

Tilings and Fractals from Pisot substitutions

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1 Definition of Pisot Unit Substitutions

$$\mathcal{A} := \{1, 2, \dots, d\} \quad (\text{alphabet})$$

$$\mathcal{A}^* := \bigcup_{n=0}^{\infty} \mathcal{A}^n \quad (\text{free monoid, i.e., the set of finite words})$$

($G\{1, \dots, d\}$: a free group of rank d)

Definition 1.1 $\sigma : \mathcal{A}^* \longrightarrow \mathcal{A}^*$ is a substitution if

$$\begin{aligned} (1) \quad \sigma(i) &= W^{(i)} \in \mathcal{A}^*, \quad W^{(i)} \neq \emptyset \\ &= w_1^{(i)} \dots w_k^{(i)} \dots w_{l_i}^{(i)}, \quad w_k^{(i)} \in \mathcal{A} \\ &= P_k^{(i)} w_k^{(i)} S_k^{(i)}; \end{aligned}$$

$$(2) \quad \sigma(w_1 \dots w_k) := \sigma(w_1) \dots \sigma(w_k) \text{ for } w_1 \dots w_k \in \mathcal{A}^*.$$

(A substitution σ is invertible if σ is an automorphism on $G\{1, 2, \dots, d\}$.)

Let L_σ be a matrix of σ , that is,

$$L_\sigma(i, j) := \text{the number of the letter } i \text{ contained in } \sigma(j).$$

Example 1.1 (Fibonacci substitution)

$$\sigma_F : \begin{array}{l} 1 \rightarrow 12 \\ 1 \rightarrow 1 \end{array}, \quad L_{\sigma_F} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Example 1.2 (Rauzy substitution)

$$\sigma_R : \begin{array}{l} 1 \rightarrow 12 \\ 1 \rightarrow 13 \\ 1 \rightarrow 1 \end{array}, \quad L_{\sigma_R} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Assumption For the substitution σ ,

- (1) L_σ is *primitive*, that is, $\exists N : L_\sigma^N > 0$;
- (2) L_σ is *unimodular*, that is, $\det L_\sigma = \pm 1$;
- (3) L_σ is *Pisot type*, that is, for eigenvalues $\lambda = \lambda_1, \lambda_2, \dots, \lambda_d$ of L_σ ,

$$\lambda = \lambda_1 > 1 > |\lambda_i|, \quad i = 2, \dots, d;$$

- (4) the characteristic polynomial $\Phi_\sigma(x)$ of L_σ is *irreducible*;
- (5) $\sigma(1) = 1W'$.

We say the substitution σ satisfying Assumption the *Pisot Unit substitution*.

On Assumption (5),

$$w_\sigma := \lim_{n \rightarrow \infty} \sigma^n(1) = s_1 s_2 \cdots s_m \cdots$$

is a fixed point of σ , that is, $\sigma(w_\sigma) = w_\sigma$.

$f : \mathcal{A}^* \rightarrow \mathbf{Z}^d$ is a homomorphism given by

$$f(i) := \mathbf{e}_i \quad \text{and} \quad f(w_1 \cdots w_k) := \sum_{j=1}^k f(w_j).$$

Lemma 1.1 *The following relation holds:*

$$\begin{array}{ccc} \mathcal{A}^* & \xrightarrow{\sigma} & \mathcal{A}^* \\ f \downarrow & & \downarrow f \\ \mathbf{Z}^d & \xrightarrow{L_\sigma} & \mathbf{Z}^{d*} \end{array}.$$

Let $\mathbf{v} > \mathbf{0}$ be a positive eigenvector of λ and P be L_σ -invariant contractive plain, that is,

$$\mathbf{R}^d := \mathcal{L}(\mathbf{v}) \oplus P.$$

The projection π is given by

$$\pi : \mathbf{Z}^d \rightarrow P \text{ along } \mathbf{v}.$$

Lemma 1.2 *The following relation holds:*

$$\begin{array}{ccc} \mathbf{Z}^d & \xrightarrow{L_\sigma} & \mathbf{Z}^d \\ \pi \downarrow & & \downarrow \pi \\ P & \xrightarrow{L_\sigma} & P \end{array}.$$

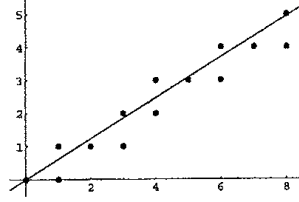


Figure 1: The figure of $\{f(s_0s_1 \cdots s_k) \mid k = 0, 1, 2, \dots\}$ on σ_F .

For the fixed point $w_\sigma = s_1s_2 \cdots s_n \cdots$,

$$\begin{aligned} Y &:= \pi \{f(s_0s_1 \cdots s_k) \mid k = 0, 1, \dots\}, \\ Y_i &:= \pi \{f(s_0s_1 \cdots s_{k-1}) \mid \exists k : s_k = i, k = 1, 2, \dots\}, \\ Y'_i &:= \pi \{f(s_0s_1 \cdots s_k) \mid \exists k : s_k = i, k = 0, 1, \dots\} \end{aligned}$$

where $s_0 = \varepsilon$ (the empty word).

Definition 1.2 $X_i :=$ the closure of πY_i is called atomic surfaces ($X'_i :=$ the closure of $\pi Y'_i$) and $X := \bigcup_{i=1}^d X_i$ ($= \bigcup_{i=1}^d X'_i$) is called atomic surface of the substitution σ .

On Example 1.1 and Example 1.2, we have the figures (see Figure 2 and Figure 3 respectively).

Remark There is the theorem. Theorem ([E-I]): On $d = 2$, X_i is interval is the interval iff σ is invertible.

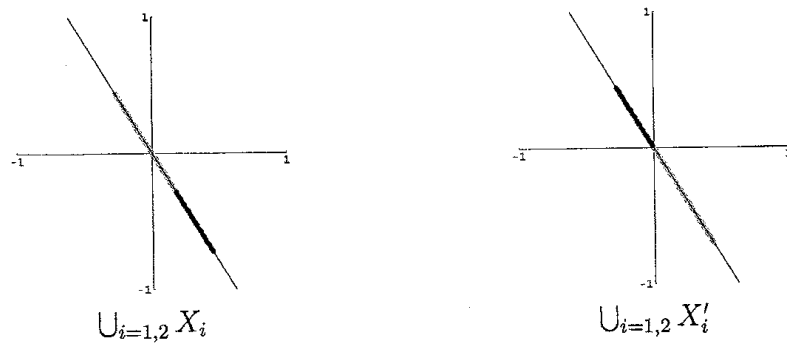


Figure 2: The figure of the atomic surface on $\sigma_F : 1 \mapsto 12, 2 \mapsto 1$.

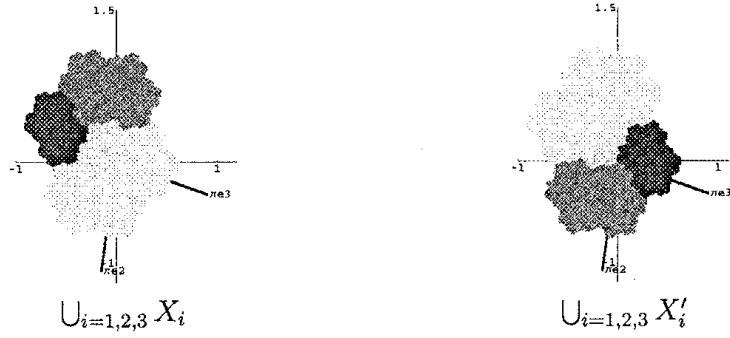


Figure 3: The figure of the atomic surface on $\sigma_R : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$.

Theorem 1 ([A-I], [I-R]) Atomic surfaces satisfy

- (1) X, X_i, X'_i are compact sets;
- (2) $\overline{\text{int}.X} = X$;
- (3) $L_\sigma^{-1}X_i = \bigcup_{j=1}^d \bigcup_{k: W_k^{(j)}=i} \left(X_j + L_\sigma^{-1} \left(f \left(P_k^{(j)} \right) \right) \right)$ (non-overlapping);
- (4) ${}^t(|X_1|, \dots, |X_d|)$ is the eigenvector of L_σ with respect to $\lambda = \lambda_1 > 1$ where $|B|$ is the volume of the set B .

Sketch of proof. On the notation,

$$\begin{aligned} (\mathbf{x}, i) &:= \{ \mathbf{x} + \lambda \mathbf{e}_i \mid 0 \leq \lambda \leq 1 \}, \\ \sigma(\mathbf{0}, i) &:= \sum_{k=1}^{l_i} \left(f \left(P_k^{(i)} \right), W_k^{(i)} \right), \quad 1 \leq i \leq d, \\ \overline{w}_\sigma &:= \lim_{n \rightarrow \infty} \sigma^n(\mathbf{0}, 1) \quad (\text{the broken line starting from } \mathbf{0}), \end{aligned}$$

we define

$$\begin{aligned} (Y_i, i) &:= \{ (\mathbf{y}, i) \mid (\mathbf{y}, i) \in \overline{w} \}, \quad \overline{w} = \bigcup_{i=1}^d (Y_i, i) \\ &= \left\{ (\mathbf{y}, i) \mid (\mathbf{y}, i) \in \sigma \left(\bigcup_{j=1}^d (Y_j, j) \right) \right\} \\ &= \bigcup_{j=1}^d \{ (\mathbf{y}, i) \mid (\mathbf{y}, i) \in \sigma(Y_j, j) \} \\ &= \bigcup_{j=1}^d \bigcup_{k: W_k^{(j)}=i} \{ (L_\sigma \mathbf{y} + f(P_k^{(j)}), i) \mid \mathbf{y} \in Y_j \}. \end{aligned}$$

Taking only the starting points of line segments, we get Theorem 1 (3).

Question Are X_i , $i = 1, 2, \dots, d$ non-overlapping?

Definition 1.3 Substitution σ satisfies the coincidence condition if $\exists n, k$:

$$(1) f(P_k^{(n,1)}) = f(P_k^{(n,2)}) = \dots = f(P_k^{(n,d)})$$

$$(2) w_k^{(n,1)} = w_k^{(n,2)} = \dots = w_k^{(n,d)}$$

$$\sigma^n(1) = w_1^{(n,1)} \dots w_k^{(n,1)} \dots w_{l(n,1)}^{(n,1)}$$

where $\vdots = \vdots$

$$\sigma^n(d) = w_1^{(n,d)} \dots w_k^{(n,d)} \dots w_{l(n,d)}^{(n,d)}.$$

Proposition 1.1 If σ satisfies the coincidence condition, then X_i , $i = 1, 2, \dots, d$ are non-overlapping.

Conjecture Any Pisot unimodular substitutions satisfy the coincidence condition.

Remark On $d = 2$, the conjecture is proved by Barge and Diamond ([B-D]).

If X_i , $i = 1, 2, \dots, d$ are non-overlapping, then we have two dynamical systems on X :

$$(1) T : X \rightarrow X,$$

$$Tx = L_\sigma^{-1}x - L_\sigma^{-1}\pi f(P_k^{(j)}) \quad \text{if } x \in X_i, \exists j, k : L_\sigma^{-1}x \in X_j + L_\sigma^{-1}\pi f(P_k^{(j)}).$$

Therefore, T is Markov endomorphism with the structure matrix ${}^tL_\sigma$.

$$(2) W : X \rightarrow X$$

$$\begin{matrix} \Psi & \Psi \\ \mathbf{x} & \mapsto \mathbf{x} - \pi \mathbf{e}_i \end{matrix} \quad \text{if } \mathbf{x} \in X_i \quad \text{is well-defined}$$

and W is called the domain exchange transformation (later, we will see $W \simeq$ the rotation on $\mathbf{T}^{d-1}$).

From Markov endomorphism (1), we have the following numerical expression.

Corollary 1.1 Using Markov endomorphism, X_i is presented by

$$X_i = \left\{ \pi_c f(P_{k_0}^{(j_0)}) + \pi_c L_\sigma f(P_{k_1}^{(j_1)}) + \pi_c L_\sigma^2 f(P_{k_2}^{(j_2)}) + \dots + \pi_c L_\sigma^n f(P_{k_n}^{(j_n)}) + \dots \mid (*) \right\}$$

where $(*)$ is defined by $\begin{pmatrix} j_0 & j_1 & \dots & j_n & \dots \\ k_0 & k_1 & \dots & k_n & \dots \end{pmatrix}$ is given by $w_{k_n}^{(j_n)} = j_{n-1}$ and $w_{k_0}^{(j_0)} = i$.

(see Figure 4).

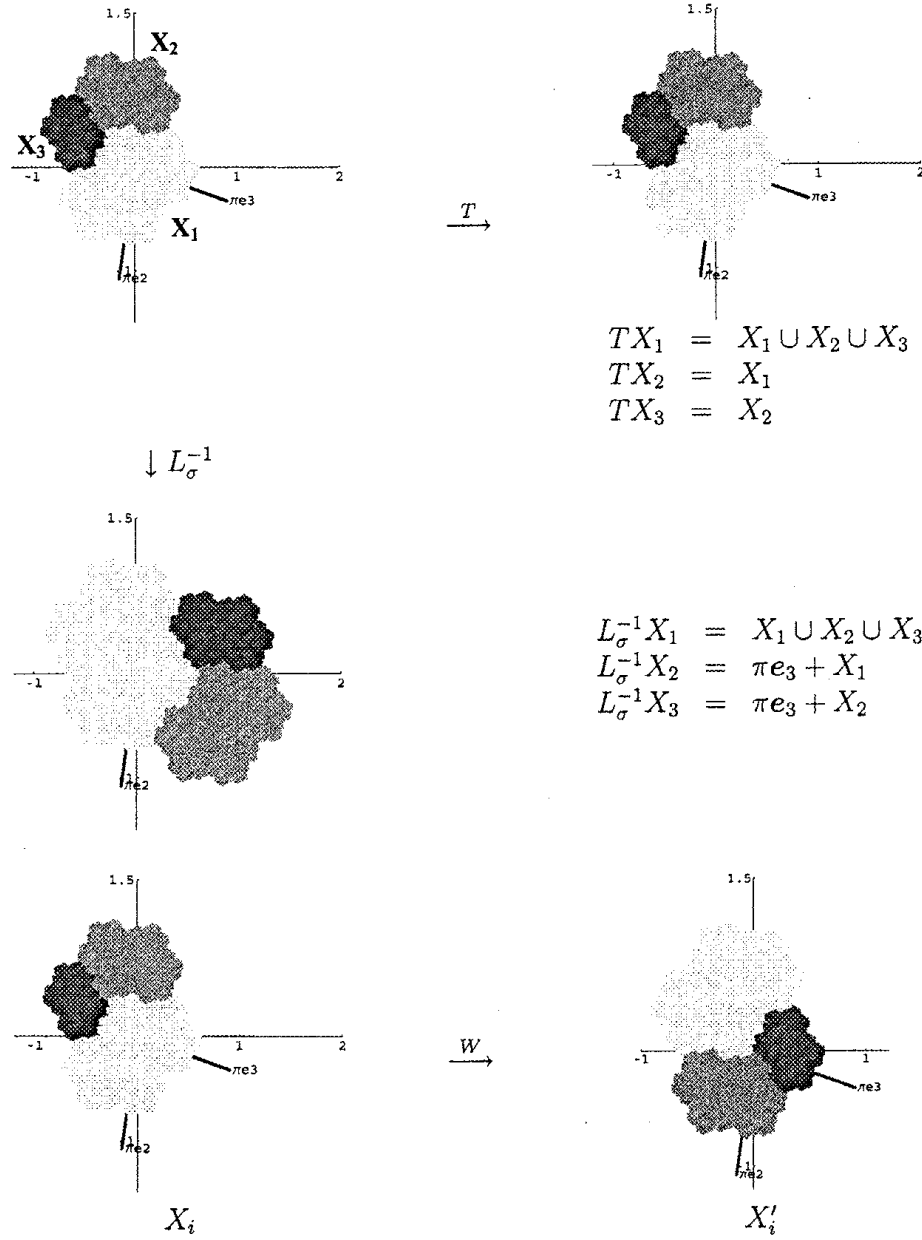


Figure 4: Figures of Markov endomorphism T and the domain exchange transformation W on Example 1.2.

2 Stepped Surfaces and Tiling Substitutions

For

$$(\mathbf{x}, i^*) \in \mathbb{Z}^d \times \{1^*, \dots, d^*\},$$

we give a geometrical meaning such that

$$(\mathbf{x}, i^*) := \left\{ \sum_{\substack{j=1, \dots, d, \\ j \neq i}} \mu_j \mathbf{e}_j \mid 0 \leq \mu_j \leq 1 \right\}$$

(see Figure 5).

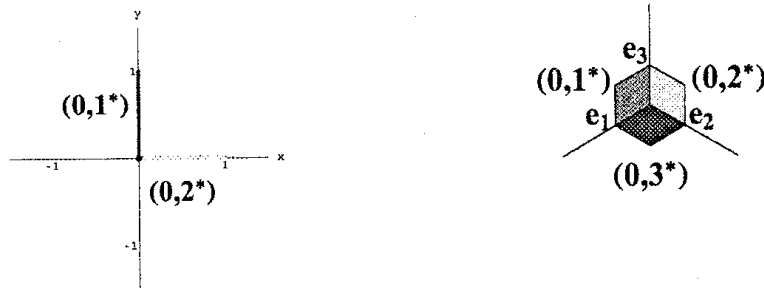


Figure 5: The figures of $(0, i^*)$.

Definition 2.1 $S := \{(\mathbf{x}, i^*) \mid (\mathbf{x}, \mathbf{u}) \geq 0, (\mathbf{x} - \mathbf{e}_i, \mathbf{u}) < 0\}$ is called the stepped surface of the contract plain P where the row vector $\mathbf{u} \geq 0$ is the eigenvector such that $\mathbf{u}L_\sigma = \lambda\mathbf{u}$ and the contract plain P is given by $P = \{\mathbf{x} \mid (\mathbf{x}, \mathbf{u}) = 0\}$.

Definition 2.2 $\pi S := \{\pi(\mathbf{x}, i^*) \mid (\mathbf{x}, i^*) \in S\}$ is called a tiling of P from the stepped surface S (see Figure 6).

3 Dual Substitution σ^* (Tiling Substitution)

$S^* := \{ \text{the finite sum of elements of } S \} \simeq \{ \text{the patches of tiles of the tiling } \pi S \}$.

Let us define the dual (tiling) substitution σ^* by

$$\sigma^* \pi(\mathbf{x}, i^*) = L_\sigma^{-1} \pi \mathbf{x} + \sum_{j=1}^d \sum_{\binom{j}{k}: W_k^{(j)} = i} \pi_c \left(L_\sigma^{-1} f \left(P_k^{(j)} \right), j^* \right)$$

(see Figure 7 and Figure 8).

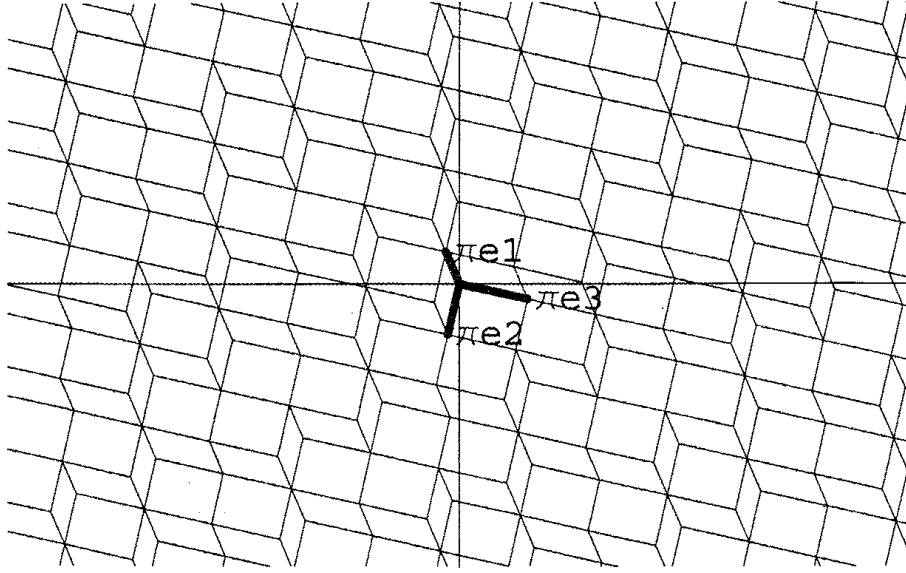


Figure 6: The figure of the tiling πS of P from the stepped surface S on σ_R in Example 1.2.

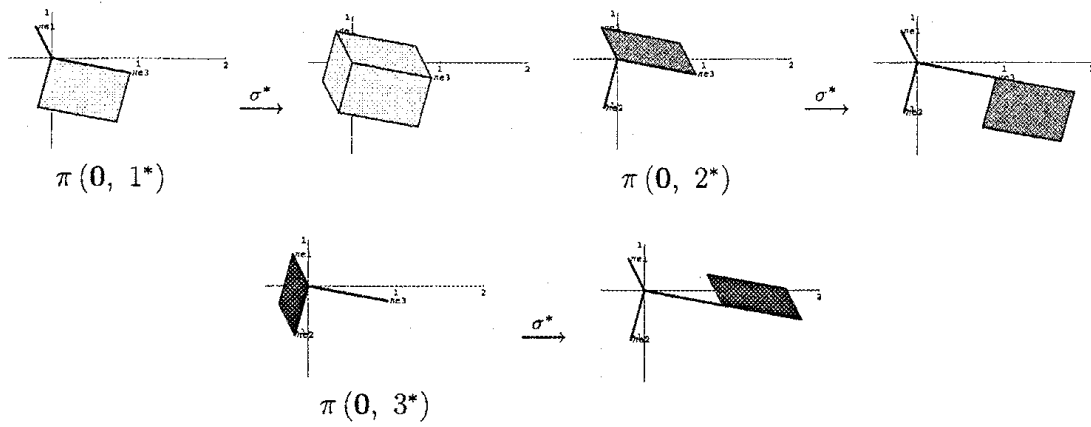


Figure 7: The figure of $\sigma_R^* \pi(0, i^*)$.

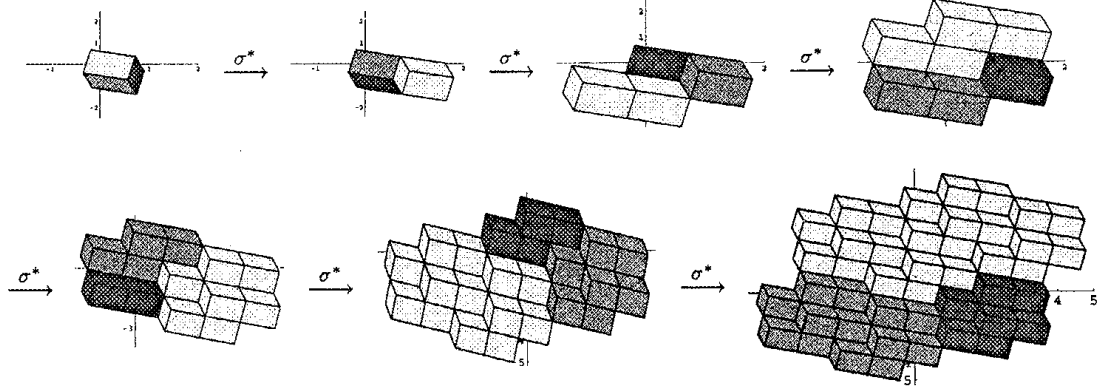


Figure 8: The figure of $\sigma_R^{*n} \bigcup_{i=1,2,3} \pi(e_i, i^*)$ on σ_R .

Theorem 2 ([A-I]) σ : An Pisot Unit substitution, then

- (1) $\sigma^* : S^* \rightarrow S^*$ is well-defined;
- (2) $\forall (x, i^*) \in S, \exists (y, j^*) \in S: (x, i^*) \in \sigma^*(y, j^*)$;
- (3) $(x, i^*) \neq (y, j^*) \in S \Rightarrow \sigma^*(x, i^*) \cap \sigma^*(y, j^*) = \emptyset$.

Theorem 3 ([I-R]) Let $\mathcal{U} = \{\pi(e_i, i^*) \mid i = 1, 2, \dots, d\}$, then,

- (1) $\sigma^*\mathcal{U} \succ \mathcal{U}$;
- (2) if $d(\partial(\sigma^{*n}\mathcal{U}), 0) \rightarrow \infty$ ($n \rightarrow \infty$), then

$$\tau' := \{\pi(x, j^*) \mid \pi(x, j^*) \in \sigma^{*n}\pi(e_i, i^*) \text{ for some } n \text{ and } j^*\}$$

coincides with πS and a quasi-periodic tiling;

- (3) $-X_i = \lim_{n \rightarrow \infty} L_\sigma^n \sigma^{*n} \pi(e_i, i^*)$;
- (4) $\tau := \{\pi x - X_j \mid \pi(x, j^*) \in \tau'\}$ is also a quasi-periodic tiling of P .

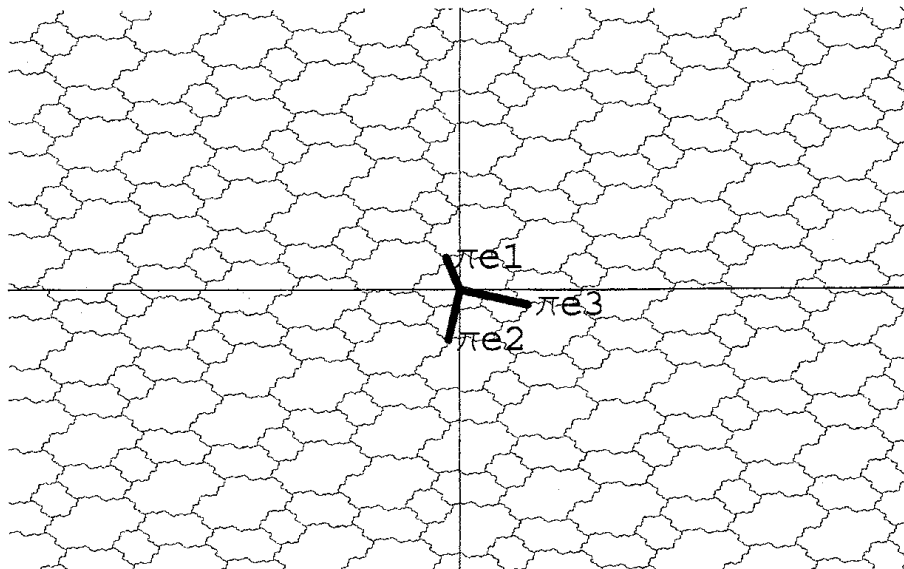


Figure 9: The figure of the quasi-periodic tiling τ .

References

- [A-I] P.ARNOUX and S.ITO, *Pisot substitutions and Rauzy fractals*, Bull. Belg. Math. Soc., **8**(2001), no.2, 181–207.
- [B-D] M.BARGE and B.DIAMOND, *Coincidence for substitutions of Pisot type*, Bull. Soc. Math. France **130** (2002), no.4, 619–626.
- [E] H.EI, *Some properties of invertible substitutions of rank d , and higher dimensional substitutions*, Osaka J. Math. **40** (2003), 543–562.
- [E-I] H.EI and S.ITO *Decomposition theorem on invertible substitutions*, Osaka J. Math. **35** (1998) no.4, 821–834.
- [E-I] H.EI and S.ITO *Tiling from some non-irreducible Pisot substitutions*, to appear in Discrete Math. & Th. Computer Science.
- [I-R] S.ITO and H.RAO, *Atomic surfaces, tilings and coincidence I. Irreducible case*, to appear in Israel J..
- [S-A-I] Y.SANO, P.ARNOUX and S.ITO, *Higher dimensional extensions of substitutions and their dual maps*, J. Anal. Math. **83** (2001), 183–206