

A note on lowness for Robinson theories

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Abstract

We show following two theorems. Theorem A: for thick simple existentially universal domain, the equality of Lascar strong types is definable by an existential type. Theorem B: for thick low existentially universal domain, Lascar strong types equal strong types. Theorem A is already proved by Ben-Yaacov [2].

1 Preliminaries

Definition 1.1 We say that an L -structure M is κ -existentially universal domain (e.u.domain) if

- if $\Sigma(x)$ is a partial existential type over A ($|A| < \kappa$) which is finitely satisfiable in M , then Σ is satisfiable in M , and
- for $|A|, |B| < \kappa$, and $f : A \rightarrow B$: a bijection such that $\text{etp}(a) \subset \text{etp}(f(a))$ for all tuples a from A , f extends to an automorphism of M .

Remark 1.1 An e.u.domain M is an existentially closed model for the universal theory of M , $\text{Th}(M)_\forall$.

Let $\mathcal{M}$ be a κ -e.u.domain for a enoughly big cardinal κ . Put $T = \text{Th}_\forall(\mathcal{M})$. $M, N, \dots$ denote existentially closed models of T , $a, b, \dots$ denote finite tuples in $\mathcal{M}$, and $A, B, \dots$ denote small subsets of $\mathcal{M}$.

Definition 1.2 Let $\Sigma(x, B)$ be an existential type over B .

1. We say that $\Sigma(x, B)$ divides over A if there exists an existentially indiscernible sequence $(B_i : i < \omega)$ over A with $B_0 = B$ such that $\bigcup_{i < \omega} \Sigma(x, B_i)$ is not realized in $\mathcal{M}$.

2. We say that $\Sigma(x)$ forks over A if there exists a small set of dividing ($/A$) existential formulas Ψ (with parameters) such that $\mathcal{M} \models \Sigma \rightarrow \bigvee \Psi$.

Remark 1.2 • If $\Sigma(x)$ divides over A , then there is an existential formula $\varphi(x)$ such that $\Sigma \vdash \varphi(x)$ and $\varphi(x)$ divides over A .

- It is not known whether that if Σ forks over A , then there are an existential formula θ where $\Sigma \vdash \theta$ and dividing ($/A$) existential formulas $\psi_1, \dots, \psi_n$ such that $\mathcal{M} \models \theta \rightarrow \bigvee_{i=1}^n \psi_i$.

Definition 1.3 We say that $\mathcal{M}$ is simple if for all $a \in \mathcal{M}$, $A \subset \mathcal{M}$, there exists $B \subset A$ with $|B| \leq |T| + \aleph_0$ such that $\text{etp}(a/A)$ does not fork over B .

Fact 1.1 [3] Suppose that $\mathcal{M}$ is simple. Then, Σ forks over A if and only if Σ divides over A .

Definition 1.4 1. We say that $\text{lstp}(a) = \text{lstp}(b)$ if for any bounded $\emptyset$ -invariant equivalence relation $E(x, y)$, $E(a, b)$ holds.

2. We say that $d(a, b) \leq 1$ if there is an existentially indiscernible sequence I such that $a, b \in I$.
3. We say that $d(a, b) \leq n$ if there exist $a_0, \dots, a_n$ with $a_0 = a, a_n = b$ such that $d(a_i, a_{i+1}) \leq 1$ for any $i < n$.
4. We say that $d(a, b) < \omega$ if $d(a, b) \leq n$ for some $n < \omega$.

Fact 1.2 [3] $\text{lstp}(a) = \text{lstp}(b)$ if and only if $d(a, b) < \omega$.

Fact 1.3 [3] If $(a_i : i < \lambda)$ is an enough long sequence and $A \subset \mathcal{M}$, then there is an existentially indiscernible sequence $(b_i : i < \omega)$ such that for any $n < \omega$, there are $i_0 < \dots < i_{n-1} < \lambda$ such that $\text{etp}(b_0, \dots, b_{n-1}/A) = \text{etp}(a_{i_0}, \dots, a_{i_{n-1}}/A)$.

Fact 1.4 [3] Suppose that $\mathcal{M}$ is simple. Then, for all a , $A \subset B$, there exists a' such that

- $\text{lstp}(a'/A) = \text{lstp}(a/A)$ and
- $\text{etp}(a'/B)$ does not fork over A .

We write $a \perp b$ to mean that $\text{etp}(a/b)$ does not fork over $\emptyset$.

Fact 1.5 (Independence theorem for simple e.u.domain, [3]) *Suppose that $\mathcal{M}$ is simple and a_1, a_2, b_1, b_2 satisfy the following:*

- $\text{lstp}(a_1) = \text{lstp}(a_2)$,
- $a_1 \perp b_1, a_2 \perp b_2, b_1 \perp b_2$.

Then, there exists a such that

- $a \models \text{etp}(a_1/b_1) \cup \text{etp}(a_2/b_2)$
- $a \perp b_1 b_2$.

2 Proof of Theorem A

In this section, we prove Theorem A. For simplicity, we show over $\emptyset$.

Definition 2.1 We say that $\mathcal{M}$ is thick if " $d(x, y) \leq 1$ " is definable by an existential type. If $\mathcal{M}$ is thick, then we assume that $q_1(x, y)$ defines " $d(x, y) \leq 1$ ".

Lemma 2.1 *Suppose that $\mathcal{M}$ is thick. Then, " $d(x, y) \leq 2$ " is definable by an existential type.*

Proof: It is defined by $\{\exists z \varphi(x, z) \wedge \varphi(z, y) \mid \varphi(x, y) \in q_1(x, y)\}$.

Lemma 2.2 *Suppose that $\mathcal{M}$ is thick and simple. Then, the following are equivalent:*

1. $\text{lstp}(a) = \text{lstp}(b)$
2. $d(a, b) \leq 2$
3. $q_1(x, a) \cup q_1(x, b)$ does not fork over $\emptyset$

Proof: $(3 \rightarrow 2 \rightarrow 1)$ is trivial. $(1 \rightarrow 2)$ Let c be a tuple such that $\text{lstp}(c) = \text{lstp}(a) = \text{lstp}(b)$ and $c \perp ab$. Take a' such that $\text{etp}(a'a) = \text{etp}(ac)$. Then $\text{lstp}(a') = \text{lstp}(a)$ and $a' \perp a$. So, by independence theorem, we can get a_2 such that $a_2 \models \text{etp}(a/c) \cup \text{etp}(a'/a)$ and $a_2 \perp ac$.

Iterating this, we can get a sequence $(a_i : i < \omega)$ such that $\text{etp}(a_i a_j) = \text{etp}(ac)$ for each $j < i < \omega$. By compactness and Fact 1.3, we can assume this sequence is existentially indiscernible. So, we get existentially indiscernible sequences I, J such that $a, c \in I$ and $b, c \in J$.

Theorem A [2] Suppose that $\mathcal{M}$ is thick and simple. Then, " $\text{lstp}(x) = \text{lstp}(y)$ " is definable by an existential type.

Proof: By above lemmas.

3 Proof of Theorem B

In this section, we prove Theorem B. Again for simplicity, we show over $\emptyset$.

Definition 3.1 We say that $\text{stp}(a) = \text{stp}(b)$ if for any definable (by an existential formula over $\emptyset$) finite equivalence relation $E(x, y)$, $E(a, b)$ holds.

Definition 3.2 1. Let $\varphi(x, y)$ be an existential formula. An existential formula $\psi(y_0, \dots, y_{k-1})$ where $\text{lh}(y_i) = \text{lh}(y)$ for each $i < k$ is said to be a k -inconsistency witness for φ if $\mathcal{M} \models \forall y_0 \dots y_{k-1} (\psi(y_0, \dots, y_{k-1}) \rightarrow \neg \exists x \bigwedge_{i < k} \varphi(x, y_i))$.

2. Let $\Sigma(x)$ be an existential type and $\varphi(x, y)$ be an existential formula.

- We say that $D(\Sigma, \varphi) \geq 0$ if Σ is satisfiable.
- We say that $D(\Sigma, \varphi) \geq n + 1$ if there is a natural number k , a k -inconsistency witness ψ , and an existentially indiscernible sequence $(b_i : i < \omega)$ such that $D(\Sigma(x) \cup \{\varphi(x, b_i)\}, \varphi) \geq n$ for each $i < \omega$ and $\mathcal{M} \models \psi(b_{i_0}, \dots, b_{i_{k-1}})$ for all $i_0, \dots, i_{k-1} < \omega$.

3. We say that $\mathcal{M}$ is low if

- $\mathcal{M}$ is simple and
- $D(x = x, \varphi) < \omega$ for any existential formula φ .

Lemma 3.1 Suppose that $\mathcal{M}$ is thick and low. Then,

1. $\{a : \varphi(x, a) \text{ divides over } \emptyset\}$ is definable by an existential type.
2. $\{(a, b) : \varphi(x, a) \wedge \varphi(x, b) \text{ does not divide over } \emptyset\}$ is definable by an existential type if it is restricted to $(p \otimes p)^\mathcal{M} = \{(a, b) : a, b \models p, a \perp b\}$. So, it is definable by an existential universal formula if it is restricted to $(p \otimes p)^\mathcal{M}$.

Proof: (1) Note that by lowness, for any $\varphi(x, y)$ there is an existential formula ψ such that for all a , if $\varphi(x, a)$ divides over $\emptyset$, then φ divides by an existentially indiscernible sequence in which any k -elements satisfies ψ .

(2) For $a, b \models p$ where $a \perp b$, the following are equivalent:

1. $\varphi(x, a) \wedge \varphi(x, b)$ does not divide over $\emptyset$
2. there exist a^* and b^* such that
 - $\mathcal{M} \models \varphi(a^*, a)$ and $a^* \perp a$;
 - $\mathcal{M} \models \varphi(b^*, b)$ and $b^* \perp b$;
 - $\text{lstp}(a^*) = \text{lstp}(b^*)$

By Theorem A, " $\text{lstp}(a^*) = \text{lstp}(b^*)$ " is expressible by an existential type. " $a^* \perp a$ " is expressible by " $D(\text{etp}(a/a^*), \varphi, \psi) \geq D(p, \varphi, \psi)$ " for any φ, ψ .

We sat that $E_{p(x), \varphi(x, y)}(b, c)$ if for all $a \models p$ with $a \perp bc$, $\varphi(x, a) \wedge \varphi(x, b)$ does not divide over $\emptyset$ if and only if $\varphi(x, a) \wedge \varphi(x, c)$ does not divide over $\emptyset$.

Lemma 3.2 *Suppose that $\mathcal{M}$ is thick and low. For any $a \models p$ where $\varphi(x, a)$ does not divide over $\emptyset$, $E_{p(x), \varphi(x, y)}$ is a definable (by an existential formula) finite equivalence relation on $(p^2)^\mathcal{M}$.*

Proof: We can check that $E_{p, \varphi}$ is a bounded equivalence relation boundedness is by " $\text{lstp}(x) = \text{lstp}(y) \Rightarrow E_{p, \varphi}(x, y)$ ". On the other hand, by the above lemma $\neg E_{p, \varphi}$ is definable by an existential type. So, $E_{p, \varphi}$ is a finite equivalence relation. Let $a_1, \dots, a_n$ be representations of classes. Then $\bigcup \{\neg E(x, a_i) : i \leq n\}$ is not satisfiable. For simplicity, we assume $n = 3$. There exists an existential formula $\varphi(x, y)$ such that

1. $\neg E(x, a_i) \vdash \varphi(x, a_i)$ for each $i \leq 3$
2. $\mathcal{M} \models \neg \exists x \varphi(x, a_1) \wedge \varphi(x, a_2) \wedge \varphi(x, a_3)$.

Put $\psi(x, y) = \neg \varphi(x, y)$. Note that $\mathcal{M} \models \forall x (\psi(x, a_1) \leftrightarrow \varphi(x, a_2) \wedge \varphi(x, a_3))$. So, $\psi(x, a_1)$ is also existential. By a symmetric argument, $\psi(x, a_2), \psi(x, a_3)$ are all existential. Then we have

$$E(x, y) \leftrightarrow \bigwedge_{i \leq 3} (\psi(x, a_i) \leftrightarrow \psi(y, a_i)).$$

We can omit parameters a_i 's because this does not depend on a choice of representations and $\psi(x, a_i)$ is existential universal.

Theorem B *Suppose that $\mathcal{M}$ is thick and low. Then, $\text{stp} = \text{lstp}$*

Proof: If $\text{stp}(a) = \text{stp}(b)$, then by the above lemma $a, b \models E_{p, \varphi}$ for any φ . Take c such that $\text{lstp}(c) = \text{lstp}(a)$ and $c \perp ab$. Then, $q_1(x, a) \cup q_1(x, c)$ does not divide by Lemma 3. Then, $q_1(x, b) \cup q_1(x, c)$ does not divide by $E_{p, \varphi}(a, b)$. Again by Lemma 3, we have $\text{lstp}(b) = \text{lstp}(c)$.

References

- [1] Itai Ben-Yaacov, Simplicity in compact abstract theories, *Journal of Mathematical Logic* 3 (2003), no. 2, 163-191
- [2] Itai Ben-Yaacov, Thickness, and a categorical view of type space functors, *Fundamenta Mathematicae* 179, 2003, 199-224.
- [3] Anand Pillay, Forking in the category of existentially closed structures. Connections between model theory and algebraic and analytic geometry, 23-42, *Quad. Mat.*, 6, Dept. Math., Seconda Univ. Napoli, Caserta, 2000.
- [4] Ziv Shami, Definability in Low Simple Theories, *J. Symbolic Logic* 65, No. 4 (2000), 1481-1490.
- [5] Akito Tsuboi, Some consideration for Shami's paper on simplicity, unpublished note.