

ON THE JAMES CONSTANT OF EXTREME ABSOLUTE NORMS ON $\mathbb{R}^2$

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Abstract

The set of all absolute normalized norms on $\mathbb{R}^2$ (denoted by AN_2) and the set of all convex functions ψ on $[0, 1]$ satisfying $\max\{1-t, t\} \leq \psi(t) \leq 1$ for $t \in [0, 1]$ (denoted by Ψ_2) have convex structures and they are isomorphic by the one to one correspondence $\psi(t) = \|(1-t, t)\|_\psi$ ($t \in [0, 1]$). In [5], the set of all extreme points of AN_2 is determined. In this note, we will report the calculation of the James constants of $(\mathbb{R}^2, \|\cdot\|_\psi)$ and its dual space $(\mathbb{R}^2, \|\cdot\|_\psi)^*$ where ψ is an arbitrary extreme point of Ψ_2 . Moreover, we will consider the relation of the James constants of these spaces.

1. PRELIMINARIES

A norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be absolute if $\|(x, y)\| = \||x|, |y|\|$ for all $(x, y) \in \mathbb{R}^2$, and normalized if $\|(1, 0)\| = \|(0, 1)\| = 1$. The set of all absolute normalized norms on $\mathbb{R}^2$ is denoted by AN_2 . Let Ψ_2 be the set of all convex functions ψ on $[0, 1]$ satisfying $\max\{1-t, t\} \leq \psi(t) \leq 1$ for $t \in [0, 1]$. Ψ_2 and AN_2 can be identified by a one to one correspondence $\psi \rightarrow \|\cdot\|_\psi$ with the relation

$$(1.1) \quad \psi(t) = \|(1-t, t)\|_\psi$$

for $t \in [0, 1]$. For $1 \leq p \leq \infty$, we denote

$$\psi_p(t) = \begin{cases} \{(1-t)^p + t^p\}^{\frac{1}{p}} & (1 \leq p < \infty) \\ \max\{1-t, t\} & (p = \infty). \end{cases}$$

Then $\psi_p \in \Psi_2$ ($1 \leq p \leq \infty$), and they correspond to the l_p -norms $\|\cdot\|_p$ on $\mathbb{R}^2$.

We call a norm $\|\cdot\| \in AN_2$ (resp. $\psi \in \Psi_2$) an extreme point of AN_2 (resp. Ψ_2) if $\|\cdot\| = \frac{1}{2}(\|\cdot\|' + \|\cdot\|'')$ and $\|\cdot\|', \|\cdot\|'' \in AN_2$ imply $\|\cdot\|' = \|\cdot\|''$ (resp. $\psi = \frac{1}{2}(\psi' + \psi'')$ and $\psi', \psi'' \in \Psi_2$ imply $\psi' = \psi''$).

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Let $0 \leq \alpha \leq \frac{1}{2} \leq \beta \leq 1$. For the case $(\alpha, \beta) \neq (\frac{1}{2}, \frac{1}{2})$, we define

$$\psi_{\alpha, \beta}(t) = \begin{cases} 1-t & (t \in [0, \alpha]) \\ \frac{\alpha + \beta - 1}{\beta - \alpha}t + \frac{\beta - 2\alpha\beta}{\beta - \alpha} & (t \in [\alpha, \beta]) \\ t & (t \in [\beta, 1]) \end{cases},$$

$$E = \{\psi_{\alpha, \beta} \mid 0 \leq \alpha \leq \frac{1}{2} < \beta \leq 1\}.$$

Proposition 1 ([5]). The following conditions are equivalent.

- (1) $\|\cdot\|_\psi$ is an extreme point of AN_2 .
- (2) ψ is an extreme point of Ψ_2 .
- (3) $\psi \in E$.

Let $\widehat{\Psi}_2 = \{\psi \in \Psi_2 \mid \psi(1-t) = \psi(t) \ (t \in [0, 1])\}$. If $\psi \in \Psi_2$, then $\psi \in \widehat{\Psi}_2$ if and only if $\|(x_1, x_2)\|_\psi = \|(x_2, x_1)\|_\psi$ for $(x_1, x_2) \in \mathbb{R}^2$. $\widehat{\Psi}_2$ also has a convex structure, and by an analogy of Proposition 1, we have

Corollary 2. Let $\widehat{E} = E \cap \widehat{\Psi}_2 = \{\psi_{\alpha, 1-\alpha} \in E \mid 0 \leq \alpha \leq \frac{1}{2}\}$. Then ψ is an extreme point of $\widehat{\Psi}_2$ if and only if $\psi \in \widehat{E}$.

2. KNOWN FACTS ON JAMES CONSTANT OF $(\mathbb{R}^2, \|\cdot\|_\psi)$

For a Banach space $(X, \|\cdot\|)$, the James constant is defined by

$$J((X, \|\cdot\|)) = \sup\{\min\{\|x+y\|, \|x-y\|\} \mid x, y \in X, \|x\| = \|y\| = 1\}.$$

$\sqrt{2} \leq J((X, \|\cdot\|)) \leq 2$ holds and $J((X, \|\cdot\|)) = \sqrt{2}$ if X is a Hilbert space. (The converse is not true.) For $1 \leq p \leq \infty$, $J(L_p) = \max\{2^{\frac{1}{p}}, 2^{\frac{1}{q}}\}$ holds where $\frac{1}{p} + \frac{1}{q} = 1$ and $\dim L_p \geq 2$. It is known that $J(X) < 2$ if and only if X is uniformly non-square, that is, there exists $\delta > 0$ such that $\|(x+y)/2\| \leq 1 - \delta$ holds whenever $\|(x-y)/2\| \geq 1 - \delta$, $\|x\| \leq 1$, $\|y\| \leq 1$. Moreover, $J(X^{**}) = J(X)$ holds and

$$(2.1) \quad 2J(X) - 2 \leq J(X^*) \leq \frac{J(X)}{2} + 1.$$

There are some Banach spaces which do not satisfy $J(X^*) = J(X)$.

For the 2-dimensional spaces with absolute normalized norms, we know the following facts on the James constant.

Proposition 3 ([8]).

- (1) If $\psi_2 \leq \psi \in \widehat{\Psi}_2$ and $\max_{t \in [0, 1]} \frac{\psi(t)}{\psi_2(t)}$ is taken at $t = \frac{1}{2}$, then

$$J((\mathbb{R}^2, \|\cdot\|_\psi)) = 2\psi(\frac{1}{2}).$$

(2) If $\psi_2 \geq \psi \in \widehat{\Psi}_2$ and $\max_{t \in [0,1]} \frac{\psi_2(t)}{\psi(t)}$ is taken at $t = \frac{1}{2}$, then

$$J((\mathbb{R}^2, \|\cdot\|_\psi)) = \frac{1}{\psi(\frac{1}{2})}.$$

(3) For $\beta \in [\frac{1}{2}, 1]$,

$$J((\mathbb{R}^2, \|\cdot\|_{\psi_{1-\beta,\beta}})) = \begin{cases} \frac{1}{\beta} & (\beta \in [\frac{1}{2}, \frac{1}{\sqrt{2}}]) \\ 2\beta & (\beta \in [\frac{1}{\sqrt{2}}, 1]). \end{cases}$$

The results in Proposition 3 are obtained by the following proposition. Also in [9] and [10], the James constants of 2 dimensional Lorentz sequence spaces and their dual spaces were culculated by using the following proposition.

Proposition 4([8]). If $\psi \in \widehat{\Psi}_2$, then

$$J((\mathbb{R}^2, \|\cdot\|_\psi)) = \max_{0 \leq t \leq \frac{1}{2}} \frac{2-2t}{\psi(t)} \psi(\frac{1}{2-2t}).$$

We have only few results on the James constants of $(\mathbb{R}^2, \|\cdot\|_\psi)$ when $\psi \in \Psi_2 \setminus \widehat{\Psi}_2$. In this note we focus our consideration on the James constants of $(\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})$ and its dual space $(\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})^*$ where $\psi_{\alpha,\beta} \in E$. There is a unique $\psi^* \in \Psi_2$ such that $(\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})^* = (\mathbb{R}^2, \|\cdot\|_{\psi^*})$, and it is obvious that $\psi_{\alpha,\beta}, \psi^* \notin \widehat{\Psi}_2$ whenever $\alpha + \beta \neq 1$.

3. JAMES CONSTANTS FOR EXTREME NORMS IN AN_2 .

In this section we consider $J((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}}))$ where $\|\cdot\|_{\psi_{\alpha,\beta}}$ is the extreme norm of AN_2 . Since $J((\mathbb{R}^2, \|\cdot\|_{\tilde{\psi}})) = J((\mathbb{R}^2, \|\cdot\|_\psi))$ where $\tilde{\psi}(t) = \psi(1-t)$, it is sufficient to culcate James constant in the case that $\alpha + \beta \leq 1$.

Theorem 5([4]). Suppose that $\alpha + \beta \leq 1$, then

$$J((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})) = \begin{cases} \frac{1}{\psi(1/2)} & (\text{if } \psi(\frac{1}{2}) \leq \frac{1}{2(1-\alpha)}) \\ 1 + \frac{1}{2\psi(1/2) + \gamma} & (\text{if } \frac{1}{2(1-\alpha)} \leq \psi(\frac{1}{2}) \leq \frac{1}{4(1-\alpha)}(1 + \frac{1}{\gamma})) \\ 2\psi(1/2) & (\text{if } \frac{1}{4(1-\alpha)}(1 + \frac{1}{\gamma}) \leq \psi(\frac{1}{2})), \end{cases}$$

where $\gamma = \frac{2\beta - 1}{\beta - \alpha}$.

Corollary 6. If $\beta \leq 1 - \alpha \leq \frac{1}{\sqrt{2}}$, then

$$J((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})) = \frac{1}{\psi(1/2)}.$$

We have some other formulations of Theorem 5. Put

$$\gamma = \gamma(\alpha, \beta) = \begin{cases} \frac{2\beta - 1}{\beta - \alpha} & (\alpha + \beta \leq 1) \\ \frac{1 - 2\alpha}{\beta - \alpha} & (\alpha + \beta \geq 1) \end{cases},$$

$$f = f(\gamma) = \frac{1}{4}\{1 - \gamma + \sqrt{(1 + \gamma)^2 + 4\gamma}\},$$

$$g = g(\gamma) = \frac{1}{4}\{1 - \gamma + \sqrt{(1 + \gamma)^2 + 4}\},$$

$$M = 1 + \frac{1}{2\psi(1/2) + \gamma}.$$

It can be shown by a simple calculation that f is increasing with respect to γ while g is decreasing and that $\frac{1}{2} \leq f(\gamma) \leq \frac{1}{\sqrt{2}} \leq g(\gamma) \leq \frac{1+\sqrt{5}}{4}$ ($\gamma \in [0, 1]$).

Theorem 7([4]).

- (1) If $\psi(1/2) \leq f(\gamma)$, then
 $2\psi(1/2) \leq M \leq \frac{1}{\psi(1/2)}$, and $J((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})) = \frac{1}{\psi(1/2)}$.
- (2) If $f(\gamma) \leq \psi(1/2) \leq g(\gamma)$, then
 $2\psi(1/2), \frac{1}{\psi(1/2)} \leq M$, and $J((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})) = M$.
- (3) If $g(\gamma) \leq \psi(1/2)$, then
 $\frac{1}{\psi(1/2)} \leq M \leq 2\psi(1/2)$, and $J((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})) = 2\psi(1/2)$.

Theorem 7'. For $\psi_{\alpha,\beta}$, put $\gamma = \gamma(\alpha, \beta) = \begin{cases} \frac{2\beta - 1}{\beta - \alpha} & (\alpha + \beta \leq 1) \\ \frac{1 - 2\alpha}{\beta - \alpha} & (\alpha + \beta \geq 1) \end{cases}$, then

$$J((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})) = \max\left\{\frac{1}{\psi(1/2)}, 1 + \frac{1}{2\psi(1/2) + \gamma}, 2\psi\left(\frac{1}{2}\right)\right\}.$$

It is known that $J((\mathbb{R}^2, \|\cdot\|_{\psi})) = \sqrt{2}$ holds for $\psi \in [\psi_2, \psi_{1-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}}] = \{(1 - \lambda)\psi_2 + \lambda\psi_{1-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}} \mid \lambda \in [0, 1]\}$. By Theorem 7 or Theorem 7' we can prove that

Corollary 8. $\|\cdot\|_{\psi_{1-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}}}$ is the only extreme point of AN_2 whose James constant is $\sqrt{2}$, that is,

$$\{\psi_{\alpha,\beta} \in E \mid J((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})) = \sqrt{2}\} = \{\psi_{1-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}}\}.$$

4. JAMES CONSTANTS FOR THE DUAL NORMS.

Let $C_{r,s}$ be the convex hull of the set consisting of eight points $(\pm 1, 0)$, $(0, \pm 1)$, and $(\pm r, \pm s)$ with $r, s \in [0, 1]$, $r + s \geq 1$. $C_{r,s}$ is an octagon whenever $1 < r + s$, $r < 1$, and $s < 1$. In the exceptional cases, it is a hexagon or a square. Let $\psi_{r,s}^* \in \Psi_2$ be

such that the unit sphere of the norm $\|\cdot\|_{\psi_{r,s}^*}$ is $C_{r,s}$. Then $\psi_{r,s}^*$ and $\|\cdot\|_{\psi_{r,s}^*}$ are given by:

$$\psi_{r,s}^*(t) = \begin{cases} 1 - \frac{r+s-1}{s}t & (t \in [0, \frac{s}{r+s}]) \\ \frac{1-s}{r} + \frac{s}{r+s-1}t & (t \in [\frac{s}{r+s}, 1]), \end{cases}$$

$$\|(x_1, x_2)\|_{\psi_{r,s}^*} = \begin{cases} x_1 - \frac{r-1}{s}x_2 & (0 \leq rx_2 \leq sx_1) \\ \frac{1-s}{r}x_1 + x_2 & (0 \leq sx_1 \leq rx_2). \end{cases}$$

It is easy to find that $\|\cdot\|_{\psi_{r,s}^*}$ is the dual norm of $\|\cdot\|_{\psi_{\alpha,\beta}}$ if and only if

$$(4.1) \quad \begin{cases} \alpha &= \frac{1-r}{1-r+s} \\ \beta &= \frac{r}{1+r-s}. \end{cases}$$

It is easy to see that for each $\psi \in \Psi_2$

$$J((\mathbb{R}^2, \|\cdot\|_{\psi})) = J((\mathbb{R}^2, \|\cdot\|_{\tilde{\psi}}))$$

where $\tilde{\psi}$ is defined by $\tilde{\psi}(t) = \psi(1-t)$ ($t \in [0, 1]$). Since $\tilde{\psi}_{r,s}^* = \psi_{s,r}^*$ holds, it follows that $J((\mathbb{R}^2, \|\cdot\|_{\psi_{r,s}^*})) = J((\mathbb{R}^2, \|\cdot\|_{\psi_{s,r}^*}))$ for all $r, s \in [0, 1]$ with $r+s \geq 1$. Hence it is sufficient to consider the case that $r \leq s$.

Theorem 9. Suppose that $r \leq s$, then

$$(4.2) \quad J((\mathbb{R}^2, \|\cdot\|_{\psi_{r,s}^*})) = \begin{cases} 1 + \frac{1-r}{s} & (f(r, s) \leq 0) \\ \frac{2r(2rs - 3s - r + 1)}{2r^2 - 3r - s + 1} & (f(r, s) \geq 0), \end{cases}$$

where $f(r, s) = -4r^2s^2 - 2r^3 + 4r^2s + 6rs^2 + 5r^2 - 4rs - s^2 - 4r + 1$.

By a simple calculation we find that there is an implicit function $s = h(r)$ of f , such that h is decreasing on $[\frac{1}{2}, \frac{1}{\sqrt{2}}]$ and $h(\frac{1}{2}) = 1$, $h(\frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}}$, and $f(r, h(r)) = 0$ for $r \in [\frac{1}{2}, \frac{1}{\sqrt{2}}]$. Moreover we can see that

$$f(r, s) \begin{cases} \leq 0 & (0 \leq r \leq \frac{1}{2}, \text{ or } \frac{1}{2} \leq r \leq \frac{1}{\sqrt{2}}, s \leq h(r)) \\ \geq 0 & (\frac{1}{2} \leq r \leq \frac{1}{\sqrt{2}}, s \geq h(r), \text{ or } \frac{1}{\sqrt{2}} \leq r \leq 1). \end{cases}$$

We have another formulation of (4.2) which is written by the function $\psi_{r,s}^*$.

Theorem 9'. Suppose that $r \leq s$, then

$$J((\mathbb{R}^2, \|\cdot\|_{\psi_{r,s}^*})) = \begin{cases} 2\omega & (r(r-2) + \omega + (2r-1)\omega^2 \leq 0) \\ \frac{2r(r-2+\omega)}{(1-2\omega)r-1+\omega} & (r(r-2) + \omega + (2r-1)\omega^2 \geq 0), \end{cases}$$

where $\omega = \psi_{r,s}^*(\frac{1}{2})$. In particular, if $r = s$, then $\omega = \frac{1}{2r}$, and

$$J((\mathbb{R}^2, \|\cdot\|_{\psi_{r,s}^*})) = \begin{cases} 2\psi_{r,s}^*(1/2) & (\frac{1}{2} \leq r \leq \frac{1}{\sqrt{2}}) \\ \frac{1}{\psi_{r,s}^*(1/2)} & (\frac{1}{\sqrt{2}} \leq r). \end{cases}$$

As stated in Section 2, $J(X^*) = J(X)$ does not always hold. We will give a partial result on the relation between $J((\mathbb{R}^2, \|\cdot\|_{\psi_{r,s}^*}))$ and $J((\mathbb{R}^2, \|\cdot\|_{\psi_{r,s}^*})^*)$. $(\mathbb{R}^2, \|\cdot\|_{\psi_{r,s}^*})^*$ is given by $(\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})$ where (α, β) satisfies (4.1).

Theorem 10. Suppose that (4.1) holds, then

- (1) If $r = s$ ($\frac{1}{2} \leq r \leq 1$), or $(r, s) = (\frac{1}{2}, 1)$,
then $J((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})) = J((\mathbb{R}^2, \|\cdot\|_{\psi_{r,s}^*}))$.
- (2) If $r \in (0, 1) \setminus \{\frac{1}{2}\}$, $s = 1$, or $r = \frac{1}{2}$, $\frac{1}{2} \leq s < 1$,
then $J((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})) \neq J((\mathbb{R}^2, \|\cdot\|_{\psi_{r,s}^*}))$.

Combining Corollary 2 and Theorem 10, we have

Corollary 11. Suppose that $\psi \in E \cap \widehat{\Psi}_2$, then $J((\mathbb{R}^2, \|\cdot\|_{\psi})) = J((\mathbb{R}^2, \|\cdot\|_{\psi})^*)$.

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