

BRAUER CORRESPONDENCE AND GREEN CORRESPONDENCE *

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1 Introduction

Let k be an algebraically closed field of prime characteristic p . Let G be a finite group of order divisible by p . We are concerned with cohomology algebras of block ideals which are in Brauer correspondence and block varieties of modules in Green correspondence.

2 Cohomology of blocks and Brauer correspondence

Let B be a block ideal of kG . Proposition 2.3 of Kessar, Linckelmann and Robinson [5] implies

$$H^*(G, B) \subseteq H^*(H, C),$$

where C is a suitably taken block ideal of a suitably chosen subgroup H of G . To understand such an inclusion via transfer map between the Hochschild cohomology algebras of the block ideals B and C we discussed in Kawai and Sasaki [4] under the following situation.

- B has D as a defect group
- H is a subgroup of G and C is a block ideal of kH
- Brauer correspondent C^G is defined and $C^G = B$ and D is also a defect group of C

We had considered the (C, B) -bimodule $M = CB$ and gave a necessary and sufficient condition for M to induce the transfer map from $HH^*(B)$ to $HH^*(C)$ which restricts to the inclusion map of $H^*(G, B)$ into $H^*(H, C)$.

Here we discuss under the following situation:

Situation (BC)

- B has D as a defect group
- H is a subgroup of G such that $DC_G(D) \leq H$ and C is a block ideal of kH
- $C^G = B$ and D is also a defect group of C

*The detailed version of this note will be submitted for publication elsewhere.

We shall denote by G^{op} the opposite group of the group G and consider the group algebra kG as a $k[G \times G^{\text{op}}]$ -module through

$$(x, y)\alpha = x\alpha y \text{ for } x, y \in G \text{ and } \alpha \in kG.$$

We have a $k[G \times G^{\text{op}}]$ -isomorphism

$$kG \simeq k[G \times G^{\text{op}}] \otimes_{k[\Delta G]} k,$$

where $\Delta G = \{(g, g^{-1}) \mid g \in G\}$.

Definition 2.1. Under Situation (BC), the Green correspondent of C with respect to $(G \times H^{\text{op}}, \Delta D, H \times H^{\text{op}})$ is defined, which turns out to be a (B, C) -bimodule; we denote it by $L(B, C)$.

The module $L(B, C)$ will play crucial role, depending on the following fact.

Theorem 2.1. Under Situation (BC) let $L = L(B, C)$.

- (i) *The relatively projective elements $\pi_L \in Z(B)$ and $\pi_{L^*} \in Z(C)$ are both invertible.*
- (ii) *Every (B, A) -bimodule is relatively L -projective; every (C, A) -bimodule is relatively L^* -projective, where A is a symmetric k -algebra.*

Following Alperin, Linckelmann and Rouquier [1], we recall the definition of source modules of block ideals.

Definition 2.2. ([1, Definition 2]) There exists an indecomposable direct summand X of ${}_{G \times D^{\text{op}}}B$ having ΔD as a vertex. The $k[G \times D^{\text{op}}]$ -module X is called a *source module* of the block B .

We shall write $H^*(G, B; X)$ for the block cohomology of B with respect to the defect group D and the source idempotent i such that $X = kGi$.

Green correspondence between indecomposable $k[G \times D^{\text{op}}]$ -modules and indecomposable $k[H \times D^{\text{op}}]$ -modules relates source modules of the blocks B and C in the following way.

Proposition 2.2. Under Situation (BC) let Y be a source module of C . Then the Green correspondent X of Y with respect to $(G \times D^{\text{op}}, \Delta D, H \times D^{\text{op}})$ is a source module of B .

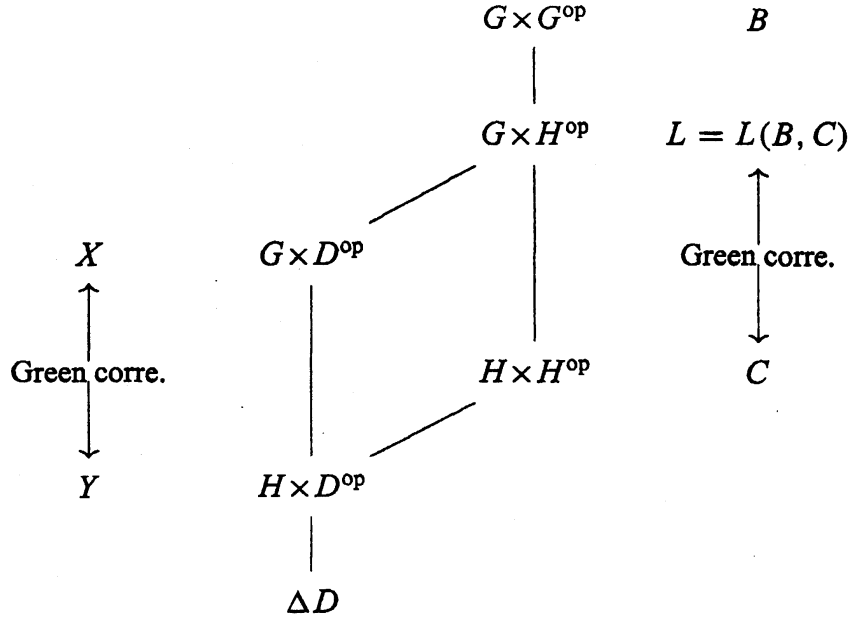
Proposition 2.3. Under Situation (BC) take a source module X of B as a direct summand of ${}_{G \times D^{\text{op}}}L(B, C)$. Then the Green correspondent Y of X with respect to $(G \times D^{\text{op}}, \Delta D, H \times D^{\text{op}})$ is a source module of C .

Thus, under Situation (BC) we can take a source module X of the block B and a source module Y of the block C in order that X and Y are in the Green correspondence with respect to $(G \times D^{\text{op}}, \Delta D, H \times D^{\text{op}})$. We refer to such situation as Situation (XY).

Situation (XY)

- B has D as a defect group
- H is a subgroup of G such that $DC_G(D) \leq H$ and C is a block ideal of kH

- $C^G = B$ and D is also a defect group of C
- a source module X of the block B and a source module Y of the block C are in the Green correspondence with respect to $(G \times D^{\text{op}}, \Delta D, H \times D^{\text{op}})$



Then the (B, C) -bimodule $L = L(B, C)$ links the source modules X and Y in a similar way to induction and restriction of modules.

Theorem 2.4. *Under Situation (XY) the following hold.*

- (i) $L^* \otimes_B X \equiv Y \oplus O(\mathcal{Y}(G \times D^{\text{op}}, \Delta D, H \times D^{\text{op}}))$.
- (ii) $L \otimes_C Y \equiv X \oplus O(\mathcal{X}(G \times D^{\text{op}}, \Delta D, H \times D^{\text{op}}))$.
- (iii) *If $D \triangleleft H$, then $L \otimes_C Y \simeq X$.*

The (B, C) -bimodule $L(B, C)$ has already appeared in some works. In particular, Alperin, Linckelmann and Rouquier [1] treated the case of $H = N_G(D, b_D)$, where (D, b_D) is a Sylow B -subpair. Theorem 5 in [1] corresponds to our theorem above.

Theorem 2.5. *Under Situation (XY) the module $L(B, C)$ is splendid with respect to X and Y , namely*

$$L(B, C) \mid X \otimes_{kD} Y^*.$$

The theorem above and the following, which states that the relatively projective elements associated with tensor products of the bimodules L , X and Y , including such as $X^* \otimes_B L \otimes_C Y$, are all invertible, lead us Theorem 2.8, which is one of our main theorems.

Theorem 2.6. *Under Situation (XY) the relatively projective elements*

- (i) $\pi_{L \otimes_C Y} \in Z(B)$, $\pi_{Y^* \otimes_C L^*} \in Z(kD)$
- (ii) $\pi_{X^* \otimes_B L \otimes_C Y} \in Z(kD)$, $\pi_{X^* \otimes_B L} \in Z(kD)$
- (iii) $\pi_{Y^* \otimes_C L^* \otimes_B X} \in Z(kD)$, $\pi_{L^* \otimes_B X} \in Z(C)$

are all invertible.

Proposition 2.7. *Under Situation (XY) we have the following commutative diagram:*

$$\begin{array}{ccccc}
 H^*(G, B; X) & \xrightarrow{\delta_D} & HH_{X^*}^*(kD) & \xrightleftharpoons[R_{X^*}]{R_X} & HH_X^*(B) \\
 & & \uparrow & & \uparrow \\
 & & HH_{X^* \otimes_B L \otimes_C Y}^*(kD) & \xrightleftharpoons[R_{X^*}]{R_X} & HH_{L \otimes_C Y}^*(B) \hookrightarrow HH_L^*(B) \\
 & & \parallel & & \uparrow R_L \quad \downarrow R_L^* \\
 & & HH_{Y^* \otimes_C L^* \otimes_B X}^*(kD) & \xrightleftharpoons[R_{Y^*}]{R_Y} & HH_{L^*}^*(C) \cap HH_Y^*(C) \hookrightarrow HH_{L^*}^*(C) \\
 & & \downarrow & & \downarrow \\
 H^*(H, C; Y) & \xrightarrow{\delta_D} & HH_{Y^*}^*(kD) & \xrightleftharpoons[R_{Y^*}]{R_Y} & HH_Y^*(C)
 \end{array}$$

Theorem 2.8. *Let B be a block ideal of kG and $D \leq G$ a defect group of B . Assume that a subgroup H of G containing $DC_G(D)$ normalizes a subgroup Q of D and contains $QC_G(Q)$. Let (D, b_D) be a Sylow B -subpair and let $(Q, b_Q) \leq (D, b_D)$. Let C be a unique block ideal of kH covering the block ideal b_Q of $kQC_G(Q)$. Then $C^G = B$ and D is a defect group of C ; hence (D, b_D) is also a Sylow C -subpair.*

Let j be a source idempotent of C such that $\text{Br}_D(j)e_D = \text{Br}_D(j)$, where $e_D \in kC_G(D)$ is the block idempotent of the block b_D ; let $Y = kHj$. Let X be a source module of B which is the Green correspondent of Y with respect to $(G \times D^{\text{op}}, \Delta D, H \times D^{\text{op}})$. We let $L = L(B, C)$. Then the following diagram commutes:

$$\begin{array}{ccccc}
 H^*(G, B; X) & \xrightarrow{\delta_D} & HH_{X^* \otimes_B L \otimes_C Y}^*(kD) & \xrightleftharpoons[R_{X^*}]{R_X} & HH_{L \otimes_C Y}^*(B) \\
 \downarrow & & \downarrow & & \downarrow R_L^* \\
 H^*(H, C; Y) & \xrightarrow{\delta_D} & HH_{Y^*}^*(kD) & \xrightleftharpoons[R_{Y^*}]{R_Y} & HH_Y^*(C)
 \end{array}$$

3 Block varieties of modules and Green correspondence

If $H^*(G, B; X) \subseteq H^*(H, C; Y)$, then Kawai and Sasaki [4, Theorem 1.3 (i)] says that the inclusion map $\iota : H^*(G, B; X) \hookrightarrow H^*(H, C; Y)$ induces a surjective map $\iota^* : V_{H,C} \rightarrow V_{G,B}$ of varieties.

Throughout this section we let $P \leq D$ and assume that $H \geq N_G(P)$. We investigate relationship between the varieties of modules in blocks B and C which are under Green correspondence.

We first note Under Situation (BC) that to tensor with $L(B, C)$ and $L(B, C)^*$ induces the Green correspondence.

Proposition 3.1. *Under Situation (BC), we let $L = L(B, C)$. If an indecomposable B -module U and an indecomposable C -module V have vertices in $\mathcal{A}(G, P, H)$ and are in the Green correspondence with respect to (G, P, H) , then*

$$\begin{aligned} L \otimes_C V &\equiv U \oplus O(\mathcal{X}(G, P, H)) \\ L^* \otimes_B U &\equiv V \oplus O(\mathcal{Y}(G, P, H)). \end{aligned}$$

The block variety of an indecomposable module is determined by particular vertex and a particular source by Benson and Linckelmann [2].

Definition 3.1. (Benson and Linckelmann [2, Proposition 2.5]) Let X be a source module of a block ideal B . Let U be an indecomposable B -module. There exists a vertex Q of U such that

$$Q \leq D, U \mid X \otimes_{kQ} X^* \otimes_B U.$$

We would like to call such a vertex Q of U an X -vertex. For an X -vertex Q of U we can take a Q -source S of U such that

$$S \mid {}_{kQ}X^* \otimes_B U, U \mid X \otimes_{kQ} S$$

We would like to call such a source a (Q, X) -source.

[2, Theorem 1.1] says that the block variety $V_{G,B}(U)$ in the block cohomology $H^*(G, B; X)$ is the pull back of the variety $V_Q(S)$ of S , where Q is an X -vertex and S is a (Q, X) -source of U .

Proposition 3.2. *Under Situation (XY), let U and V be as in Proposition 3.1. Then the following hold.*

- (i) *If $Q \in \mathcal{A}(G, P, H)$ is a Y -vertex of V and S is a (Q, Y) -source of V , then Q is an X -vertex of U and S is a (Q, X) -source of U .*
- (ii) *If $Q \in \mathcal{A}(G, P, H)$ is an X -vertex of U and S is a (Q, X) -source of U , then Q is a Y -vertex of V and S is a (Q, Y) -source of V .*

It is well known that the Green correspondent of an indecomposable module lies in a block ideal of a subgroup of G lies in its Brauer correspondent. The following is a partial converse to this fact.

Proposition 3.3. *Under Situation (XY), assume that an indecomposable B -module U has an X -vertex belonging to $\mathcal{A}(G, P, H)$. Then the Green correspondent V of U with respect to (G, P, H) lies in the block C .*

The following is our main theorem.

Theorem 3.4. *Under Situation (XY) assume that $H^*(G, B; X) \subseteq H^*(H, C; Y)$.*

- (i) *Assume that an indecomposable B -module U has an X -vertex belonging to $\mathcal{A}(G, P, H)$. Then the Green correspondent V of U with respect to (G, P, H) lies in the block C and*

$$V_{G,B}(U) = \iota^* V_{H,C}(V).$$

- (ii) Assume that an indecomposable C -module V has a Y -vertex belonging to $\mathcal{A}(G, P, H)$. Then the Green correspondent U of V with respect to (G, P, H) lies in the block B and

$$V_{G,B}(U) = \iota^* V_{H,C}(V).$$

Example. (cf [2, Corollary 1.4]) Let B be a block ideal of kG and $D \leq G$ a defect group of B . Let X be a source module of B . Let U be an indecomposable B -module and Q an X -vertex of U and S a (Q, X) -source of U . Assume that the X -vertex Q of U is normal in D and let $H = N_G(Q)$. Let $P \leq D$ and assume that $H \geq N_G(P)$ and that $Q \in \mathcal{A}(G, P, H)$.

Let (D, b_D) be a Sylow B -subpair such that $b_D X(D) = X(D)$ and let $(Q, b_Q) \leq (D, b_D)$. Let C be a unique block of kH covering the block b_Q .

Then we have

- (i) $H^*(G, B) \subseteq H^*(H, C)$;
- (ii) Q is a Y -vertex of V and S is a (Q, Y) -source of V ;
- (iii) V lies in C and $V_{G,B}(U) = \iota^* V_{H,C}(V)$.

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