

ARCHIMEDEAN ZETA INTEGRALS FOR FURUSAWA'S PULLBACK FORMULA

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Given a positive integer n , define the algebraic group $\mathrm{GSp}(2n)$ over $\mathbb{Z}$ as

$$\mathrm{GSp}(2n, R) = \left\{ g \in \mathrm{GL}(2n, R) : {}^t g \begin{bmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{bmatrix} g = \nu_g \begin{bmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{bmatrix}, \nu_g \in R^\times \right\}$$

for all $\mathbb{Z}$ -algebra R . When $n = 1$, we write $\mathrm{GL}(2) = \mathrm{GSp}(2)$. Fix an imaginary quadratic field $\mathcal{K}$. Define the algebraic group $\mathrm{GU}(n, n)$ over $\mathbb{Z}$ as

$$\mathrm{GU}(n, n)(R) = \left\{ g \in \mathrm{GL}(2n, \mathcal{O}_{\mathcal{K}} \otimes R) : {}^t \bar{g} \begin{bmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{bmatrix} g = \nu_g \begin{bmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{bmatrix}, \nu_g \in R^\times \right\},$$

where for $\alpha \in \mathcal{K}$, $\bar{\alpha}$ denotes its image under the nontrivial element in $\mathrm{Gal}(\mathcal{K}/\mathbb{Q})$.

Let $Q_{\mathrm{GU}(3,3)} \subset \mathrm{GU}(3, 3)$ be the standard Siegel parabolic subgroup. Let Ξ be a Hecke character of $\mathcal{K}$ and χ be a Dirichlet character, and $I_v(s, \chi, \Xi)$ be the degenerate principal series on $\mathrm{GU}(3, 3)(\mathbb{Q}_v)$ consisting of smooth functions $\mathbf{f}_v(s, \chi, \Xi) : \mathrm{GU}(3, 3)(\mathbb{Q}_v) \rightarrow \mathbb{C}$ such that

$$\mathbf{f}_v(s, \chi, \Xi) \left(\begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ 0 & \mathfrak{D} \end{bmatrix} g \right) = \Xi_v(\det \mathfrak{A}) \chi_v(\det \mathfrak{A} \mathfrak{D}^{-1}) |\det \mathfrak{A} \mathfrak{D}^{-1}|_v^{s+\frac{3}{2}} \mathbf{f}_v(s, \chi, \Xi)(g)$$

for all $g \in \mathrm{GU}(3, 3)(\mathbb{Q}_v)$ and $\begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ 0 & \mathfrak{D} \end{bmatrix} \in Q_{\mathrm{GU}(3,3)}(\mathbb{Q}_v)$. If v is a finite place where χ_v, Ξ_v are unramified, we denote by $\mathbf{f}_v^{\mathrm{sph}}(s, \chi, \Xi)$ the unique section in $I_v(s, \chi, \Xi)$ invariant under the right translation by $\mathrm{GU}(3, 3)(\mathbb{Z}_v)$ and taking value 1 on $\mathrm{GU}(3, 3)(\mathbb{Z}_v)$. Put $I(s, \chi, \Xi) = \bigotimes'_v I_v(s, \chi, \Xi)$, the restricted tensor product with respect to $\mathbf{f}_v^{\mathrm{sph}}(s, \chi, \Xi)$. Given a section $\mathbf{f}(s, \chi, \Xi) \in I(s, \chi, \Xi)$ analytic in the variable s , the associated Siegel Eisenstein series is defined as

$$E^{\mathrm{Sieg}}(g; \mathbf{f}(s, \chi, \Xi)) = \sum_{\gamma \in Q_{\mathrm{GU}(3,3)}(\mathbb{Q}) \backslash \mathrm{GU}(3,3)(\mathbb{Q})} \mathbf{f}(s, \chi, \Xi)(\gamma g).$$

Let Π (resp. π) be a cuspidal automorphic representation of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ (resp. $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$). Given $f \in \pi$, thanks to the isomorphism

$$\begin{aligned} (\mathrm{Res}_{\mathcal{K}/\mathbb{Q}} \mathrm{GL}(1)) \times \mathrm{GL}(2) / \{(a, a^{-1} \cdot \mathbf{1}_2) : a \in \mathrm{GL}(1)\} &\xrightarrow{\sim} \mathrm{GU}(1, 1) \\ (\mathfrak{a}, g) &\longmapsto \mathfrak{a}g, \end{aligned}$$

by choosing a Hecke character $\Upsilon : \mathcal{K}^\times \backslash \mathbb{A}_{\mathcal{K}}^\times \rightarrow \mathbb{C}^\times$ with $\Upsilon_{\mathbb{Q}} = \Upsilon|_{\mathbb{A}_{\mathbb{Q}}^\times} = \omega_\pi$, we can define an automorphic form f^Υ on $\mathrm{GU}(1, 1)$ as

$$f^\Upsilon(\mathfrak{a}g) = \Upsilon(\mathfrak{a})f(g), \quad \mathfrak{a} \in \mathbb{A}_{\mathcal{K}}^\times, g \in \mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}}).$$

The space $\pi^\Upsilon = \{f^\Upsilon : f \in \pi\}$ is an irreducible cuspidal automorphic representation of $\mathrm{GU}(1, 1)(\mathbb{A}_{\mathbb{Q}})$. Put

$$\mathrm{GSp}(4) \times_{\mathrm{GL}(1)} \mathrm{GU}(1, 1) = \{(g, h) \in \mathrm{GSp}(4) \times \mathrm{GU}(1, 1) : \nu_g = \nu_h\},$$

and fix an embedding

$$(1) \quad \iota : \mathrm{GSp}(4) \times_{\mathrm{GL}(1)} \mathrm{GU}(1, 1) \longrightarrow \mathrm{GU}(3, 3)$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \times \begin{bmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathfrak{c} & \mathfrak{d} \end{bmatrix} \mapsto \begin{bmatrix} A & B & \\ C & \mathfrak{a} & \mathfrak{b} \\ & \mathfrak{c} & \mathfrak{d} \end{bmatrix}.$$

We consider the integration of the restriction of $E^{\mathrm{Sieg}}(g; \mathbf{f}(s, \chi, \Xi))$ via ι to $\mathrm{GSp}(4) \times_{\mathrm{GL}(1)} \mathrm{GU}(1, 1)$ against a pair of automorphic forms on $\mathrm{GSp}(4)$ and $\mathrm{GU}(1, 1)$. By Furusawa's pullback formula [Fur93], the integral factorizes into a product of local zeta integrals and gives an integral representation for the L -function of $\Pi \times \pi$.

Theorem 1. *Let $\Lambda = (\Xi^c \Upsilon)^{-1}$ and we assume that $\Lambda_{\mathbb{Q}} = \Lambda|_{\mathbb{A}_{\mathbb{Q}}^\times} = \omega_{\Pi}$. Fix isomorphisms $\Pi \cong \bigotimes_v \Pi_v$, $\pi \cong \bigotimes_v \pi_v$, and suppose that the images of $\varphi \in \Pi$, $f \in \pi$ under these isomorphisms are pure tensors, denoted by $\bigotimes_v \varphi_v$, $\bigotimes_v f_v$.*

$$\int_{[\mathrm{GSp}(4) \times_{\mathrm{GL}(1)} \mathrm{GU}(1, 1)]} E^{\mathrm{Sieg}}(\iota(g, h); \mathbf{f}(s, \chi, \Xi)) \cdot \varphi(g) \cdot f^{\Upsilon}(h) \Xi^{-1}(\det h) dh dg$$

$$= B_{\mathbb{S}, \Lambda}(\varphi) W_{\mathfrak{c}}(f) \cdot \prod_v Z_v \left(\mathbf{f}_v(s, \chi, \Xi), \mathcal{B}_{\mathbb{S}, \Lambda_v}^{\Pi_v}(\varphi_v), \mathcal{W}_{\mathfrak{c}}^{\pi_v, \Upsilon_v}(f_v) \right).$$

Here

$$\mathbb{S} = \begin{bmatrix} \mathfrak{a} & \frac{\mathfrak{b}}{2} \\ \frac{\mathfrak{b}}{2} & \mathfrak{c} \end{bmatrix} \in \mathrm{Sym}_2(\mathbb{Q}) \text{ such that } \mathcal{K} = \mathbb{Q} \left(\alpha_{\mathbb{S}} = \frac{\mathfrak{b} + \sqrt{\mathfrak{b}^2 - 4\mathfrak{a}\mathfrak{c}}}{2\mathfrak{c}} \right),$$

$$B_{\mathbb{S}, \Lambda}(\varphi) = \int_{[R_{\mathbb{S}}]} \Lambda_{\mathbb{S}}^{-1}(r) \cdot \varphi(r) dr, \quad W_{\mathfrak{c}}(f) = \int_{\mathbb{Q} \backslash \mathbb{A}_{\mathbb{Q}}} f \left(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \right) \cdot \psi_{\mathbb{A}_{\mathbb{Q}}}(-\mathfrak{c}x) dx,$$

is the global Bessel period (resp. Whittaker period), where

$$R_{\mathbb{S}} = \left\{ \begin{bmatrix} \iota_{\mathbb{S}}(\mathfrak{z}) & \\ & {}_t \iota_{\mathbb{S}}(\bar{\mathfrak{z}}) \end{bmatrix} \begin{bmatrix} \mathbf{1}_2 & X \\ & \mathbf{1}_2 \end{bmatrix} : \mathfrak{z} \in \mathrm{Res}_{\mathcal{K}/\mathbb{Q}} \mathrm{GL}(1), X \in \mathrm{Sym}_2 \right\},$$

$$\iota_{\mathbb{S}}(\mathfrak{z}) = \begin{bmatrix} \alpha_{\mathbb{S}} & 1 \\ \bar{\alpha}_{\mathbb{S}} & 1 \end{bmatrix}^{-1} \begin{bmatrix} \mathfrak{z} & \\ & \bar{\mathfrak{z}} \end{bmatrix} \begin{bmatrix} \alpha_{\mathbb{S}} & 1 \\ \bar{\alpha}_{\mathbb{S}} & 1 \end{bmatrix},$$

$$\Lambda_{\mathbb{S}} \left(\begin{bmatrix} \iota_{\mathbb{S}}(\mathfrak{z}) & \\ & {}_t \iota_{\mathbb{S}}(\bar{\mathfrak{z}}) \end{bmatrix} \begin{bmatrix} \mathbf{1}_2 & X \\ & \mathbf{1}_2 \end{bmatrix} \right) = \Lambda(\mathfrak{z}) \psi_{\mathbb{A}_{\mathbb{Q}}}(\mathrm{Tr} \mathbb{S} X).$$

- If φ_v, f_v are both spherical, Υ, Ξ, χ are all unramified at v , $\mathbb{S} \in M_2(\mathbb{Z}_v)$, $\mathfrak{c} \in \mathbb{Z}_v^\times$, $4 \det \mathbb{S}$ generates $\mathrm{disc}(\mathcal{K}_v/\mathbb{Q}_v)$, the local zeta integral $Z_v \left(\mathbf{f}_v^{\mathrm{sph}}(s, \chi, \Xi), \mathcal{B}_{\mathbb{S}, \Lambda_v}^{\Pi_v}(\varphi_v), \mathcal{W}_{\mathfrak{c}}^{\pi_v, \Upsilon_v}(f_v) \right)$ equals

$$\left(\prod_{1 \leq j \leq 3} L_v \left(2s + j, \Xi_{\mathbb{Q}} \chi^2 \eta_{\mathcal{K}/\mathbb{Q}}^{n-j} \right) \right)^{-1} L_v \left(s + \frac{1}{2}, \tilde{\Pi} \times \tilde{\pi} \times \chi \right)$$

with $\eta_{\mathcal{K}/\mathbb{Q}}$ the quadratic character of $\mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times$ associated to the extension $\mathcal{K}/\mathbb{Q}$.

- In general, taking nonzero $B_{\mathbb{S}, \Lambda_v}^{\Pi_v} \in \mathrm{Hom}_{R_{\mathbb{S}}(\mathbb{Q}_v)}(\Pi_v, \Lambda_{\mathbb{S}, v})$, $W_{\mathfrak{c}}^{\pi_v} \in \mathrm{Hom}_{U_{\mathrm{GL}(2)}(\mathbb{Q}_v)}(\pi_v, \psi_{\mathfrak{c}, v})$ and putting $\mathcal{B}_{\mathbb{S}, \Lambda_v}^{\Pi_v}(\varphi_v)$ the function on $\mathrm{GSp}(4, \mathbb{Q}_v)$ sending g_v to $B_{\mathbb{S}, \Lambda_v}^{\Pi_v}(g_v \cdot \varphi_v)$, $\mathcal{W}_{\mathfrak{c}}^{\pi_v, \Upsilon_v}(f_v)$

the function on $\mathrm{GU}(1,1)(\mathbb{Q}_v)$ sending $\mathfrak{a}g$, $\mathfrak{a} \in \mathcal{K}$, $g \in \mathrm{GL}(2, \mathbb{Q}_v)$, to $\Upsilon_v(\mathfrak{a}) \cdot W_{\mathfrak{e}}^{\pi_v}(g \cdot f_v)$,

$$(2) \quad \begin{aligned} & Z_v \left(\mathfrak{f}_v(s, \chi, \Xi), \mathcal{B}_{\mathbb{S}, \Lambda_v}^{\Pi_v}(\varphi_v), \mathcal{W}_{\mathfrak{e}}^{\pi_v, \Upsilon_v}(f_v) \right) \\ &= B_{\mathbb{S}, \Lambda_v}^{\Pi_v}(\varphi_v)^{-1} W_{\mathfrak{e}}^{\pi_v}(f_v)^{-1} \int_{(R'_{\mathbb{S}} \setminus \mathrm{GSp}(4) \times_{\mathrm{GL}(1)} \mathrm{GU}(1,1))(\mathbb{Q}_v)} \mathfrak{f}_v(s, \chi, \Xi) (S^{-1} \iota(\eta_{\mathbb{S}} g, h)) \\ & \quad \times \mathcal{B}_{\mathbb{S}, \Lambda_v}^{\Pi_v}(\varphi_v)(g) \mathcal{W}_{\mathfrak{e}}^{\pi_v, \Upsilon_v}(f_v) \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} h \right) \Xi_v^{-1}(\det h) dh dg, \end{aligned}$$

with

$$\eta_{\mathbb{S}} = \begin{bmatrix} 1 & & & \\ \alpha_{\mathbb{S}} & 1 & & \\ & & 1 & -\bar{\alpha}_{\mathbb{S}} \\ & & & 1 \end{bmatrix}, \quad \mathcal{S} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \\ & 1 & & & & 1 \end{bmatrix}.$$

$$R'_{\mathbb{S}} = \left\{ \left(\begin{bmatrix} \iota_{\mathbb{S}}(\mathfrak{z}) & \\ & \iota_{\mathbb{S}}(\bar{\mathfrak{z}}) \end{bmatrix} \begin{bmatrix} \mathbf{1}_2 & X \\ & \mathbf{1}_2 \end{bmatrix}, \mathfrak{z} \cdot \mathbf{1}_2 \right) : \mathfrak{z} \in \mathrm{Res}_{\mathcal{K}/\mathbb{Q}} \mathrm{GL}(1), X \in \mathrm{Sym}_2 \right\}.$$

We consider the computation of the local zeta integral (2) at the archimedean place in a case that is of arithmetic interest. Suppose that $\chi_{\infty}, \Xi_{\infty}$ are both trivial on $\mathbb{R}_{>0}$ and the ∞ -type of Ξ_{∞} is $(\frac{r}{2}, -\frac{r}{2})$. Given a positive integer t , we define the weight- t -classical section $\mathfrak{f}_{\infty}^t(s, \chi, \Xi)$ as

$$g = \begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix} \mapsto \chi_{\infty}(\nu_g) \nu_g^{\frac{3}{2}(r-t)} |\nu_g|^{\frac{3}{2}(s+\frac{3}{2}-r)} (\det g)^{\frac{r+t}{2}} \det(\mathfrak{C}i + \mathfrak{D})^{-t} |\det(\mathfrak{C}i + \mathfrak{D})|^{-2s-3+t}.$$

(This section is of scalar weight $(\frac{t+r}{2}, \frac{t+r}{2}, \frac{t+r}{2}; -\frac{t+r}{2}, -\frac{t+r}{2}, -\frac{t+r}{2})$.) Denote the ∞ -type of Λ by $(\frac{r_{\Lambda}}{2}, -\frac{r_{\Lambda}}{2})$. We consider the evaluation of (2), with $v = \infty$, at $s = k + \frac{l_1+l_2+l-1}{2}$ in case where

- $\Pi_{\infty} = \mathcal{D}_{l_1, l_2}$ (resp. $\pi_{\infty} \cong \mathcal{D}_l$), the holomorphic discrete series of weight (l_1, l_2) (resp. weight l) satisfying the condition

$$(3) \quad \min\{-l_1 + l_2 + l, l_1 + l_2 - l\} \geq 3,$$

and $\varphi_{\infty} \in \Pi_{\infty}$ (resp. $f_{\infty} \in \pi_{\infty}$) is a vector of weight (l_1, l_2) (resp. weight l) inside the lowest K_{∞} -type (where K_{∞} is the connected component of the identity inside the maximal compact subgroup of $\mathrm{GSp}(4, \mathbb{R})$ (resp. $\mathrm{GL}(2, \mathbb{R})$)),

- $-l_1 + l_2 \leq r_{\Lambda} \leq l_1 - l_2$,
- the section $\mathfrak{f}_{\infty}(s, \chi, \Xi)$ is given as

$$D_{k, l, l_1, l_2, r_{\Lambda}} \cdot \mathfrak{f}_{\infty}^{|2k+l_1+l_2+l-1|+3}(s, \chi, \Xi)$$

with

$$\begin{aligned} D_{t, r_{\Lambda}, l_1, l_2, l} &= (2\pi i)^{\frac{3(|2k+l_1+l_2+l-1|+3)-l_1-l_2-l}{2}} \cdot \left(\mu_{\mathrm{GU}(3,3),31}^+ \right)^{\frac{l_1-l_2+r_{\Lambda}}{2}} \left(\mu_{\mathrm{GU}(3,3),13}^+ \right)^{\frac{l_1-l_2-r_{\Lambda}}{2}} \\ & \quad \times \det \begin{bmatrix} \mu_{\mathrm{GU}(3,3),13}^+ & \mu_{\mathrm{GU}(3,3),23}^+ \\ \mu_{\mathrm{GU}(3,3),31}^+ & \mu_{\mathrm{GU}(3,3),32}^+ \end{bmatrix}^{\frac{-l_1+l_2+l-(|2k+l_1+l_2+l-1|+3)}{2}} \\ & \quad \times \left(\frac{\mu_{\mathrm{GU}(3,3),21}^+ - \mu_{\mathrm{GU}(3,3),12}^+}{2} \right)^{\frac{l_1+l_2-l-(|2k+l_1+l_2+l-1|+3)}{2}}, \end{aligned}$$

a differential operator constructed as a polynomial in the Lie algebra operators $\mu_{\mathrm{GU}(3,3),ij}^+ \in \mathrm{Lie} \, \mathrm{GU}(3,3)(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ defined as

$$\begin{aligned} \mu_{\mathrm{GU}(3,3),ij}^+ &= \mu_{\mathrm{GU}(3,3),X}^+, & X &= \frac{E_{ij} + E_{ji}}{2} \otimes 1 + \frac{E_{ij} - E_{ji}}{2\sqrt{-1}} \otimes \sqrt{-1} \\ \mu_{\mathrm{GU}(3,3),X}^+ &= \begin{bmatrix} X & \\ & -X \end{bmatrix} \otimes 1 + \begin{bmatrix} & X \\ X & \end{bmatrix} \otimes \sqrt{-1}, & X &\in \mathrm{Her}_3(\mathcal{K} \otimes_{\mathbb{Q}} \mathbb{R}). \end{aligned}$$

Theorem 2. *In the case described as above,*

$$\begin{aligned} & Z_{\infty} \left(D_{k,l,l_1,l_2,r_{\Lambda}} \cdot \mathbf{f}_{\infty}^{|2k+l_1+l_2+l-1|+3}(s, \chi, \Xi), \mathcal{B}_{S, \Lambda_{\infty}}^{\mathcal{D}_{l_1,l_2}} \left(\begin{bmatrix} & \mathbf{1}_2 \end{bmatrix} \varphi_{\infty} \right), \mathcal{W}_{\mathbf{e}}^{\mathcal{D}_l, \Upsilon_{\infty}} \left(\begin{bmatrix} & 1 \end{bmatrix} \right) f_{\infty} \right) \Big|_{s=k+\frac{l+l_1+l_2-1}{2}} \\ &= \Gamma(k+l_1+l_2+l-2) \Gamma(k+l_1+l_2-1) \Gamma(k+l_2+l-1) \Gamma(k+l_1+l) \\ & \times \mathbf{e}^{k+l_1+1} (\det S)^{-k-l_1-\frac{l_2+l+1}{2}} \begin{cases} \frac{2^{-3l_1-2l_2-2l+10} \pi^{2k+7} i^{2k+l_1+2l_2-2l-r_{\Lambda}-1}}{\prod_{j=0}^2 \Gamma(2k+l_1+l_2+l+2-j)}, & k + \frac{l_1+l_2+l}{2} \geq \frac{1}{2} \\ 2^{-12k-9l_1-8l_2-8l+15} \pi^{-4k-3l_1-3l_2-3l+10} i^{2k-l_1+l-r_{\Lambda}}, & k + \frac{l_1+l_2+l}{2} \leq \frac{1}{2} \end{cases} \end{aligned}$$

The proof makes use of the existence of a four-variable p -adic L -function with partial interpolation properties. We first introduce some notation that will be used in the proof. We fix an odd prime p and a sufficiently large finite extension F of $\mathbb{Q}_p$. Denote by $\mathcal{O}$ the ring of integers of F . Let

$$\begin{aligned} T_{\mathrm{GSp}(2n)} &= \left\{ \mathrm{diag}(a_1, \dots, a_n, \nu a_1^{-1}, \dots, \nu a_n^{-1}) \in \mathrm{GSp}(2n) \right\}, \\ T_{\mathrm{GSp}(2n)}^1 &= T_{\mathrm{GSp}(2n)} \cap \mathrm{Sp}(2n). \end{aligned}$$

and

$$\tilde{\Lambda}_{\mathrm{GSp}(2n)} = \mathcal{O}[[T_{\mathrm{GSp}(2n)}^1(\mathbb{Z}_p)]].$$

Given $m_1 \geq m_2 \geq \dots \geq m_n \geq 0$, we define the $\mathbb{U}_p$ -operator $U_{p,m_1,\dots,m_n}$ as

$$\int_{U_{\mathrm{GSp}(2n)}(\mathbb{Z}_p)} \text{action by } u \, \mathrm{diag}(p^{m_1}, \dots, p^{m_n}, p^{-m_1}, \dots, p^{-m_n}) \, du.$$

We define the following p -modified Petersson inner products

$$\mathbf{P} : \mathcal{A}_{\mathrm{GSp}(4),\omega,\mathbb{U}_p \neq 0}^0 \times \mathcal{A}_{\mathrm{GSp}(4),\omega} \longrightarrow \mathbb{C}, \quad \mathbf{P} : \mathcal{A}_{\mathrm{GL}(2),\omega,\mathbb{U}_p \neq 0}^0 \times \mathcal{A}_{\mathrm{GL}(2),\omega} \longrightarrow \mathbb{C},$$

on the product of the subspace of cuspidal automorphic forms on $\mathrm{GSp}(4)$ (resp. $\mathrm{GL}(2)$) of central character ω on which $\mathbb{U}_p$ -operators act invertibly and the whole space of automorphic forms on $\mathrm{GSp}(4)$ of central character ω (resp. $\mathrm{GL}(2)$) as

$$\begin{aligned} \mathbf{P}(\varphi_1, \varphi_2) &= \int_{[\mathrm{GSp}(4)]} U_{p,m_1,m_2}^{-1} \varphi_1(g) \\ & \times \varphi_2 \left(g \begin{bmatrix} & \mathbf{1}_2 \end{bmatrix}_{p^{\infty}} \begin{bmatrix} p^{m_1} & & \\ & p^{m_2} & \\ & & p^{-m_1} \\ & & & p^{-m_2} \end{bmatrix}_p \right) \cdot \omega(\nu_g)^{-1} dg, & m_1 \gg m_2 \gg 0 \\ \mathbf{P}(f_1, f_2) &= \int_{[\mathrm{GL}(2)]} U_{p,m}^{-1} f_1(g) f_2 \left(g \begin{bmatrix} & 1 \end{bmatrix}_{p^{\infty}} \begin{bmatrix} p^m & \\ & p^{-m} \end{bmatrix}_p \right) \cdot \omega(\det g)^{-1} dg, & m \gg 0. \end{aligned}$$

One can check that the right hand sides of the above formulas are independent of m_1, m_2, m , and if φ_1, φ_2 (resp. f_1, f_2) both belong to $\mathcal{A}_{\mathrm{GSp}(4),\omega,\mathbb{U}_p \neq 0}^0$ (resp. $\mathcal{A}_{\mathrm{GL}(2),\omega,\mathbb{U}_p \neq 0}^0$), then $\mathbf{P}(\varphi_1, \varphi_2) = \mathbf{P}(\varphi_2, \varphi_1)$, $\mathbf{P}(f_1, f_2) = \mathbf{P}(f_2, f_1)$.

Proof of Theorem 2. Choose $\mathbb{L}_{\mathcal{B}}$ -adic and $\mathbb{L}_{\mathcal{B}'}$ -adic Hida families $\mathcal{B}, \mathcal{B}'$ on $\mathrm{GL}(2)$ of tame level $K_{\mathrm{GL}(2)}^p$ such that both admit a Jacquet–Langlands transfer to Hida families of a quaternion algebra D definite at ∞ . We also require $\mathcal{B}$ to be primitive. Define

$$\lambda_{\theta(\mathcal{B}, \mathcal{B}')} : \mathbb{T}_{\mathrm{GSp}(4), p} \longrightarrow \mathbb{L}_{\theta(\mathcal{B}, \mathcal{B}')} = \tilde{\Lambda}_{\mathrm{GSp}(4)} \otimes_{\tilde{\Lambda}_{\mathrm{GL}(2)} \hat{\otimes} \tilde{\Lambda}_{\mathrm{GL}(2)}} (\mathbb{L}_{\mathcal{B}} \hat{\otimes} \mathbb{L}_{\mathcal{B}'})$$

such that the central character equals $\omega_{\mathcal{B}} \otimes \omega_{\mathcal{B}'}$ and

$$\begin{aligned} \lambda_{\theta(\mathcal{B}, \mathcal{B}')} \left(\mathrm{GSp}(4, \mathbb{Z}_v) \begin{bmatrix} \varpi_v & & & \\ & \varpi_v & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{GSp}(4, \mathbb{Z}_v) \right) &= \lambda_{\mathcal{B}} \left(\mathrm{GL}(2, \mathbb{Z}_v) \begin{bmatrix} \varpi_v & \\ & 1 \end{bmatrix} \mathrm{GL}(2, \mathbb{Z}_v) \right), \\ \lambda_{\theta(\mathcal{B}, \mathcal{B}')} \left(\mathrm{GSp}(4, \mathbb{Z}_v) \begin{bmatrix} \varpi_v & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi_v \end{bmatrix} \mathrm{GSp}(4, \mathbb{Z}_v) \right) &= \omega_{\mathcal{B}}^{\frac{1}{2}} \omega_{\mathcal{B}'}^{-\frac{1}{2}}(\varpi_v) \lambda_{\mathcal{B}'} \left(\mathrm{GL}(2, \mathbb{Z}_v) \begin{bmatrix} \varpi_v & \\ & 1 \end{bmatrix} \mathrm{GL}(2, \mathbb{Z}_v) \right), \end{aligned}$$

for places v where $K_{\mathrm{GL}(2), v}^p = \mathrm{GL}_2(\mathbb{Z}_v)$ and $D(\mathbb{Q}_v) \cong \mathrm{GL}(2, \mathbb{Q}_v)$, and $\mathbb{T}_{\mathrm{GSp}(4), p}$ is the Hecke algebra containing unramified Hecke operators at such places and $\mathbb{U}_p$ -operators at p . Here we view $\tilde{\Lambda}_{\mathrm{GL}(2)} \hat{\otimes} \tilde{\Lambda}_{\mathrm{GL}(2)}$ as a subalgebra of $\tilde{\Lambda}_{\mathrm{GSp}(4)}$ via the embedding induced by

$$\begin{aligned} T_{\mathrm{GL}(2)}^1 \times T_{\mathrm{GL}(2)}^1 &\longrightarrow T_{\mathrm{GSp}(4)}^1, \\ (\mathrm{diag}(a_1, a_1^{-1}), \mathrm{diag}(a_2, a_2^{-1})) &\longmapsto \mathrm{diag}(a_1 a_2, a_1 a_2^{-1}, a_1^{-1} a_2^{-1}, a_1^{-1} a_2). \end{aligned}$$

We can take a tame level $K_{\mathrm{GSp}(4)}^p$ with $\det(K_{\mathrm{GSp}(4)}^p) = \widehat{\mathbb{Z}}_p^\times$ such that at a set of classical points of $\mathcal{B}, \mathcal{B}'$ that is Zariski dense, there are Yoshida lifts from $D^\times \times_{\det} D^\times$ to $\mathrm{GSp}(4)$ belonging to the Hecke eigensystem $\lambda_{\theta(\mathcal{B}, \mathcal{B}')}$. Then the spectrum of $\mathbb{T}_{\mathrm{GSp}(4), \mathrm{ord}}$, the Hecke algebra acting on Hida families on $\mathrm{GSp}(4)$ of tame level $K_{\mathrm{GSp}(4)}^p$, has an irreducible component $\theta(\mathcal{B}, \mathcal{B}')$ with $\lambda_{\theta(\mathcal{B}, \mathcal{B}')}$ as the corresponding Hecke eigensystem. Choose another $\mathbb{L}_{\mathcal{C}}$ -adic primitive Hida family $\mathcal{C}$ of tame level $K_{\mathrm{GL}(2)}^{p, \prime}$. The convention for p -nebentypus we use for the Hida families is

$$(4) \quad \begin{aligned} \begin{bmatrix} a & * \\ & d \end{bmatrix} &\longmapsto [\mathrm{diag}(a, a^{-1})] \in \mathcal{O}[\![T_{\mathrm{GL}(2)}^1(\mathbb{Z}_p)]\!], \\ \begin{bmatrix} a_1 & * & * & * \\ & a_2 & * & * \\ & & a_1^{-1} \nu & * \\ & & * & a_2^{-1} \nu \end{bmatrix} &\longmapsto [\mathrm{diag}(a_1, a_2, a_1^{-1}, a_2^{-1})] \in \mathcal{O}[\![T_{\mathrm{GSp}(4)}^1(\mathbb{Z}_p)]\!]. \end{aligned}$$

We fix a finite set S containing p, ∞ and all places where there is ramification.

By constructing a p -adic family of Siegel Eisenstein series on $\mathrm{GU}(3, 3)$ [Liu24], we can show that there exists

$$\begin{aligned} \mu_{\mathcal{C}, \theta(\mathcal{B}, \mathcal{B}')}^S &\in \mathcal{M}eas \left(\mathbb{Z}_p^\times, \mathcal{M}'_{\mathrm{GL}(2), \mathrm{ord}} \hat{\otimes}_{\mathcal{O}} \mathcal{M}_{\mathrm{GSp}(4), \mathrm{ord}} \right) \\ &\quad \otimes_{\tilde{\Lambda}_{\mathrm{GL}(2)} \hat{\otimes} \mathcal{O} \tilde{\Lambda}_{\mathrm{GSp}(4)}} (F_{\mathcal{C}} \hat{\otimes}_{\mathcal{O}} F_{\theta(\mathcal{B}, \mathcal{B}')}), \end{aligned}$$

where $\mathcal{M}'_{\mathrm{GL}(2),\mathrm{ord}}$ (resp. $\mathcal{M}_{\mathrm{GSp}(4),\mathrm{ord}}$) denotes the space of Hida families of tame level $K_{\mathrm{GL}(2)}^{p,\prime}$ (resp. $K_{\mathrm{GSp}(4)}^p$) on $\mathrm{GL}(2)$ (resp. $\mathrm{GSp}(4)$), and $F_{\mathcal{C}} = \mathrm{Frac}(\mathbb{L}_{\mathcal{C}}), F_{\theta(\mathcal{B},\mathcal{B}')} = \mathrm{Frac}(\mathbb{L}_{\theta(\mathcal{B},\mathcal{B}')}))$, such that

$$\begin{aligned}
 (5) \quad & \left(\int_{\mathbb{Z}_p^\times} \chi(y) y^k d\mu_{\mathcal{C},\theta(\mathcal{B},\mathcal{B}')}^S(y) \right) (x) \\
 &= \mathfrak{c} \sqrt{\det \mathbb{S}} 2^{-3l+2} i^l \frac{D^S\left(k + \frac{3l}{2}, \sigma_x \times \pi_x \times \chi\right) D^S\left(k + \frac{3l}{2}, \pi_x \times \sigma'_x \times \omega_{\sigma_x}^{\frac{1}{2}} \omega_{\sigma'_x}^{-\frac{1}{2}} \times \chi\right)}{\mathbf{P}(f_x, f_x)} \\
 &\quad \times f_{x,\mathfrak{c}} f_x \sum_{\varphi \in \mathcal{S}_{\mathrm{GSp}(4),x}} \frac{B_{\mathbb{S},\Lambda}^\dagger(\varphi) \varphi}{\mathbf{P}(\varphi, \varphi)}
 \end{aligned}$$

where

- $x \in \mathbb{L}_{\theta(\mathcal{B},\mathcal{B}')}(\overline{\mathbb{Q}}_p) \times \mathbb{L}_{\mathcal{C}}(\overline{\mathbb{Q}}_p)$ is a classical point with weight (l, l) , l étale over the weight space, where $l \geq 3$ and $-2l + 2 \leq k \leq -l - 1$
- π_x (resp. σ_x, σ'_x) is an irreducible automorphic representation of $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$ generated by a form belonging to the eigensystem parameterized by the projection of x in $\mathcal{C}$ (resp. $\mathcal{B}, \mathcal{B}'$), and f_x is a nonzero p -ordinary form in $\pi_x^{K_{\mathrm{GL}(2)}^{p,\prime}}$ with $f_{x,\mathfrak{c}}$ the $\mathfrak{c}$ -th coefficient in its q -expansion,
- $\mathcal{S}_{\mathrm{GSp}(4),x}$ is an orthogonal basis, with respect to the above defined modified Petersson inner product $\mathbf{P}(\cdot, \cdot)$, of the space spanned by ordinary cuspidal holomorphic forms on $\mathrm{GSp}(4)$ of weight $(l_1, l_2) = (l, l)$ and tame level $K_{\mathrm{GSp}(4)}^p$ with nebentypus at p given by (4) specialized at the character associated to the projection of x inside the weight space, belonging to the Hecke eigenspace parameterized by x ,
- $B_{\mathbb{S},\Lambda}^\dagger$ denotes the p -modified Bessel period defined as

$$B_{\mathbb{S},\Lambda}^\dagger(\varphi) = \lambda_{p,m_1,m_2}(\varphi)^{-1} \cdot B_{\mathbb{S},\Lambda} \left(\left[\begin{array}{ccc} p^{m_1} & & \\ & p^{m_2} & \\ & & p^{-m_1} \end{array} \right]_p \varphi \right), \quad m_1 \gg m_2 \gg 0.$$

One can check that the right hand does not depend on the choice of m_1, m_2 [Liu23, Proposition 2.7.1],

- $D^S(s, \cdot)$ denotes the product of the S -imprimitive L -function and the modified Euler factors at p, ∞ for p -adic interpolation (as defined in [CPR89, Coa91]),
- $\mathbb{S} \in \mathrm{Sym}_2(\mathbb{Q}) > 0$ is chosen such that $f_{x,\mathfrak{c}}, B_{\mathbb{S},\Lambda}^\dagger(\varphi)$ are nonzero at some $x, \varphi \in \mathcal{S}_{\mathrm{GSp}(4),x}$.

For more general classical points $x \in \mathbb{I}_{\theta(\mathcal{B}, \mathcal{B}')}(\overline{\mathbb{Q}}_p) \times \mathbb{I}_{\mathcal{C}}(\overline{\mathbb{Q}}_p)$ with weight (l_1, l_2) , l étale over the weight space and k satisfying (3), we have

$$\begin{aligned}
 (6) \quad & \left(\int_{\mathbb{Z}_p^\times} \chi(y) y^k d\mu_{\mathcal{C}, \theta(\mathcal{B}, \mathcal{B}')}^S(y) \right) (x) \\
 &= i^{r_\Lambda} \frac{L^S \left(k + \frac{l_1+l_2+l}{2}, \sigma_x \times \pi_x \times \chi \right) L^S \left(k + \frac{l_1+l_2+l}{2}, \pi_x \times \sigma'_x \times \omega_{\sigma_x}^{\frac{1}{2}} \omega_{\sigma'_x}^{-\frac{1}{2}} \times \chi \right)}{\mathbf{P}(f_x, f_x)} \\
 & \times E_p \left(k + \frac{l_1+l_2+l}{2}, \sigma_x \times \pi_x \times \chi \right) E_p \left(k + \frac{l_1+l_2+l}{2}, \pi_x \times \sigma'_x \times \omega_{\sigma_x}^{\frac{1}{2}} \omega_{\sigma'_x}^{-\frac{1}{2}} \times \chi \right) \\
 & \times \mathfrak{c}^{k+l+l_1+l_2} \left(-(\alpha_{\mathbb{S}} - \bar{\alpha}_{\mathbb{S}})^2 \right)^{k+l+\frac{l_1+l_2}{2}} \begin{cases} \frac{\prod_{j=0}^2 \Gamma(2k+l_1+l_2+l+2-j)}{2^{3(2k+l_1+l_2+l)} \pi^{3(2k+l_1+l_2+l+1)} i^{2k+l_1+l_2+l+2}}, & k + \frac{l_1+l_2+l}{2} \geq \frac{1}{2} \\ 2^{3(2k+l_1+l_2+l-2)+1} \pi^{-6} i^{2k+l_1+l_2+l}, & k + \frac{l_1+l_2+l}{2} \leq \frac{1}{2} \end{cases} \\
 & \times (\alpha_{\mathbb{S}} - \bar{\alpha}_{\mathbb{S}})^{l_1-l} Z_{k, l, l_1, l_2, r_\Lambda} \cdot f_x \mathfrak{c} f_x \sum_{\varphi \in \mathcal{I}_{\mathrm{GSp}(4), x}} \frac{B_{\mathbb{S}, \Lambda}^\dagger(\varphi) \varphi}{\mathbf{P}(\varphi, \varphi)}
 \end{aligned}$$

with $Z_{k, l, l_1, l_2, r_\Lambda}$ the left hand side of the formula stated in Theorem 2.

On the other hand, by the results on p -adic L -functions for Dirichlet characters, Rankin–Selberg products [Hid88] (Theorem 5.1d.) [CH20, Theorem A] and symplectic groups [Liu20], there exist

$$\mathcal{L}_{\mathcal{B}, \mathcal{C}}^{S,*}, \mathcal{L}_{\mathcal{C}, \mathcal{B}'}^{S,*} \in \mathcal{M}eas(\mathbb{Z}_p^\times, \Lambda_{\mathrm{GL}(2)} \hat{\otimes} \mathcal{O} \Lambda_{\mathrm{GSp}(4)}) \otimes_{\tilde{\Lambda}_{\mathrm{GL}(2)} \hat{\otimes} \mathcal{O} \tilde{\Lambda}_{\mathrm{GSp}(4)}} (F_{\mathcal{C}_1} \hat{\otimes} \mathcal{O} F_{\theta(\mathcal{B}, \mathcal{B}')})$$

and

$$\mathcal{F}_{\theta(\mathcal{B}, \mathcal{B}')} \in \left(\mathcal{M}_{\mathrm{GSp}(4), \mathrm{ord}} \otimes_{\tilde{\Lambda}_{\mathrm{GSp}(4)}} \mathcal{M}_{\mathrm{GSp}(4), \mathrm{ord}} \right) \otimes_{\tilde{\Lambda}_{\mathrm{GSp}(4)}} F_{\theta(\mathcal{B}, \mathcal{B}')})$$

satisfying the following interpolation properties: for $x \in \mathbb{I}_{\theta(\mathcal{B}, \mathcal{B}')}(\overline{\mathbb{Q}}_p) \times \mathbb{I}_{\mathcal{C}}(\overline{\mathbb{Q}}_p)$ at which (6) is proved,

$$(7) \quad \mathcal{L}_{\mathcal{B}, \mathcal{C}}^{S,*}(\kappa, x) = \frac{D^S \left(k + \frac{l+l_1+l_2}{2}, \sigma_x \times \pi_x \times \chi \right)}{(-2i)^{l_1+l_2-1} \mathbf{P}(h_x, h_x)}$$

$$(8) \quad \mathcal{L}_{\mathcal{C}, \mathcal{B}'}^{S,*}(\kappa, x) = \frac{D^S \left(k + \frac{l+l_1+l_2}{2}, \pi_x \times \sigma'_x \times \omega_{\sigma_x}^{\frac{1}{2}} \omega_{\sigma'_x}^{-\frac{1}{2}} \chi \right)}{(-2i)^{l+1} \mathbf{P}(f_x, f_x)}$$

$$\mathcal{F}_{\theta(\mathcal{B}, \mathcal{B}')} (x) = 2^{-1} i^{-l_1-l_2+1} \mathbf{P}(h_x, h_x) \sum_{\varphi \in \mathcal{I}_{\mathrm{GSp}(4), x}} \frac{\varphi \boxtimes \varphi}{\mathbf{P}(\varphi, \varphi)}$$

where $h_x \in \sigma_x$ the unique normalized ordinary form fixed by $K_{\mathrm{GL}(2)}^p$. (Note that the weights of the archimedean components of σ_x, σ'_x are l_1+l_2-2, l_1-l_2+2 .) By p -adically interpolating the Bessel periods, from $\mathcal{F}_{\theta(\mathcal{B}, \mathcal{B}')}$, one can construct

$$\mathcal{F}_{\theta(\mathcal{B}, \mathcal{B}'), \mathbb{S}, \Lambda} \in \mathcal{M}_{\mathrm{GSp}(4), \mathrm{ord}} \otimes_{\tilde{\Lambda}_{\mathrm{GSp}(4)}} F_{\theta(\mathcal{B}, \mathcal{B}')})$$

such that for almost all x as above,

$$(9) \quad \mathcal{F}_{\theta(\mathcal{B}, \mathcal{B}'), \mathbb{S}, \Lambda} (x) = 2^{-1} i^{-l_1-l_2+1} \mathbf{P}(h_x, h_x) \sum_{\varphi \in \mathcal{I}_{\mathrm{GSp}(4), x}} \frac{B_{\mathbb{S}, \Lambda}^\dagger(\varphi) \varphi}{\mathbf{P}(\varphi, \varphi)}.$$

Denote by $\mathbf{f}$ the primitive Hida family associated to $\mathcal{C}$. Combining (5)(7)(8)(9), we see that

$$\begin{aligned} & \mathfrak{c}\sqrt{\det \mathbb{S}} 2^3 i^{-1} \cdot \mathcal{L}_{\mathcal{B}, \mathcal{C}}^{S,*} \mathcal{L}_{\mathcal{C}, \mathcal{B}'}^{S,*} \cdot \mathcal{F}_{\theta(\mathcal{B}, \mathcal{B}'), \mathbb{S}, \Lambda} \cdot \mathbf{f}_{\mathfrak{c}} \mathbf{f} \\ & \in \text{Meas} \left(\mathbb{Z}_p^\times, \mathcal{M}'_{\text{GL}(2), \text{ord}} \hat{\otimes} \mathcal{M}_{\text{GSp}(4), \text{ord}} \right) \otimes_{\tilde{\Lambda}_{\text{GL}(2)} \hat{\otimes} \tilde{\Lambda}_{\text{GSp}(4)}} (F_{\mathcal{C}} \hat{\otimes} F_{\theta(\mathcal{B}, \mathcal{B}')}) \end{aligned}$$

have the same evaluation as $\mu_{\mathcal{C}, \theta(\mathcal{B}, \mathcal{B}')}^S$ at points for which (5) are proved. The set of such points is Zariski dense in $\mathbb{L}_{\mathcal{C}} \hat{\otimes}_{\mathbb{L}_{\theta(\mathcal{B}, \mathcal{B}')}} [\mathbb{Z}_p^\times]$. Thus, we have

$$\mu_{\mathcal{C}, \theta(\mathcal{B}, \mathcal{B}')}^S = \mathfrak{c}\sqrt{\det \mathbb{S}} 2^3 i^{-1} \cdot \mathcal{L}_{\mathcal{B}, \mathcal{C}}^{S,*} \mathcal{L}_{\mathcal{C}, \mathcal{B}'}^{S,*} \cdot \mathcal{F}_{\theta(\mathcal{B}, \mathcal{B}'), \mathbb{S}, \Lambda} \cdot \mathbf{f}_{\mathfrak{c}} \mathbf{f}.$$

Comparing their valuations at points for which (6) is proved, (noting that for all $k, l, (l_1, l_2)$ considered in the theorem we want to prove here, there exist such points with $k, l, (l_1, l_2)$ as the algebraic part of their projections in the weight space), we get that

$$\begin{aligned} Z_{k, l, l_1, l_2, r_\Lambda} &= \mathfrak{c}^{-k-l-l_1-l_2+1} 2^{-l_1-l_2-l+2} i^{-r_\Lambda-l_1+l} \left(-(\alpha_{\mathbb{S}} - \bar{\alpha}_{\mathbb{S}})^2 \right)^{-k-l_1-\frac{l_2+l}{2}} \sqrt{\det \mathbb{S}} \\ &\times \begin{cases} \frac{2^{3(2k+l_1+l_2+l)} \pi^{3(2k+l_1+l_2+l+1)} i^{2k+l_1+l_2+l+2}}{\prod_{j=0}^2 \Gamma(2k+l_1+l_2+l+2-j)}, & k + \frac{l_1+l_2+l}{2} \geq \frac{1}{2} \\ 2^{-3(2k+l_1+l_2+l-2)-1} \pi^6 i^{-(2k+l_1+l_2+l)}, & k + \frac{l_1+l_2+l}{2} \leq \frac{1}{2} \end{cases} \\ &\times E_\infty \left(k + \frac{l+l_1+l_2}{2}, \mathcal{D}_{l_1+l_2-2} \times \mathcal{D}_l \right) E_\infty \left(k + \frac{l+l_1+l_2}{2}, \mathcal{D}_l \times \mathcal{D}_{l_1-l_2+2} \right), \end{aligned}$$

and we can rewrite it as

$$\begin{aligned} Z_{k, l, l_1, l_2, r_\Lambda} &= \mathfrak{c}^{k+l_1+1} (\det \mathbb{S})^{-k-l_1-\frac{l_2+l+1}{2}} \\ &\times \begin{cases} \frac{2^{4k+l_2+l+2} \pi^{6k+3l_1+3l_2+3l+3} i^{2k+l_2+2l-r_\Lambda+2}}{\prod_{j=0}^2 \Gamma(2k+l_1+l_2+l+2-j)}, & k + \frac{l_1+l_2+l}{2} \geq \frac{1}{2} \\ 2^{-8k-6l_1-5l_2-5l+7} \pi^6 i^{-2k-2l_1-l_2-r_\Lambda}, & k + \frac{l_1+l_2+l}{2} \leq \frac{1}{2} \end{cases} \\ &\times 2^{-4k-3l_1-3l_2-3l+8} \pi^{-4k-3l_1-3l_2-3l+4} i^{l_1+l_2+l} \Gamma(k+l_1+l_2+l-2) \Gamma(k+l_1+l_2-1) \\ &\times \Gamma(k+l_2+l-1) \Gamma(k+l_1+l) \end{aligned}$$

□

REFERENCES

- [CH20] Shih-Yu Chen and Ming-Lun Hsieh. On primitive p -adic Rankin-Selberg L -functions. In *Development of Iwasawa theory—the centennial of K. Iwasawa’s birth*, volume 86 of *Adv. Stud. Pure Math.*, pages 195–242. Math. Soc. Japan, Tokyo, [2020] ©2020. 7
- [Coa91] John Coates. Motivic p -adic L -functions. In *L -functions and arithmetic (Durham, 1989)*, volume 153 of *London Math. Soc. Lecture Note Ser.*, pages 141–172. Cambridge Univ. Press, Cambridge, 1991. 6
- [CPR89] John Coates and Bernadette Perrin-Riou. On p -adic L -functions attached to motives over $\mathbf{Q}$. In *Algebraic number theory*, volume 17 of *Adv. Stud. Pure Math.*, pages 23–54. Academic Press, Boston, MA, 1989. 6
- [Fur93] Masaaki Furusawa. On L -functions for $\text{GSp}(4) \times \text{GL}(2)$ and their special values. *J. Reine Angew. Math.*, 438:187–218, 1993. 2
- [Hid88] Haruzo Hida. A p -adic measure attached to the zeta functions associated with two elliptic modular forms. II. *Ann. Inst. Fourier (Grenoble)*, 38(3):1–83, 1988. 7
- [Liu20] Zheng Liu. p -adic L -functions for ordinary families on symplectic groups. *J. Inst. Math. Jussieu*, 19(4):1287–1347, 2020. 7
- [Liu21] Zheng Liu. The doubling archimedean zeta integrals for p -adic interpolation. *Math. Res. Lett.*, 28(1):145–173, 2021.
- [Liu23] Zheng Liu. p -adic L -functions for $\text{GSp}(4) \times \text{GL}(2)$. 2023. [arXiv:2308.08533](https://arxiv.org/abs/2308.08533). 6
- [Liu24] Zheng Liu. p -adic L -functions for $\text{GSp}(4) \times \text{GL}(2)$ II. 2024. [arXiv:2406.05847](https://arxiv.org/abs/2406.05847). 5