

On an univalence criterion

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Abstract

The method of subordination chains is used to establish an univalence criterion for holomorphic mappings in the open unit ball B^n in $\mathbb{C}^n$. The authors consider an univalence criterion for holomorphic functions in B^n .

1 Introduction

Let $\mathbb{C}^n$ be the space of n -complex variables $z = (z_1, \dots, z_n)$ with Euclidean inner product $\langle z, w \rangle = \sum_{k=1}^n z_k \bar{w}_k$ and the norm $\|z\| = \langle z, z \rangle^{\frac{1}{2}}$.

Let B^n be the open unit ball in $\mathbb{C}^n$, i.e. $B^n = \{z \in \mathbb{C}^n : \|z\| < 1\}$. We denote by $\mathcal{L}(\mathbb{C}^n)$ the space of continuous linear operators from B^n into $\mathbb{C}^n$, i.e. the $n \times n$ complex matrices $A = (A_{jk})$ with the standard operator norm

$$\|A\| = \sup \{\|Az\| : \|z\| < 1\}, \quad A \in \mathcal{L}(\mathbb{C}^n)$$

and

$I = (I_{jk})$ denotes the identity in $\mathcal{L}(\mathbb{C}^n)$.

Let $H(B^n)$ be the class of holomorphic mappings

$$f(z) = (f_1(z), \dots, f_n(z)), \quad z \in B^n$$

from B^n into $\mathbb{C}^n$. We say that $f \in H(B^n)$ is *locally biholomorphic* in B^n if f has local holomorphic inverse at each point in B^n or equivalently, if the derivative

$$Df(z) = \left(\frac{\partial f_k(z)}{\partial z_j} \right)_{1 \leq j, k \leq n}$$

is nonsingular at each point $z \in B^n$.

A mapping $v \in H(B^n)$ is called a *Schwarz function* if $\|v(z)\| \leq \|z\|$, for all $z \in B^n$.

Let $f, g \in H(B^n)$. Then f is said to be *subordinate* to g (written by $f \prec g$) in B^n if there exists a Schwarz function v such that $f(z) = g(v(z))$, $z \in B^n$. A function $L : B^n \times [0, \infty) \rightarrow \mathbb{C}^n$ is said to be the *subordination chain* if $L(\cdot, t)$ is holomorphic and univalent in B^n , $L(0, t) = 0$ for all $t \in [0, \infty)$, and $L(z, s) \prec L(z, t)$ whenever $0 \leq s \leq t < \infty$.

The subordination chain $L(z, t)$ is called a *normalized subordination chain* if $DL(0, t) = e^t I$ for all $t \geq 0$.

The following result concerning subordination chains is due to J. A. Pfaltzgraff [5].

Theorem 1 Let $L(z, t) = e^t z + \dots$ be a function from $B^n \times [0, \infty)$ into $\mathbb{C}^n$ such that:

(i) $L(\cdot, t) \in H(B^n)$ for all $t \in [0, \infty)$.

(ii) $L(z, t)$ is a locally absolutely continuous function of t , locally uniformly with respect to $z \in B^n$.

Let $h(z, t)$ be a function from $B^n \times [0, \infty)$ into $\mathbb{C}^n$ which satisfies the following condition:

(iii) $h(\cdot, t) \in H(B^n)$, $h(0, t) = 0$, $Dh(0, t) = I$ and $\operatorname{Re} \langle h(z, t), z \rangle \geq 0$ for all $t \geq 0$ and $z \in B^n$.

(iv) For each $T > 0$ and $r \in (0, 1)$, there is a number $K = K(r, T)$ such that $\|h(z, t)\| \leq K(r, T)$ when $\|z\| \leq r$ and $t \in [0, T]$.

(v) For each $z \in B^n$, $h(\cdot, t)$ is a measurable function on $[0, \infty)$.

Suppose $h(z, t)$ satisfies

$$\frac{\partial L(z, t)}{\partial t} = DL(z, t) h(z, t) \quad \text{a.e. } t \geq 0, \quad \text{for all } z \in B^n.$$

Further, suppose there is a sequence $\{t_m\}_{m \geq 0}$, $t_m > 0$ which is increasing to ∞ such that

$$\lim_{m \rightarrow \infty} e^{-t_m} L(z, t_m) = F(z)$$

locally uniformly in B^n .

Then for each $t \in [0, \infty)$, $L(\cdot, t)$ is univalent in B^n .

A version of Theorem 1 for subordination chains which are not normalized is due to P. Curt [2].

Theorem 2 Let $L(z, t) = a_1(t)z + \dots$, $a_1(t) \neq 0$ be a function from $B^n \times [0, \infty)$ into $\mathbb{C}^n$ such that:

- (i) For each $t \geq 0$, $L(\cdot, t) \in H(B^n)$.
 - (ii) $L(z, t)$ is a locally absolutely continuous function of $t \in [0, \infty)$, locally uniformly with respect to $z \in B^n$.
 - (iii) $a_1(t) \in C^1[0, \infty)$ and $\lim_{t \rightarrow \infty} |a_1(t)| = \infty$.
 - (iv) $h(\cdot, t) \in H(B^n)$ for all $t \in [0, \infty)$.
 - (v) For each $z \in B^n$, $h(z, \cdot)$ is a measurable function on $[0, \infty)$.
 - (vi) $h(0, t) = 0$ and $\operatorname{Re} \langle h(z, t), z \rangle \geq 0$ for all $t \geq 0$ and $z \in B^n$.
 - (vii) For each $T > 0$ and $r \in (0, 1)$, there exists a number $K = K(r, T)$ such that $\|h(z, t)\| \leq K(r, T)$ when $\|z\| \leq r$ and $t \in [0, T]$.
- Suppose that $h(z, t)$ satisfies

$$\frac{\partial L(z, t)}{\partial t} = DL(z, t) h(z, t) \quad \text{a.e. } t \geq 0, \text{ for all } z \in B^n. \quad (1)$$

Further suppose that there exists a sequence $\{t_m\}_{m \geq 0}$, $t_m > 0$ which is increasing to ∞ such that

$$\lim_{m \rightarrow \infty} \frac{L(z, t_m)}{a_1(t_m)} = F(z) \quad (2)$$

locally uniformly in B^n .

Then for each $t \in [0, \infty)$, $L(\cdot, t)$ is univalent in B^n .

2 Main results

By using Theorem 2 we obtain an univalence criterion which generalize two univalence criteria for holomorphic mappings in B^n .

Theorem 3 *Let f, g be holomorphic functions in B^n satisfying the conditions:*

1° g is locally univalent in B^n .

2° $f(0) = g(0) = 0$ and $Df(0) = Dg(0) = I$.

Let α be a real number with $\alpha \geq 2$. If

$$\left\| (Dg(z))^{-1} Df(z) - \frac{\alpha}{2} I \right\| < \frac{\alpha}{2} \quad (3)$$

and

$$\begin{aligned} & \left\| \|z\|^2 [Dg(z)^{-1} Df(z) - I] \right. \\ & \quad \left. + (1 - \|z\|^\alpha) (Dg(z))^{-1} D^2g(z)(z, \cdot) + \left(1 - \frac{\alpha}{2}\right) I \right\| \leq \frac{\alpha}{2} \quad (4) \end{aligned}$$

for all $z \in B^n$, then f is an univalent function in B^n .

Remark 1

The second derivative of function $g \in H(B^n)$ is a symmetric bilinear operator $D^2g(z)(\cdot, \cdot)$ on $\mathbb{C}^n \times \mathbb{C}^n$ and $D^2g(z)(w, \cdot)$ is the linear operator obtained by restricting $D^2g(z)$ to $\{w\} \times \mathbb{C}^n$. The linear operator $D^2g(z)(z, \cdot)$ has the matrix representation

$$D^2g(z)(z, \cdot) = \left(\sum_{m=1}^n \frac{\partial^2 g_k(z)}{\partial z_j \partial z_m} z_m \right)_{1 \leq j, k \leq n}.$$

Proof. We define the function $L(z, t)$ by

$$L(z, t) = f(e^{-t}z) + (e^{\alpha t} - 1)e^{-t}Dg(e^{-t}z)(z), \quad (z, t) \in B^n \times [0, \infty). \quad (5)$$

We have to show that $L(z, t)$ satisfies the conditions of Theorem 2 and hence $L(\cdot, t)$ is univalent in B^n , for all $t \in [0, \infty)$. It is easy to check that $a_1(t) = e^{(\alpha-1)t}$ and hence $a_1(t) \neq 0$, $\lim_{t \rightarrow \infty} |a_1(t)| = \infty$ and $a_1 \in C^1[0, \infty)$.

We have

$$L(z, t) = a_1(t)z + (\text{holomorphic term}).$$

Thus $\lim_{t \rightarrow \infty} \frac{L(z, t)}{a_1(t)} = z$, locally uniform with respect to B^n and hence (2) holds with $F(z) = z$. Obviously $L(z, t)$ satisfies the absolute continuity requirements of Theorem 2. From (5) we obtain

$$DL(z, t) = \frac{\alpha}{2} e^{(\alpha-1)t} Dg(e^{-t}z) [I - E(z, t)], \quad (6)$$

where, for all $(z, t) \in B^n \times [0, \infty)$, $E(z, t)$ is the linear operator defined by

$$\begin{aligned} E(z, t) = & \left(1 - \frac{2}{\alpha}\right) I - \frac{2}{\alpha} e^{-\alpha t} \left[(Dg(e^{-t}z))^{-1} Df(e^{-t}z) - I \right] \\ & - \frac{2}{\alpha} (1 - e^{-\alpha t}) (Dg(e^{-t}z))^{-1} D^2g(e^{-t}z)(e^{-t}z, \cdot). \end{aligned} \quad (7)$$

For $t = 0$, $E(z, 0) = -\frac{2}{\alpha} [(Dg(z))^{-1} Df(z) - \frac{\alpha}{2} I]$, it follows

$$\|E(z, 0)\| < 1, \text{ for all } z \in B^n. \quad (8)$$

For $t > 0$, $E(\cdot, t) : \overline{B^n} \rightarrow \mathcal{L}(C^n)$ is holomorphic. By using the weak maximum modulus theorem[4], we obtain that $\|E(z, t)\|$ can have no maximum in B^n unless $\|E(z, t)\|$ is of constant value throughout $\overline{B^n}$.

If $z = 0$ and $t > 0$, then we have

$$\|E(0, t)\| = \left\| \left(1 - \frac{2}{\alpha}\right) I \right\| = \left|1 - \frac{2}{\alpha}\right| < 1. \quad (9)$$

We also have

$$\|E(z, t)\| < \max_{\|w\|=1} \|E(w, t)\|.$$

If we let $u = e^{-t}w$, with $\|w\| = 1$, then $\|u\| = e^{-t}$ and from (4) we obtain

$$\begin{aligned} \|E(w, t)\| &= \frac{2}{\alpha} \left\| \|u\|^\alpha \cdot [(Dg(u))^{-1} Df(u) - I] \right. \\ &\quad \left. + (1 - \|u\|^\alpha) (Dg(u))^{-1} D^2g(u)(u, \cdot) + \left(1 - \frac{\alpha}{2}\right) I \right\| \leq 1. \end{aligned}$$

Since $\|E(z, t)\| < 1$ for all $(z, t) \in B^n \times [0, \infty)$, it follows $I - E(z, t)$ is an invertible operator.

Further the calculation shows that

$$\begin{aligned} \frac{\partial L(z, t)}{\partial t} &= \frac{\alpha}{2} e^{(\alpha-1)t} Dg e^{-t} z [I + E(z, t)](z) \\ &= DL(z, t) [I - E(z, t)]^{-1} [I + E(z, t)](z). \end{aligned}$$

Hence $L(z, t)$ satisfies the differential equation (1) for all $t \geq 0$ and $z \in B^n$, where

$$h(z, t) = [I - E(z, t)]^{-1} [I + E(z, t)](z). \quad (10)$$

It remains to show that $h(z, t)$ satisfies the conditions (iv) (v) (vi) and (vii) of Theorem 2. Clearly, $h(z, t)$ satisfies the holomorphy and measurability requirements and $h(0, t) = 0$. The inequality

$$\begin{aligned} \|h(z, t) - z\| &= \|E(z, t)(h(z, t) + z)\| \leq \|E(z, t)\| \|(h(z, t) + z)\| \\ &< \|(h(z, t) + z)\| \end{aligned}$$

implies $\operatorname{Re} \langle h(z, t), z \rangle \geq 0$, for all $z \in B^n$ and $t \geq 0$.

By using the inequality

$$\|[I - E(z, t)]^{-1}\| \leq [1 - \|E(z, t)\|]^{-1},$$

we obtain

$$\|h(z, t)\| \leq \frac{1 + \|E(z, t)\|}{1 - \|E(z, t)\|} \|z\|.$$

Since the conditions of Theorem 2 are satisfied it follows that the function $L[z, t], t \geq 0$ is univalent in B^n . In particular $f(z) = L(z, 0)$ is univalent in B^n .

Remark 2

- 1) If $g = f$ and $\alpha = 2$, then Theorem 3 becomes the n -dimensional version of Beker's univalence criterion [5].
- 2) For $\alpha = 2$ we obtain an univalence criterion due to P. Curt and D. Răducanu [3].

References

- [1] J. Becker, *Loewnersche Differentialgleichung und Schichtheits - Kriteri-
onn*, Math. Ann. 202(1973), 321-335.
- [2] P. Curt, *A generalization in n -dimensional complex space of Ahlfos and
Becker criterion for univalence*, Studia Math. 49(1994), 31-38.
- [3] P. Curt and D. Răducanu, *On an univalence condition*, (to appear).
- [4] E. Hille and R.S. Phillips, *Functional analysis and semigroups*, Amer.
Math. Soc. Collaq. Publ. 31(1957).
- [5] J.A. Pfaltzgraff, *Subordination chains and univalence of holomorphic
mappings in C^n* , Math. Ann. 210(1974), 55-68.

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