

Proof of the Mordell-Lang Conjecture for function fields: 20 years later

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Abstract

Hrushovski's celebrated Theorem 1.1 in [H96] proved the function field Mordell-Lang conjecture. In his proof highly sophisticated model theoretic arguments of Zariski geometry played crucial roles. As a continuous effort of removing Zariski geometry arguments, the authors of [BBPa], [BBPb], [BBPc] succeeded recently to prove Hrushovski's theorem by reducing to the function field Manin-Mumford conjecture without the dichotomy theorem of Zariski geometry. I report here outline of their arguments.

1 Hrushovski's Theorem

We start with a few definitions.

Definition 1. (1) A group Γ is p' -finitely generated

- if $\text{ch} = 0$ and $\mathbb{Q} \otimes \Gamma$ is finitely generated as a $\mathbb{Q}$ -module,
- if $\text{ch} > 0$ and $\mathbb{Q}_p \otimes \Gamma$ is finitely generated as a $\mathbb{Q}_p$ -module where $\mathbb{Q}_p = \{m/n \in \mathbb{Q} : n \text{ prime to } p\}$. (This condition is valid for finitely generated Abelian groups, and for prime-to- p -torsion groups.)

(2) A semi-Abelian variety is an extension of an Abelian variety by an algebraic torus.

Hrushovski's Theorem 1.1 in [H96] is the following statement. Suppose K/k is a field extension and k is an algebraically closed field. Assume:

- S is a semi-Abelian variety defined over K ,
- X is a subvariety of S .
- Γ is a p' -finitely generated subgroup of S .

Assume further $X \cap \Gamma$ is Zariski dense in X .

Then there exists

- S_0 a semi-Abelian variety defined over k ,
- X_0 a subvariety of S_0 defined over k , and
- a rational homomorphism h from a group subvariety of S into S_0 , such that X is a translate of $h^{-1}(X_0)$.

2 Proof without Zariski geometries

In the introduction of [BBPb], the authors explain the motivation of proving Model-Lang conjecture for function fields without appealing Zariski geometries;

... But in the positive characteristic case, *type-definable* Zariski geometries are the relevant objects. The needed dichotomy in this case is a combination of a complicated axiomatic account of a field construction in [17]¹, together with arguments in [15]² showing that the axioms are satisfied for the particular minimal types in separably closed fields that we are interested in.

...

In both cases, the dichotomy theorem is difficult and its proof impenetrable for non model theory experts (as well as for many model-theorists). ...

2.1 fML and fMM

Here we state four conjectures; function field Mordel-Lang conjectures in characteristic zero and positive characteristic, and similarly function field Manin-Mamford conjecture.

2.1.1 Function field Mordell-Lang in $\text{ch} = 0$

Let $K = \mathbb{C}(t)^{\text{alg}}$. Let A be an abelian variety over K with $\mathbb{C}$ -trace 0, i.e., with no nonzero homomorphic images defined over $\mathbb{C}$.

Assume:

- X : an irreducible subvariety of A (defined over K),
- Γ : a “finite-rank” subgroup of $A(K)$, i.e., Γ is contained in the division points of a finitely generated subgroup of $A(K)$.
- Suppose $X \cap \Gamma$ is Zariski-dense in X .

¹E. Hrushovski, B. Zilber, Zariski Geometries, J. AMS, 1996

²[Hr96]

Then X is a translate of an abelian subvariety of A .

2.1.2 Function field Manin-Mumford in $\text{ch} = 0$

Let $K = \mathbb{C}(t)^{\text{alg}}$. Let A be an abelian variety over K with $\mathbb{C}$ -trace 0, i.e, with no nonzero homomorphic images defined over $\mathbb{C}$.

Assume:

- X : an irreducible subvariety of A (defined over K),
- $\Gamma \subseteq \text{Tor}(A)$
- Suppose $X \cap \Gamma$ is Zariski-dense in X .

Then X is a translate of an abelian subvariety of A .

2.1.3 Function field Mordell-Lang in $\text{ch } p > 0$.

Let $K = (\mathbb{F}_p^{\text{alg}}(t))^{\text{sep}}$, the separable closure of $\mathbb{F}_p^{\text{alg}}(t)$. Let A be an abelian variety over K with $\mathbb{F}_p^{\text{alg}}$ -trace 0.

Assume:

- X : an irreducible subvariety of A , defined over K ,
- Γ : a subgroup of $A(K)$ contained in the prime-to- p -division points of a finitely generated subgroup.
- Suppose that $X \cap \Gamma$ is Zariski-dense in X .

Then X is a translate of an abelian subvariety of A .

2.1.4 Function field Manin-Mumford in $\text{ch } p > 0$.

Let $K = (\mathbb{F}_p^{\text{alg}}(t))^{\text{sep}}$, the separable closure of $\mathbb{F}_p^{\text{alg}}(t)$. Let A be an abelian variety over K with $\mathbb{F}_p^{\text{alg}}$ -trace 0.

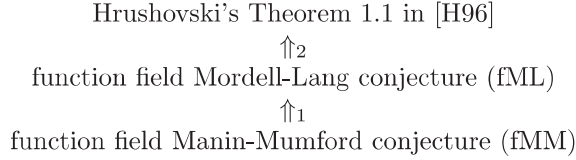
Assume:

- X : an irreducible subvariety of A , defined over K ,
- $\Gamma \subseteq \text{Tor}(A)$
- Suppose that $X \cap \Gamma$ is Zariski-dense in X .

Then X is a translate of an abelian subvariety of A .

2.2 Strategy

In [BBPa], [BBPb], the authors reduce first the proof of Theorem 1.1 [H96] to fML then secondly the proof of fML to the proof of fMM by model theoretic arguments without the dichotomy theorem of Zariski geometry.



Finally note that fMM has been proved without model theory, by Pink, Corpit, Rössler [PR], [R1, R2].

2.3 Embed the algebraic-geometric set-up in a differential algebraic one.

The function field Mordell-Lang conjecture is an algebraic-geometric statement. Inspired by Buium's approach [Bu] to the conjecture in characteristic zero, Hrushovski used model theory of differentially closed fields and separably closed fields together with the dichotomy theorem of Zariski geometry to prove function field Mordell-Lang conjecture.

In [BBPb] to prove function field Mordell-Lang conjecture from function field Manin-Mumford conjecture, the authors use in addition to the basic properties of model theory of differentially closed fields and separably closed fields, and both in $\text{ch} = 0$ and $\text{ch} > 0$, they need

- the Theorem of the Kernel
- g -minimal groups of finite Morley rank.

Moreover, in $\text{ch} = 0$,

- Weak socle theorem.

In $\text{ch} > 0$,

- Quantifier elimination for $A^\# = p^\infty(A(\mathcal{U}))$.

2.4 Definitions

Let G be a commutative connected group of finite Morley rank, definable in some ambient stable structure M .

We say that G is generated (abstractly) by some sets $X_1, \dots, X_n$ if $G \subset \text{acl}(F \cup X_1 \cup \dots \cup X_n)$, where F is a finite set.

G is said to be almost strongly minimal, if $G \subset \text{acl}(F \cup X)$, where X is a strongly minimal set and F is a finite set.

Definition 2 ($\text{Socle}S(G)$). *The socle $S(G)$ of G , is the greatest connected definable subgroup of G which is generated by strongly minimal definable subsets of G , i.e., contained in $\text{acl}(F \cup X_1 \cup \dots \cup X_n)$ where each X_i is a strongly minimal definable subset and F is a finite set.*

Definition 3 (Stabilizer). *For X a definable subset of G with Morley degree 1, define the model theoretic stabilizer of X in G ,*

$$\text{Stab}_G(X) := \{g \in G : MR(X \cap (X + g)) = MR(X)\}$$

(so the stabilizer of the generic type of X over M).

Definition 4 (Rigid). *H is a definable subgroup of G , defined over some B . H is rigid if, passing to a saturated model, all connected definable subgroups of H are defined over $\text{acl}(B)$.*

2.5 The group $A^\sharp$, Def 2.1 in [BBPb]

Let $\mathcal{U}$ be a monster model.

(i) In $\text{ch} = 0$,

$A^\sharp$ denotes the “Kolchin closure of the torsion”, i.e., the smallest definable (in the sense of differentially closed fields) subgroup of $A(\mathcal{U})$ which contains the torsion subgroup ($A^\sharp$ is definable over K .)

$$\boxed{\text{Tor}(A) \subseteq A^\sharp \subseteq A(\mathcal{U})}$$

(ii) In $\text{ch} > 0$,

$A^\sharp$ denotes $p^\infty(A(\mathcal{U})) = \bigcap_n p^n(A(\mathcal{U}))$
(an infinitely definable subgroup over K).

2.6 Properties of $A^\sharp$, [H96]

Here is a list of properties of $A^\sharp$, [H96].

- (i) $A^\sharp$ is the smallest Zariski dense (infinitely) definable subgroup of $A(\mathcal{U})$.
- (ii) $A^\sharp$ is connected (no relatively definable subgroup of finite index), and of finite U -rank in $\text{ch} > p$, and finite Morley rank in $\text{ch} = 0$.
- (iii) If A is a simple abelian variety, $A^\sharp$ has no proper infinite type-definable subgroup.
- (iv) If A is the sum of simple A_i then $A^\sharp$ is the sum of the $A_i^\sharp$.

2.7 Theorem of the Kernel

In all characteristics: K/k : both algebraically closed fields.

Theorem 5. *Let A be an abelian variety over K with k -trace 0, then*

$$A^\sharp(K) \subseteq \text{Tor}(A)$$

where $\text{Tor}(A)$ is the group of torsion points of A .

2.8 g -minimal groups

Let G be a group (with additional structure). Assume $\text{RM}(G) < \infty$. commutative and connected

Definition 6. G is g -minimal if it has no proper nontrivial connected definable subgroup (equivalently, no proper infinite definable subgroup).

Theorem 7 (Wagner). *Suppose that G is g -minimal. Then any infinite algebraically closed subset of G is an elementary substructure of G .*

3 From fMM to fML in $ch = 0$

Here I explain how to prove fML from fMM in case of $ch = 0$.

Lemma 8 (Lemma 4.6 in [H96]). *The socle $S(G)$ is an almost direct sum of pairwise orthogonal definable groups G_i , where each G_i is almost strongly minimal.*

Theorem 9 (Prop. 4.3 in Hr96, Weak socle theorem). *Let G be a commutative connected group of finite Morley rank. Suppose that the socle $S(G)$ of G is rigid. Suppose further that X is a definable subset of G of Morley degree 1 with finite $\text{Stab}_G(X)$.*

Then, some translate of X is contained in $S(G)$.³

Fact 10. *If H is a connected finite Morley rank definable subgroup of $A = A(\mathcal{U})$ containing $A^\sharp$, then the socle $S(H) = A^\sharp$.*

We now prove fML from fMM. Assume:

- $K = \mathbb{C}(t)^{\text{alg}}$, and A is an abelian variety over K with $\mathbb{C}$ -trace 0.
- X : an irreducible subvariety of A (defined over K),
- Γ : a “finite-rank” subgroup of $A(K)$.
- Suppose $X \cap \Gamma$ is Zariski-dense in X .

We want to show that X is a translate of an abelian subvariety of A .

A. Recall properties of $A^\sharp$:

- $A^\sharp := A^\sharp(\mathcal{U})$, the smallest definable subgroup of $A(\mathcal{U}) \supseteq \text{Tor}(A)$.
- $A^\sharp$ is definable over K , connected with finite Morley rank.
- if A is simple, $A^\sharp$ is a g -minimal group.
- $A/A^\sharp$ definably embeds via some μ in a vector group.
- $\mu(\Gamma)$ is contained in a finite-dimensional vector space over $\mathbb{C}$, the preimage of which we call H .

³up to subtracting a definable subset X' of X with $\text{RM}(X') < \text{RM}(X)$.

- H is a definable subgroup of A , connected, $\text{RM}(H) < \infty$, containing both $A^\sharp$ and Γ , and defined over K .

$$\begin{array}{c} A^\sharp \\ \subseteq \\ \Gamma \end{array} \quad \begin{array}{c} H \\ \subseteq \\ A \end{array}$$

Claim : $A^\sharp(K) = A^\sharp(K^{\text{diff}})$

Proof: $A^\sharp(K) \subseteq A^\sharp(K^{\text{diff}})$ is clear. If $b \in A^\sharp(K^{\text{diff}})$ then $tp(b/K)$ is an isolated type with respect to DCF_0 . But then $tp(b/\mathcal{A}(K))$ is isolated in the structure $\mathcal{A}$. As

$$\mathcal{A}(K) \preceq \mathcal{A},$$

we have $b \in A^\sharp(K)$. Hence $A^\sharp(K^{\text{diff}}) \subseteq A^\sharp(K)$. ■

Note that X is now defined over K^{diff} , and $X \cap A^\sharp(K^{\text{diff}})$ is Zariski-dense in X . By the Claim, we have

$$A^\sharp(K^{\text{diff}}) = A^\sharp(K)$$

(so in fact X is defined over K) so, by the Theorem of the Kernel, we have

$$A^\sharp(K^{\text{diff}}) = \text{Tor}(A),$$

hence by function field Manin-Mumford, X is a translate of an abelian subvariety of A .

4 From fMM to fML in $ch > 0$

In this section we prove fML from fMM in case of $ch > 0$.

4.1 Quantifier elimination

Key theorem is the following:

Theorem 11 (Thm 3.1 [BBPb]). Let K be a separably closed field of imperfection 1. and $\text{ch}(K) > 0$. Assume A is an abelian variety over K , and $\mathcal{U}$ is a saturated elementary extension of K .

Let $A^\sharp = p^\infty A(\mathcal{U})$ and set

$$\mathcal{A}^\sharp := (A^\sharp, \text{with all relatively definable sets (parameters from } K))$$

Then $\text{Th}(\mathcal{A}^\sharp)$ eliminates quantifiers as $L_{\lambda,t}$ -theory.

The proof is very long and technical. As a corollary we have;

Theorem 12 (Lemma 4.6 [BBPb]). $\mathcal{A}^\sharp$ and $\mathcal{A}_i^\sharp$ are both connected and of finite Morley rank. Each $\mathcal{A}_i^\sharp$ is g -minimal. Moreover we have $\mathcal{A}(K) \preceq \mathcal{A}$.

4.2 From fMM to fML

Now we show that fML holds from fMM.

Let $K = (\mathbb{F}_p^{\text{alg}}(t))^{\text{sep}}$ be the separable closure of $\mathbb{F}_p^{\text{alg}}(t)$. Let A be a $\mathbb{F}_p^{\text{alg}}$ -trace zero abelian variety over K . Assume;

- X is an irreducible subvariety of A and defined over K ,
- There exists a finitely generated subgroup H of $A(K)$ such that $\Gamma \subseteq \text{Tor}_{p'}(H) \subseteq A(K)$.
- $X \cap \Gamma$ is Zariski dense in X .

We want to prove that X is a translate of a commutative subvariety of A .

Key point:

Construct an appropriate $\mathcal{E}_0$. Then replace $X \cap \Gamma$ with $X \cap \mathcal{E}_0$ which is Zariski dense in X . Then apply fMM.

Proof : Note that for each $n > 0$ we have $[p^n \Gamma : p^{n+1} \Gamma] < \infty$. Since $X \cap \Gamma$ is Zariski dense in X , there exists a coset D_i of $p^i \Gamma$ in Γ such that

- $i < j \implies D_i \supseteq D_j$,
- each $X \cap D_i$ is Zariski dense in X .

Hence there is a descending sequence of cosets E_i of $p^i A(K)$ in $A(K)$ with each $X \cap E_i$ is Zariski dense in X . We now work over $\mathcal{U}$, and set

$$E := \bigcap_i E_i.$$

then E is a translate of $A^\sharp$, and $X \cap E$ is Zariski dense in X .

In order to apply fMM and the Theorem of the Kernel (Theorem 5), we need that $X \cap E(K)$ is Zariski dense in X inside the field K . Since the field K is not saturated we do not know whether $E(K) \neq \emptyset$.

Thus we need to consider a translate of $A^\sharp$ rather than a translate of $E(K)$. Put

$$M := (\mathcal{A}, \mathcal{E}) = (A^\sharp(\mathcal{U}), E(\mathcal{U})) \text{ the structure} \\ \text{adding relations and functions definable over } K$$

Then by Theorem 12 which is a corollary to the quantifier elimination of $\text{Th}(\mathcal{A}^\sharp)$ we have that Morley rank of $\text{Th}(M)$ is finite.

Let $M_0 \prec M$ be a prime model (atomic model) over $\mathcal{A}(K)$. Then there exists $\mathcal{E}_0$ such that $\mathcal{E}_0 \preceq \mathcal{E}$ and $M_0 = (\mathcal{A}(K), \mathcal{E}_0)$.

Lemma 13. $X \cap \mathcal{E}_0$ is Zariski dense in X .

Proof: By contradiction. So suppose $X \cap \mathcal{E}_0$ is not Zariski dense in X . There is a proper subvariety Z of X such that $X \cap \mathcal{E}_0 \subseteq Z \subsetneq X$. By replacing Z with the Zariski closure of $X \cap \mathcal{E}_0$, we view Z as defined over $\mathcal{E}_0$. Note we have $X \cap \mathcal{E}_0 = Z \cap \mathcal{E}_0$.

Consider $Z \cap \mathcal{E}$ as a definable subset in $\mathcal{E}$. Then $Z \cap \mathcal{E}$ is definable over $\mathcal{E}_0$.

Since $\mathcal{E}_0 \prec \mathcal{E}$, we have $X \cap \mathcal{E} = Z \cap \mathcal{E}$. This contradicts with the fact that $X \cap \mathcal{E}$ is Zariski dense in X . Therefore $X \cap \mathcal{E}_0$ must be Zariski dense in X . ■

Now we complete the proof as follows, Let $a \in X \cap \mathcal{E}_0$ and set $X_1 = X - \{a\}$. Then $X_1 \cap A^\sharp(K)$ is Zariski dense in X_1 , and X_1 is definable over K . From Theorem of the Kernel we have $A^\sharp(K) = \text{Tor}(A)$, and we also have that $X_1 \cap \text{Tor}(A)$ is Zariski dense in X_1 . Hence by applying fMM to $A^\sharp(K)$, we see that X_1 is a translate of commutative subvariety of A . It follows that X is a translate of commutative subvariety of A as well. Now we are done.

5 Final remarks

In this note I only explain briefly how to prove fML from fMM, i.e., $\Uparrow_1$ [BBPb] in the following diagram:

$$\begin{array}{c}
 \text{Hrushovski's Theorem 1.1 in [H96]} \\
 \Uparrow_2 \\
 \text{function field Mordell-Lang conjecture (fML)} \\
 \Uparrow_1 \\
 \text{function field Manin-Mumford conjecture (fMM)}
 \end{array}$$

Compare the assumptions on Γ in fML and fMM:

(1) in fML (i) in $\text{ch} = 0$, $\Gamma \subseteq \text{Tor}(G)$ where G is a finitely generated subgroup of $A(K)$, (ii) in $\text{ch} > 0$, $\Gamma \subseteq \text{Tor}_{p'}(G)$ where G is a finitely generated subgroup of $A(K)$.

(2) in fMM, $\Gamma \subseteq \text{Tor}(A)$.

The condition (2) is stronger than (1). Model theory of differentially closed fields and separably closed fields is needed to fill the gap of assumptions on Γ in fML and fMM. The authors of [BBPa, BBPb, BBPc] managed to proceed their project without appealing to the dichotomy theorem of Zariski geometry.

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