

On Pfaffian systems on $\mathbb{P}_2(\mathbb{C})$
with logarithmic singularities

R. Gérard*

Department of Mathematics

University of Tokyo

Hongo, Tokyo

and present address

Institut de Recherche Mathématique

Alsacien

7 rue René Descartes

67084 Strasbourg France

0. Introduction.

Let $\mathcal{A} = \bigcup_{i=1}^n \mathcal{A}_i$ be an algebraic subset of $\mathbb{P}_2(\mathbb{C})$ where for each i , $\mathcal{A}_i$ is irreducible and given by an irreducible polynomial equation in the homogeneous coordinates on $\mathbb{P}_2(\mathbb{C})$:

$$P_i(x_1, x_2, x_3) = 0.$$

We are considering Pfaffian systems of the form

$$(1) \quad dz = \omega z$$

$$\omega = \sum_{i=1}^n A_i \frac{dP_i}{P_i}$$

* J.S.P.S. visiting Professor July 1978 to the end of October 1978.

where the A_i 's are constant square matrices of order m .

In this lecture we are going to speak about

1) the set $\mathcal{R}(1)$ of relations between the A_i 's implied by the condition

$$d\omega = \omega \wedge \omega = 0$$

2) the relations in $\pi_1(\mathbb{P}_2(\mathbb{C}) - \mathcal{A})$

3) the nature of the solutions of (1)

4) the Riemann-Hilbert problem.

In this lecture there are no deep results but only simple remarks.

1. Some examples.

1.1. $\mathcal{A} = \emptyset$, $\omega = 0$, $\pi_1(\mathbb{P}_2(\mathbb{C})) = 1$ and $\mathbb{C}^m$ is the vector space of solutions.

1.2. $\mathcal{A} = \bigcup_{i=1}^3 \mathcal{A}_i$, $\mathcal{A}_i = \{x \in \mathbb{P}_2(\mathbb{C}) \mid x_i = 0\}$

$$(1.2) \quad \omega = \sum_{i=1}^3 A_i \frac{dx_i}{x_i}.$$

Then $\mathcal{R}(1)$:

$$\begin{cases} [A_i, A_j] = 0 & i \neq j \\ A_1 + A_2 + A_3 = 0. \end{cases}$$

And

$\pi_1(\mathbb{P}_2(\mathbb{C}) - \mathcal{A})$ is abelian.

A fundamental matrix of solutions is given by:

$$\begin{pmatrix} A_1 & A_2 & A_3 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

and all solutions are elementary functions. The Riemann-Hilbert problem can easily be solved with a system of the form (1.2).

1.3. The Pfaffian system associated to the hypergeometric functions in two variables.

The hypergeometric function F_1 is given by the system of partial differential equations

$$x(1-x)(x-y) \frac{\partial^2 z}{\partial x^2} + [\gamma(x-y) - (\alpha+\beta+1)x^2 + (\alpha+\beta-\beta'+1)xy + \beta'y] \frac{\partial z}{\partial x}$$

$$- \beta y(1-y) \frac{\partial z}{\partial y} - \alpha \beta (x-y)z = 0$$

$$y(1-y)(y-x) \frac{\partial^2 z}{\partial y^2} + [\gamma(y-x) - (\alpha+\beta'+1)y^2 + (\alpha+\beta'-\beta+1)xy + \beta x] \frac{\partial z}{\partial y}$$

$$- \beta' x(1-x) \frac{\partial z}{\partial x} - \alpha \beta' (y-x)z = 0$$

$$(x-y) \frac{\partial^2 z}{\partial x \partial y} - \beta \frac{\partial z}{\partial x} + \beta \frac{\partial z}{\partial y} = 0.$$

But the map

$$z \longmapsto \begin{pmatrix} z \\ x \frac{\partial z}{\partial x} \\ y \frac{\partial z}{\partial y} \end{pmatrix}$$

transforms this "complicated" system into the following one of type (1) which is completely integrable.

$$dz = \omega z$$

$$\omega = \sum_{i=1}^3 A_i \frac{dx_i}{x_i} + \sum_{i=1}^3 B_i \frac{du_i}{u_i}$$

$$u_i = x_j - x_k \quad j \neq k, j \neq i, k \neq i,$$

where

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1-\gamma+\beta' & 0 \\ 0 & -\beta' & 0 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\alpha\beta' & -\beta' & \gamma-\alpha-\beta'-1 \end{pmatrix}$$

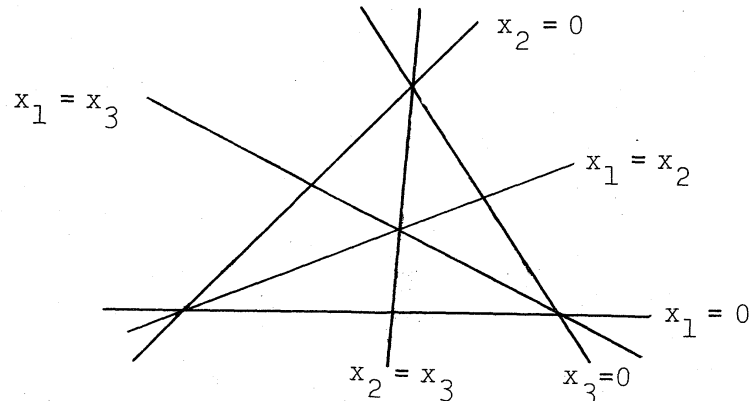
$$A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -\beta \\ 0 & 0 & 1-\gamma+\beta \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 0 & 0 & 0 \\ -\alpha\beta & \gamma-\alpha-\beta-1 & -\beta \\ 0 & 0 & 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0 & -1 & -1 \\ \alpha\beta & \alpha+\beta & \beta \\ \alpha\beta' & \beta' & \alpha+\beta' \end{pmatrix}$$

$$B_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\beta' & \beta \\ 0 & \beta' & -\beta \end{pmatrix}$$

and we have $d\omega = \omega \wedge \omega = 0$. The singular set:



The same thing can be done for the functions F_2 and F_3 by using the map

$$z \longmapsto \begin{pmatrix} z \\ x \frac{\partial z}{\partial x} \\ y \frac{\partial z}{\partial y} \\ xy \frac{\partial^2 z}{\partial x \partial y} \end{pmatrix}$$

for F_2 the singular set is

$$\mathcal{A} = \{x_1 = 0\} \cup \{x_2 = 0\} \cup \{x_1 = x_3\} \cup \{x_2 = x_3\} \cup \\ \{x_1 + x_2 - x_3 = 0\} \cup \{x_3 = 0\},$$

for F_3 the singular set is

$$\mathcal{A} = \{x_1 = 0\} \cup \{x_2 = 0\} \cup \{x_3 = 0\} \cup \{x_1 = x_3\} \cup \{x_2 = x_3\} \cup \\ \{x_2 x_3 - x_1 x_3 - x_1 x_2 = 0\}.$$

We don't know if a transformation does exist for F_4 . If it does exist it will be more complicated than for F_1, F_2, F_3 .

1.4. Let $\mathcal{A} = \bigcup_{i=1}^n \mathcal{A}_i$ an arbitrary algebraic subset of $\mathbb{P}_2(\mathbb{C})$.

Then let $\{A_i\}_{i=1,2,\dots,n}$ a set of permutable matrices satisfying

$$\sum_{i=1}^n \deg(P_i) A_i = 0.$$

Then the system

$$dz = \omega z$$

with

$$\omega = \sum_{i=1}^n A_i \frac{dP_i}{P_i}$$

is completely integrable and has $\mathcal{A}$ as singular set. But we cannot call this system "generic".

Problem I. For any algebraic set $\mathcal{A} = \bigcup_{i=1}^n \mathcal{A}_i$ in $\mathbb{P}_2(\mathbb{C})$ does there exist a completely integrable Pfaffian system of type (1) with singularities on $\mathcal{A}$ and which is not example 1.4. ?

For some singular sets the answer is yes: examples F_1 , F_2 , F_3 .

In example 1.4, we have a global fundamental matrix of solutions

$$\prod_{i=1}^n P_i^{A_i}$$

which is given by elementary functions. Then we can reformulate problem I in

Problem I'. Find the algebraic sets $\mathcal{A} = \bigcup_{i=1}^n \mathcal{A}_i$ in $\mathbb{P}_2(\mathbb{C})$

for which there exists a completely integrable Pfaffian system having singularities on $\mathcal{A}$ and which does not have a global fundamental matrix of solutions which is elementary.

2. The condition $d\omega = \omega \wedge \omega = 0$.

Let $\mathcal{A} = \bigcup_{i=1}^n \mathcal{A}_i$, $\mathcal{A}_i : P_i(x_1, x_2, x_3) = 0$ (irreducible).

2.1. The $\mathcal{A}_i$'s are normal crossing.

Consider $\omega = \sum_{i=1}^n A_i \frac{dP_i}{P_i}$ then it is easy to see that

$$d\omega = \omega \wedge \omega \iff [A_i, A_j] = 0 \text{ for all } i, j \text{ } i \neq j.$$

And ω is well defined on $\mathbb{P}_2(\mathbb{C})$ if we add the condition

$$\sum_{i=1}^n (\deg P_i) A_i = 0.$$

And in this case problems I and I' are solved and the solutions are all elementary functions.

The Riemann-Hilbert problem can be solved with a system of type (1). In fact $\pi_1(\mathbb{P}_2(\mathbb{C}) - \mathcal{A})$ is abelian, then any linear representation

$$\chi : \pi_1(\mathbb{P}_2(\mathbb{C}) - \mathcal{A}) \longrightarrow GL(m, \mathbb{C})$$

is abelian.

The fundamental group has n -generators $g_1, g_2, \dots, g_n$ related by

$$\prod_{i=1}^n g_i^{\deg P_i} = 1.$$

Set $\chi(g_i) = D_i$ and $\tilde{A}_i = \frac{1}{2\pi i} \log D_i$. Then

$$[\tilde{A}_i, \tilde{A}_j] = 0$$

and

$$\sum_{i=1}^n (\deg P_i) \tilde{A}_i = 2m\pi i I$$

and by choosing in a suitable way the determination of $\log D_i$, it is possible to find the set of A_i 's satisfying

$$[A_i, A_j] = 0$$

and

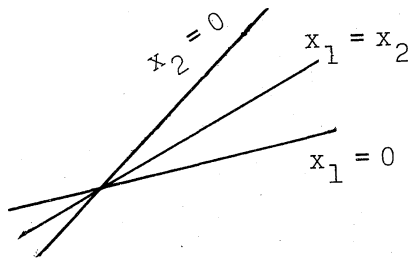
$$\sum_{i=1}^n (\deg P_i) A_i = 0.$$

2.2. The A_i 's are not normal crossing.

2.2.1. An example.

$$A = \bigcup_{i=1}^3 A_i, \quad A_1 = \{x_1 = 0\}, \quad A_2 = \{x_2 = 0\}$$

$$A_3 = \{x_1 = x_2\}.$$



$$\text{Set } \omega = A_1 \frac{dx_1}{x_1} + A_2 \frac{dx_2}{x_2} + A_3 \frac{d(x_1 - x_2)}{x_1 - x_2}.$$

And $d\omega = \omega \wedge \omega$ gives

$$[A_1, A_1 + A_2 + A_3] = 0$$

$$[A_2, A_1 + A_2 + A_3] = 0$$

$$[A_3, A_1 + A_2 + A_3] = 0$$

which are trivial consequence of

$$A_1 + A_2 + A_3 = 0$$

which has to be satisfied to have a system on $\mathbb{P}_2(\mathbb{C})$.

This means that any set of matrices A_1, A_2, A_3 such that $A_1 + A_2 + A_3 = 0$ solves the problem I, the solutions are not elementary but nearly related the hypergeometric Gauss function of one variable.

2.2.2. Let us consider a Pfaffian form

$$\omega = \sum_{i=1}^n A_i \frac{dP_i}{P_i}$$

$A_i : P_i = 0$, P_i irreducible. Then by a finite number of monoidal transformations (blowing up), we construct a manifold X with an analytic map

$$\sigma : X \longrightarrow \mathbb{P}_2(\mathbb{C}).$$

There exists in X , a divisor

$$A^* = \bigcup_{j=1}^m A_j^* \quad (m > n)$$

where the A_j^* are normal crossing and for each $i \in [1, 2, \dots, n]$ there exists $j(i) \in [1, 2, \dots, m]$ such that

$$\sigma : A_{j(i)}^* \xrightarrow{\sim} A_i.$$

But A^* contains some exceptional divisor coming from blowing up of the singularities of A . Moreover σ is an isomorphism from

$$X - A^* \text{ onto } \mathbb{P}_2(\mathbb{C}) - A.$$

This implies that

$$\pi_1(X - A^*) \simeq \pi_1(\mathbb{P}_2(\mathbb{C}) - A).$$

Denote by $\sigma^*(\omega)$ the inverse image of ω by σ , then

$\sigma^*(\omega)$ has logarithmic poles on A^* . Then we have for each $i \in [1, 2, \dots, n]$

$$\text{R\'es}_{A_i}(\sigma^*\omega) = \text{R\'es}_{A_i}(\omega) = A_i$$

and for each exceptional divisor B

$$\text{R\'es}_B(\sigma^*\omega) = \sum_{i=1}^n r_i \text{R\'es}_{A_i}(\omega)$$

where the r_i 's are integers.

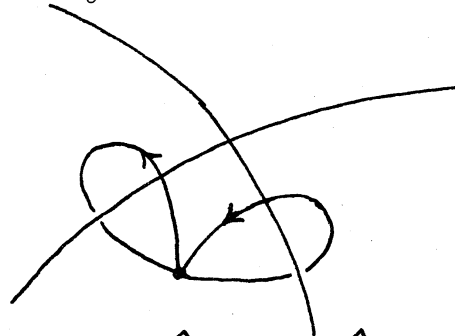
By local computations in X , we can easily prove the following

$$\omega \wedge \omega = 0 \iff \sigma^*\omega \wedge \sigma^*\omega = 0 \iff [\text{R\'es}_{A_i}^{\sigma^*}(\omega), \text{R\'es}_{A_j}^{\sigma^*}(\omega)] = 0$$

for all i, j $i \neq j$ with $A_i^* \cap A_j^* \neq \emptyset$.

This result can also be seen by using the following remark.

Locally in X near a point $M \in A_i^* \cap A_j^*$, choose a simple path γ_i surrounding A_i^* and a path γ_j surrounding A_j^* having the same origin and such that



$$\int_{\gamma_i} \frac{dP_i^*}{P_i^*} = 2\pi\sqrt{-1}$$

$$\int_{\gamma_j} \frac{dP_j^*}{P_j^*} = 2\pi\sqrt{-1}.$$

Denote by $\hat{\gamma}_i$ and $\hat{\gamma}_j$ the local homotopy classes of γ_i and γ_j . We have

$$\hat{\gamma}_i \hat{\gamma}_j = \hat{\gamma}_j \hat{\gamma}_i$$

(the local fundamental group is abelian). This means that

there exists a two cell S_{ij} homotopic to zero such that

$$\partial S_{ij} = \gamma_i \gamma_j \gamma_i^{-1} \gamma_j^{-1}$$

and

Proposition 1.

$$\int_{S_{ij}} \sigma^*(\omega) \wedge \sigma^*(\omega) = +4\pi^2 [\text{R\'es}_{A_j^*} \sigma^*(\omega), \text{R\'es}_{A_i^*} \sigma^*(\omega)]$$

and as a corollary

$$\omega \wedge \omega = 0 \implies [\text{R\'es}_{A_j^*} \sigma^*(\omega), \text{R\'es}_{A_i^*} \sigma^*(\omega)] = 0$$

for all i, j $i \neq j$.

Let us summarize the results:

The complete integrability condition is equivalent to the commutation of the r\'esidues of $\sigma^*(\omega)$ for A_i^* and A_j^* when $A_i^* \cap A_j^* \neq \emptyset$.

This means that for $\omega = \sum_{i=1}^n A_i \frac{dP_i}{P_i}$ we have

$$\omega \wedge \omega = 0 \iff \begin{cases} \sum_{i=1}^n (\deg P_i) A_i = 0 \\ [\text{R\'es}_{A_i^*} \sigma^*(\omega), \text{R\'es}_{A_j^*} \sigma^*(\omega)] = 0 \\ \text{where} \\ \text{R\'es}_{A_j^*} \sigma^*(\omega) = \text{R\'es}_{A_j} \omega \end{cases}$$

where A_j is not an exceptional divisor and

$$\text{R\'es}_{A_j^*} \sigma^*(\omega) = \sum_{i=1}^n r_i^j \text{R\'es}_{A_i} \omega$$

where r_i^j are integers.

2.3. The group of the relation $\omega \wedge \omega = 0$.

Let us construct a group G in the following way.

To each irreducible component A_i of A is associated in an abstract way a generator g_i of G . Then to each relation among the A_i 's we associate a relation between the g_i 's in the following way.

$$\text{To } \sum_{i=1}^n P_i A_i = 0 \quad P_i = \deg P_i \quad \text{let us associate}$$

$$g_1^{P_1} g_2^{P_2} \dots g_n^{P_n} = 1 \quad \text{and} \quad \tau(g_1^{P_1} g_2^{P_2} \dots g_n^{P_n}) = 1$$

for all circular permutation τ of the factors.

$$\text{To } [\text{Rés}_{A_i} \sigma^*(\omega), \text{Rés}_{A_j} \sigma^*(\omega)] = 0$$

$$[g_1^{r_1^i} g_2^{r_2^i} \dots g_n^{r_n^i}, g_1^{r_1^j} \dots g_n^{r_n^j}] = 1$$

and

$$[\tau(g_1^{r_1^i} \dots g_n^{r_n^i}), \tau(g_1^{r_1^j} \dots g_n^{r_n^j})] = 1$$

for each circular permutation τ of the factors.

Examples.

1. See 1.2.

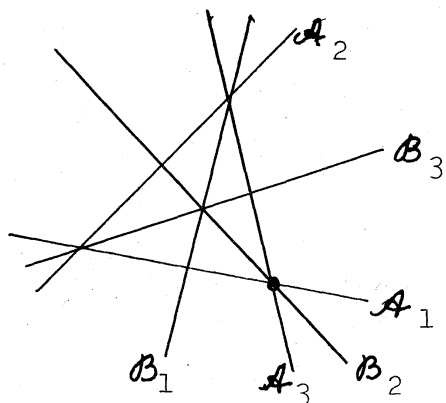
$$G = \{g_1, g_2, g_3; [g_i, g_j] = 1, g_1 g_2 g_3 = 1, \tau(g_1 g_2 g_3) = 1\}$$

and G is a free abelian with two generators.

2. See 2.1. Normal crossing case.

G is a free abelian with $(n-1)$ -generators.

3. See 1.3. Hypergeometric function F_1 .



The group G has 6 generators

$$a_1, a_2, a_3, b_1, b_2, b_3$$

corresponding to $A_1, A_2,$

A_3, B_1, B_2, B_3 .

The relations are up to cyclic

permutations as indicated above:

$$[a_1, b_1] = [a_2, b_2] = [a_3, b_3] = 1$$

$$[a_1, b_2 a_3] = [a_1, b_3 a_2] = 1$$

$$[a_2, b_3 a_1] = [a_2, b_1 a_3] = 1$$

$$[a_3, b_1 a_2] = [a_3, b_2 a_1] = 1$$

$$[b_1, b_2 b_3] = [b_1, a_2 a_3] = 1$$

$$[b_2, b_1 b_3] = [b_2, a_1 a_3] = 1$$

$$[b_3, b_1 b_2] = [b_3, a_1 a_2] = 1$$

and

$$a_1 a_2 a_3 b_1 b_2 b_3 = 1.$$

4. See 2.2.1.

The group G has three generators g_1, g_2, g_3 and up to

cyclic permutation as indicated we have the relations

$$[g_i, g_1 g_2 g_3] = 1 \quad i = 1, 2, 3.$$

but also $g_1 g_2 g_3 = 1$. As a consequence G is a free group with two generators.

3. Some problems.

I. If problem I in section 1 is solvable for an algebraic set $\mathcal{A}$, what are the relations between

- 1) the monodromy of the Pfaffian system and the group G
- 2) $\pi_1(\mathbb{P}_2(\mathbb{C}) - \mathcal{A})$ and G ?

II. In which cases is the Riemann-Hilbert problem solvable by a Pfaffian system of type (1)? Or such that $\sigma^*(\omega)$ is of type (1) in each coordinate system in X ?

Remark: In many particular examples when $\pi_1(\mathbb{P}_2(\mathbb{C}) - \mathcal{A})$ is known we have $G = \pi_1(\mathbb{P}_2(\mathbb{C}) - \mathcal{A})$.

To finish this summary let me thank Professor M. Oka of the University of Tokyo with whom I have had a very interesting discussion about fundamental groups of the complementary of an algebraic curve in $\mathbb{P}_2(\mathbb{C})$.