A = n$, $\ell_B = \ell_C = 2n - 1$, $\ell_D = 2n - 2$ and let ν_X be the sequence of the

colors over the arrows of $\mathbb{B}_X$, i.e.,

$$\begin{aligned}\nu_A &= (1, 2, \dots, n-1, n), \\ \nu_B &= \nu_C = (1, 2, \dots, n-1, n, n-1, \dots, 2, 1), \\ \nu_D &= (1, 2, \dots, n-1, n, n-2, \dots, 2, 1).\end{aligned}$$

For a segment $\mathbf{s}$, let $\mathbf{e}(\mathbf{s}) = (e_1, \dots, e_{\ell_X}) \in \mathbb{Z}^{\ell_X}$ be the ℓ_X -tuple of $\mathbb{Z}$ defined by

$$\mathbf{h}(\mathbf{s}) = \tilde{f}_{\nu_X}^{\mathbf{e}(\mathbf{s})} \mathbf{t}(\mathbf{s}), \quad e_j \neq 0 \text{ for some } 1 \leq j \leq n.$$

Here we consider $\mathbb{Z}^{\ell_X}$ as an abelian group. Let $\mathbf{e}_i = (0, \dots, 0, \overset{\text{ith}}{\underset{\downarrow}{1}}, 0, \dots, 0)$ for $i = 1, \dots, \ell_X$.

Proposition 3.1. *Let $\Upsilon : \mathcal{M}_X \rightarrow B(\infty)$ be the map defined by*

$$\Upsilon(\underline{\mathbf{m}}) = \tilde{f}_{\nu_X}^{\mathbf{e}_1(\underline{\mathbf{m}})} \dots \tilde{f}_{\nu_X}^{\mathbf{e}_n(\underline{\mathbf{m}})} 1,$$

$$\text{where } \mathbf{e}_i(\underline{\mathbf{m}}) = \begin{cases} \sum_{\mathbf{s} \in \underline{\mathbf{m}}(n+1-i)} \mathbf{e}(\mathbf{s}) & \text{if either } X = A, B, C \text{ or } i \neq 1, 2 \text{ } (X = D), \\ \# \underline{\mathbf{m}}(n-1) \mathbf{e}_{n-1} & \text{if } i = 2 \text{ } (X = D), \\ \# \underline{\mathbf{m}}(n) \mathbf{e}_n & \text{if } i = 1 \text{ } (X = D). \end{cases}$$

Then the map Υ is bijective.

Proof. We focus on the case $X = D$ since the remaining cases can be proved in a similar manner. Let $\mathcal{S}$ be the set of $\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{Z}_{\geq 0}^{n(2n-2)}$ such that

- (a) $\mathbf{t}_j = (t_{j,1}, \dots, t_{j,2n-2}) \in \mathbb{Z}_{\geq 0}^{(2n-2)}$ for $j = 1, \dots, n$,
- (b) $t_{j,n+1-j} \geq t_{j,n+2-j} \geq \dots \geq t_{j,n-1}, t_{j,n} \geq t_{j,n+1} \geq \dots \geq t_{j,n-2+j}$ for $j = 3, \dots, n$,
- (c) if $j = 3, \dots, n$, then $t_{j,k} = 0$ for either $k < n+1-j$ or $k > n-2+j$,
- (d) $t_{2,k} = 0$ for $k \neq n-1$ and $t_{1,k} = 0$ for $k \neq n$.

It follows from [15, Section 7] that the map $\mathcal{S} \rightarrow B(\infty)$ mapping $\mathbf{t} = (t_1, \dots, t_n) \in \mathcal{S}$ to $\tilde{f}_{\nu_D}^{\mathbf{t}_1} \dots \tilde{f}_{\nu_D}^{\mathbf{t}_n} 1 \in B(\infty)$ is bijective.

Let $\underline{\mathbf{m}} \in \mathcal{M}_X$ and write $\mathbf{e}_j(\underline{\mathbf{m}}) = (e_{j,1}, \dots, e_{j,2n-2})$ for $j = 1, \dots, n$. By (ii) and (iii) of (3.1), the sequences $\mathbf{e}_j(\underline{\mathbf{m}})$ satisfy the above condition (b), (c) and (d). Hence the map $\phi : \mathcal{M}_X \rightarrow \mathcal{S}$ given by

$$\phi(\underline{\mathbf{m}}) = (\mathbf{e}_1(\underline{\mathbf{m}}), \dots, \mathbf{e}_n(\underline{\mathbf{m}})) \quad (\underline{\mathbf{m}} \in \mathcal{M}_X)$$

is well-defined.

On the other hand, let

$$\hat{i} = \begin{cases} \bar{i} & \text{if } 1 \leq i \leq n, \\ 2n+1-i & \text{if } n+1 \leq i \leq 2n. \end{cases}$$

and, for a segment $\mathfrak{s}$ and a multisegment $\underline{\mathfrak{m}} \in \mathcal{M}_X$, let us denote by $\sigma(\underline{\mathfrak{m}}, \mathfrak{s})$ the multiplicity of $\mathfrak{s}$ in $\underline{\mathfrak{m}}$. Let $\psi : \mathcal{S} \rightarrow \mathcal{M}_X$ be the map mapping $\mathfrak{t} \in \mathcal{S}$ to the multisegment $\psi(\mathfrak{t}) \in \mathcal{M}_X$ such that, if $i = 1, \dots, n-2$, then

$$\sigma(\psi(\mathfrak{t}), \{\widehat{i}, \widehat{j+1}\}) = \begin{cases} t_{n+1-i,j} - t_{n+1-i,j+1} & \text{if } j \leq n-3, \\ t_{n+1-i,n-2} - \max\{t_{n+1-i,n-1}, t_{n+1-i,n}\} & \text{if } j = n-2, \\ \max\{0, t_{n+1-i,n-1} - t_{n+1-i,n}\} & \text{if } j = n-1, \\ \max\{0, t_{n+1-i,n} - t_{n+1-i,n-1}\} & \text{if } j = n, \\ \min\{t_{n+1-i,n-1}, t_{n+1-i,n}\} - t_{n+1-i,n+1} & \text{if } j = n+1, \\ t_{n+1-i,j-1} - t_{n+1-i,j} & \text{if } j \geq n+2, \end{cases}$$

and, if $i = n-1$, then

$$\sigma(\psi(\mathfrak{t}), \{\widehat{n-1}, \widehat{j+1}\}) = \begin{cases} \max\{0, t_{2,n-1} - t_{1,n}\} & \text{if } j = n-1, \\ \max\{0, t_{1,n} - t_{2,n-1}\} & \text{if } j = n, \\ \min\{t_{1,n}, t_{2,n-1}\} & \text{if } j = n+1. \end{cases}$$

Then it is straightforward to verify that $\phi \circ \psi = \text{id}_{\mathcal{S}}$ and $\psi \circ \phi = \text{id}_{\mathcal{M}_X}$. $\square$

We now return to the Khovanov-Lauda-Rouquier algebras. Let $\mathbf{1}$ be the trivial $R(0)$ -module. For a segment $\mathfrak{s}$, we define

$$\nabla(\mathfrak{s}) = \tilde{f}_{\nu(\mathfrak{s})}^{\mathbf{e}(\mathfrak{s})} \mathbf{1} \in \mathfrak{B}(\infty).$$

We give an explicit description of the module structure of $\nabla(\mathfrak{s})$ as follows.

Let $l = |\text{wt}(\mathbf{h}(\mathfrak{s})) - \text{wt}(\mathbf{t}(\mathfrak{s}))|$. If one of the following holds: $\mathbf{h}(\mathfrak{s}) \succ \mathbf{t}(\mathfrak{s})$ (A_n, C_n), either $\mathbf{t}(\mathfrak{s}) \succeq 0$ or $0 \succeq \mathbf{h}(\mathfrak{s})$ (B_n), either $\mathbf{t}(\mathfrak{s}) \succ n-1$ or $\overline{n-1} \succ \mathbf{h}(\mathfrak{s})$ (D_n), then the module $\nabla(\mathfrak{s})$ is the 1-dimensional module $\mathbb{C}v$ given by

$$x_i v = 0, \quad \tau_j v = 0, \quad e(\nu)v = \begin{cases} v & \text{if } \nu = \nu(\mathfrak{s}), \\ 0 & \text{otherwise,} \end{cases}$$

where $\nu(\mathfrak{s}) \in I^l$ such that $\mathbf{h}(\mathfrak{s}) = \tilde{f}_{\nu(\mathfrak{s})} \mathbf{t}(\mathfrak{s})$.

If $\mathbf{h}(\mathfrak{s}) \succ 0 \succ \mathbf{t}(\mathfrak{s})$ for type B_n , then $\nabla(\mathfrak{s})$ is the 2-dimensional module $\mathbb{C}u \oplus \mathbb{C}v$ with

$$\begin{aligned} x_i u = 0, \quad \tau_j u &= \begin{cases} v & \text{if } j = d, \\ 0 & \text{otherwise,} \end{cases} \quad e(\nu)u = \begin{cases} u & \text{if } \nu = \nu(\mathfrak{s}), \\ 0 & \text{otherwise,} \end{cases} \\ x_i v &= \begin{cases} u & \text{if } i = d, \\ -u & \text{if } i = d+1, \\ 0 & \text{otherwise} \end{cases}, \quad \tau_j v = 0, \quad e(\nu)v = \begin{cases} u & \text{if } \nu = \nu(\mathfrak{s}), \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where $\nu(\mathfrak{s}) \in I^l$ such that $\mathbf{h}(\mathfrak{s}) = \tilde{f}_{\nu(\mathfrak{s})} \mathbf{t}(\mathfrak{s})$, and d is an integer such that $s_d(\nu(\mathfrak{s})) = \nu(\mathfrak{s})$.

If $h(\mathfrak{s}) \geq \overline{n-1}$ and $n-1 \geq t(\mathfrak{s})$ for type D_n , then the module $\nabla(\mathfrak{s})$ is the 2-dimensional module $\mathbb{C}u \oplus \mathbb{C}v$ defined by

$$\begin{aligned} x_i u = 0, \quad \tau_j u &= \begin{cases} v & \text{if } j = d, \\ 0 & \text{otherwise,} \end{cases} & e(\nu)u &= \begin{cases} Q_{n-1,n}(x_{n-1}, x_n)u & \text{if } \nu = \nu(\mathfrak{s})^+, \\ 0 & \text{otherwise,} \end{cases} \\ x_i v = 0, \quad \tau_j v &= \begin{cases} u & \text{if } j = d, \\ 0 & \text{otherwise,} \end{cases} & e(\nu)v &= \begin{cases} v & \text{if } \nu = \nu(\mathfrak{s})^-, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where $\nu(\mathfrak{s})^+, \nu(\mathfrak{s})^- \in I^l$ such that $\nu(\mathfrak{s})^+ \neq \nu(\mathfrak{s})^-$ and $h(\mathfrak{s}) = \tilde{f}_{\nu(\mathfrak{s})^+} t(\mathfrak{s}) = \tilde{f}_{\nu(\mathfrak{s})^-} t(\mathfrak{s})$, and d is an integer such that $s_d(\nu(\mathfrak{s})^+) = \nu(\mathfrak{s})^-$. Note that $Q_{n-1,n}(x_{n-1}, x_n) \in \mathbb{C}^*$. For the description above, the character $\text{ch}\nabla(\mathfrak{s})$ is given as follows

$$(3.2) \quad \text{ch}\nabla(\mathfrak{s}) = \begin{cases} \nu(\mathfrak{s})^+ + \nu(\mathfrak{s})^- & \text{if } h(\mathfrak{s}) \geq \overline{n-1}, n-1 \geq t(\mathfrak{s}) \quad (D_n), \\ 2\nu(\mathfrak{s}) & \text{if } h(\mathfrak{s}) > 0 > t(\mathfrak{s}) \quad (B_n), \\ \nu(\mathfrak{s}) & \text{otherwise.} \end{cases}$$

For a multisegment $\underline{\mathfrak{m}} = \{\mathfrak{s}_1 \preceq \mathfrak{s}_2 \preceq \mathfrak{s}_3 \preceq \dots\}$, we define

$$\nabla(\underline{\mathfrak{m}}) = \nabla(\mathfrak{s}_1) \boxtimes \nabla(\mathfrak{s}_2) \boxtimes \nabla(\mathfrak{s}_3) \boxtimes \dots$$

Then we have the following theorem.

Theorem 3.2. [1, Theorem 3.2]

(1) For a multisegment $\underline{\mathfrak{m}}$, we have

$$\text{hdInd}\nabla(\underline{\mathfrak{m}}) = \tilde{f}_{\nu_X}^{\mathbf{e}_1(\underline{\mathfrak{m}})} \dots \tilde{f}_{\nu_X}^{\mathbf{e}_n(\underline{\mathfrak{m}})} \mathbf{1},$$

$$\text{where } \mathbf{e}_i(\underline{\mathfrak{m}}) = \begin{cases} \sum_{\mathfrak{s} \in \underline{\mathfrak{m}}(n+1-i)} \mathbf{e}(\mathfrak{s}) & \text{if either } X = A, B, C \text{ or } i \neq 1, 2 \text{ } (X = D), \\ \#\underline{\mathfrak{m}}(n-1)\mathbf{e}_{n-1} & \text{if } i = 2 \text{ } (X = D), \\ \#\underline{\mathfrak{m}}(n)\mathbf{e}_n & \text{if } i = 1 \text{ } (X = D). \end{cases}$$

(2) Let $\Psi : \mathcal{M}_X \rightarrow \mathfrak{B}(\infty)$ be the map defined by

$$\Psi(\underline{\mathfrak{m}}) = \text{hdInd}\nabla(\underline{\mathfrak{m}}) \quad \text{for } \underline{\mathfrak{m}} \in \mathcal{M}_X.$$

Then the map Ψ is bijective.

Proof. We give a sketch of the proof of [1, Theorem 3.2]. When $X = D$, without loss of generality, we may assume that

$$\sigma(\underline{\mathfrak{m}}, \{\overline{n-1}, \bar{n}\}) \geq \sigma(\underline{\mathfrak{m}}, \{\overline{n-1}, n\}),$$

where $\sigma(\underline{\mathfrak{m}}, \mathfrak{s})$ is the multiplicity of $\mathfrak{s}$ in $\underline{\mathfrak{m}}$. Note that $\sum_{\mathfrak{s} \in \underline{\mathfrak{m}}(n-1)} \mathbf{e}(\mathfrak{s}) = \#\underline{\mathfrak{m}}(n-1)\mathbf{e}_{n-1} + \#\underline{\mathfrak{m}}(n)\mathbf{e}_n$ if $X = D$. It follows from [1, Lemma 4.3] and the definition of $\underline{\mathfrak{m}}(k)$ that

$$(a) \quad \varepsilon_i(\text{Ind}\nabla(\underline{\mathfrak{m}}(k))) = 0 \text{ for } i = n+1-k, n+2-k, \dots, n,$$

$$(b) \text{hdInd}\nabla(\underline{\mathbf{m}}(k)) = \begin{cases} \tilde{f}_{\nu_X}^{\mathbf{e}_{n+1-k}(\underline{\mathbf{m}})} 1 & \text{if } k = 1, \dots, n \ (X = A, B, C), \\ k = 1, \dots, n-2 \ (X = D), \\ \tilde{f}_{\nu_X}^{\mathbf{e}_1(\underline{\mathbf{m}})} \tilde{f}_{\nu_X}^{\mathbf{e}_2(\underline{\mathbf{m}})} 1 & \text{if } k = n-1 \ (X = D), \end{cases}$$

Let $I_{(k)} = \{n+1-k, \dots, n\}$ and let $\mathcal{N}_k$ be the module given in (2.1). Then, by the crystal description given in [15], we have

$$\text{hdInd}\nabla(\underline{\mathbf{m}}(k)) \simeq \begin{cases} \mathcal{N}_{n+1-k} & \text{if } k = 1, \dots, n \ (X = A, B, C), \\ k = 1, \dots, n-2 \ (X = D), \\ \text{Ind}(\mathcal{N}_1 \boxtimes \mathcal{N}_2) & \text{if } k = n-1 \ (X = D). \end{cases}$$

Combining Proposition 2.3 and [1, Lemma 1.8] with the above conditions (a) and (b), we obtain

$$\begin{aligned} \text{hdInd}(\boxtimes_{k=1}^n \mathcal{N}_k(v)) &\simeq \text{hdInd}(\boxtimes_{k=n'}^1 \text{Ind}\nabla(\underline{\mathbf{m}}(k))) \\ &\simeq \text{hdInd}(\boxtimes_{k=n'}^1 \nabla(\underline{\mathbf{m}}(k))) \\ &\simeq \text{hdInd}\nabla(\underline{\mathbf{m}}), \end{aligned}$$

where $n' = n$ ($X = A, B, C$) and $n' = n-1$ ($X = D$). Therefore, the assertion follows from Proposition 2.3. $\square$

From Proposition 3.1 and Theorem 3.2, we have the following corollary.

Corollary 3.3. *The composition $\Psi \circ \Upsilon^{-1} : B(\infty) \longrightarrow \mathfrak{B}(\infty)$ is a crystal isomorphism.*

Let $\lambda \in P^+$ be the dominant integral weight and let $B(\lambda)$ be the crystal of the irreducible highest weight module $V(\lambda)$. It was shown in [10] that there is a unique strict crystal embedding

$$B(\lambda) \hookrightarrow B(\infty) \otimes T_\lambda \otimes C, \quad v_\lambda \mapsto 1 \otimes t_\lambda \otimes c$$

where v_λ is the highest weight vector of $B(\lambda)$. Here, $T_\lambda = \{t_\lambda\}$ (resp. $C = \{c\}$) is a crystal with $\text{wt}(t_\lambda) = \lambda$, $\varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = 0$, $\tilde{e}_i t_\lambda = \tilde{f}_i t_\lambda = -\infty$ (resp. $\text{wt}(t_\lambda) = 0$, $\varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = 0$, $\tilde{e}_i t_\lambda = \tilde{f}_i t_\lambda = 0$). We denote by ι_λ the composition of the strict embedding and the natural projection:

$$B(\lambda) \hookrightarrow B(\infty) \otimes T_\lambda \otimes C \rightarrow B(\infty).$$

Let $\mathcal{M}_X(\lambda) = \Upsilon^{-1} \circ \iota_\lambda(B(\lambda))$. By Proposition 3.1, the set $B(\lambda)$ is in 1-1 correspondence to $\mathcal{M}_X(\lambda)$ via $\Upsilon^{-1} \circ \iota_\lambda$. Then the map $\Upsilon_\lambda := (\Upsilon^{-1} \circ \iota_\lambda)^{-1} : \mathcal{M}_X(\lambda) \rightarrow B(\lambda)$ is give by

$$\Upsilon_\lambda(\underline{\mathbf{m}}) = \tilde{f}_{\nu_X}^{\mathbf{e}_1(\underline{\mathbf{m}})} \dots \tilde{f}_{\nu_X}^{\mathbf{e}_n(\underline{\mathbf{m}})} v_\lambda \quad \text{for } \underline{\mathbf{m}} \in \mathcal{M}_X(\lambda),$$

$$\text{where } \mathbf{e}_i(\underline{\mathbf{m}}) = \begin{cases} \sum_{\mathbf{s} \in \underline{\mathbf{m}}(n+1-i)} \mathbf{e}(\mathbf{s}) & \text{if either } X = A, B, C \text{ or } i \neq 1, 2 \ (X = D), \\ \#\underline{\mathbf{m}}(n-1)\mathbf{e}_{n-1} & \text{if } i = 2 \ (X = D), \\ \#\underline{\mathbf{m}}(n)\mathbf{e}_n & \text{if } i = 1 \ (X = D). \end{cases}$$

We remark that the set $\mathcal{M}_X(\lambda)$ can be described explicitly from the string parametrization of $B(\lambda)$ given in [15]. By Theorem 3.2 and Corollary 3.3, we have the following Corollary.

Corollary 3.4. *Let $\lambda \in P^+$ be a dominant integral weight.*

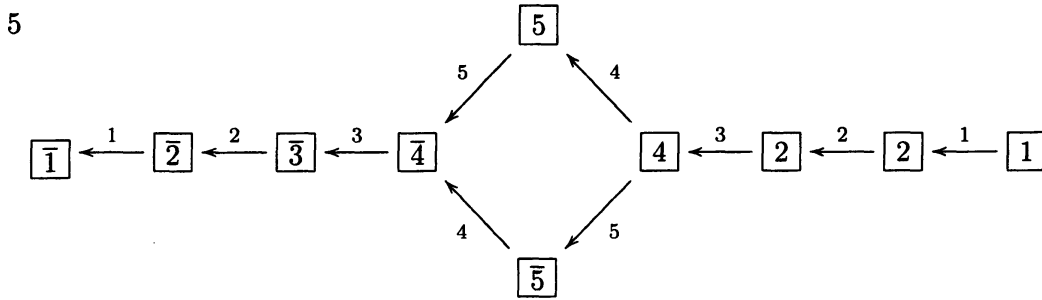
(1) *Let $\Psi^\lambda : \mathcal{M}_X(\lambda) \rightarrow \mathfrak{B}(\lambda)$ be the map defined by*

$$\Psi^\lambda(\underline{\mathbf{m}}) = \text{hdInd} \nabla(\underline{\mathbf{m}}) \quad \text{for } \underline{\mathbf{m}} \in \mathcal{M}_X(\lambda).$$

Then the map Ψ^λ is bijective.

(2) *The composition $\Psi^\lambda \circ \Upsilon_\lambda^{-1} : B(\lambda) \rightarrow \mathfrak{B}(\lambda)$ is a crystal isomorphism.*

Example 3.5. Let $U_q(\mathfrak{g})$ be of type D_5 . Then the crystal $\mathbb{B}_D$ is given as follows:



Note that $\nu_D = (1, 2, 3, 4, 5, 3, 2, 1)$. We choose the following segments $\mathfrak{s}_k$ ($k = 1, \dots, 6$) of $\mathbb{B}_X$:

$$\begin{aligned} \mathfrak{s}_1 &:= \{\bar{4}, \bar{5}\}, & \mathfrak{s}_2 &:= \{\bar{3}, 5\}, & \mathfrak{s}_3 &:= \{\bar{2}, \bar{4}\}, \\ \mathfrak{s}_4 &:= \{\bar{2}, 4\}, & \mathfrak{s}_5 &:= \{\bar{1}, \bar{4}\}, & \mathfrak{s}_6 &:= \{\bar{1}, 4\} \end{aligned}$$

and let $\underline{\mathbf{m}} = \{\mathfrak{s}_1 \preceq \mathfrak{s}_2 \preceq \mathfrak{s}_3 \preceq \mathfrak{s}_4 \preceq \mathfrak{s}_5 \preceq \mathfrak{s}_6\}$ be the multisegment consisting of $\mathfrak{s}_k$ ($k = 1, \dots, 6$). Note that

$$\begin{aligned} \text{ch} \nabla(\mathfrak{s}_1) &= (4), & \text{ch} \nabla(\mathfrak{s}_2) &= (3, 5), \\ \text{ch} \nabla(\mathfrak{s}_3) &= (2, 3), & \text{ch} \nabla(\mathfrak{s}_4) &= (2, 3, 4, 5) + (2, 3, 5, 4), \\ \text{ch} \nabla(\mathfrak{s}_5) &= (1, 2, 3), & \text{ch} \nabla(\mathfrak{s}_6) &= (1, 2, 3, 4, 5) + (1, 2, 3, 5, 4). \end{aligned}$$

Since $\underline{\mathbf{m}}(1) = \{\mathfrak{s}_5, \mathfrak{s}_6\}$, $\underline{\mathbf{m}}(2) = \{\mathfrak{s}_3, \mathfrak{s}_4\}$, $\underline{\mathbf{m}}(3) = \{\mathfrak{s}_2\}$, $\underline{\mathbf{m}}(4) = \{\mathfrak{s}_1\}$ and $\underline{\mathbf{m}}(5) = \emptyset$, we have

$$\begin{aligned} \mathbf{e}_1(\underline{\mathbf{m}}) &= (2, 2, 2, 1, 1, 0, 0, 0), & \mathbf{e}_2(\underline{\mathbf{m}}) &= (0, 2, 2, 1, 1, 0, 0, 0), \\ \mathbf{e}_3(\underline{\mathbf{m}}) &= (0, 0, 1, 0, 1, 0, 0, 0), & \mathbf{e}_4(\underline{\mathbf{m}}) &= (0, 0, 0, 1, 0, 0, 0, 0), \\ \mathbf{e}_5(\underline{\mathbf{m}}) &= (0, 0, 0, 0, 0, 0, 0, 0). \end{aligned}$$

It follows from Theorem 3.2 that

$$\begin{aligned} \mathrm{hdInd} \nabla(\underline{\mathfrak{m}}) &\simeq \mathrm{hdInd} (\nabla(\mathfrak{s}_1) \boxtimes \nabla(\mathfrak{s}_2) \boxtimes \nabla(\mathfrak{s}_3) \boxtimes \nabla(\mathfrak{s}_4) \boxtimes \nabla(\mathfrak{s}_5) \boxtimes \nabla(\mathfrak{s}_6)) \\ &\simeq \tilde{f}_4 \tilde{f}_3 \tilde{f}_5 \tilde{f}_2^2 \tilde{f}_3^2 \tilde{f}_4 \tilde{f}_5 \tilde{f}_1^2 \tilde{f}_2^2 \tilde{f}_3^2 \tilde{f}_4 \tilde{f}_5 1. \end{aligned}$$

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