

Asymptotic behaviors of singular homogeneous solutions of some partial differential operators in the complex domain

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§1 Introduction

Let $L(z, \partial_z)$ be a linear partial differential operator with holomorphic coefficients in a neighborhood of $z = 0$ in $\mathbb{C}^{d+1}$ and K be a nonsingular complex hypersurface through the origin. The coordinate of $\mathbb{C}^{d+1}$ is denoted by $(z_0, z_1, \dots, z_d)$ and chosen such that $K = \{z_0 = 0\}$. Let $u(z)$ be a solution of $L(z, \partial_z)u(z) = 0$, which is not necessarily holomorphic on K . The existence of singular solutions is studied by many mathematician (for example see [2], [3], [5] and [9]). The purpose of the present paper is to introduce a class of partial differential operators and to study the asymptotic behaviors as $z_0 \rightarrow 0$ of singular solutions of $L(z, \partial_z)u(z) = 0$ for $L(z, \partial_z)$ belonging to this class. In general there are many singular homogeneous solutions, hence we restrict solutions by adding a condition of the growth order of its singularities to them. So we treat solutions with at most some exponential order singularities on K which is given the constant γ defined by (2.2). It is the main result that we can give the asymptotic terms of solutions as $z_0 \rightarrow 0$ and the remainder term with Gevrey type estimate. The Gevrey exponent is also determined by γ . The operators considered here have useful examples, so the main result of Ōuchi [6] follows from that in this paper and those of Mandai [4] and Tahara [10] concerning the structure of homogeneous solutions of Fuchsian operators also do in some sense. So the results here are extensions of results of [4] and [10] to non-Fuchsian operators in some sense.

The details of this paper will be appeared in Ōuchi [8].

§2 Operators and Definitions

In this section let us introduce a class of operators studied in this paper and give some definitions. Let $L(z, \partial_z)$ be an m -th order linear partial differential

operator with holomorphic coefficients in a domain in $\mathbb{C}^{d+1}$ of the form:

$$(2.1) \quad \begin{cases} L(z, \partial_z) = A(z, \partial_{z_0}) + B(z, \partial_z), \\ A(z, \partial_{z_0}) = \sum_{i=0}^k a_i(z')(z_0 \partial_{z_0})^i, \\ B(z, \partial_z) = \sum_{|\alpha| \leq m} b_\alpha(z) \partial_z^\alpha, \quad z = (z_0, z_1, \dots, z_d) = (z_0, z'). \end{cases}$$

Let $j_\alpha \in \mathbb{N}$ such that $b_\alpha(z) = z_0^{j_\alpha} \tilde{b}_\alpha(z)$ with $\tilde{b}_\alpha(0, z') \not\equiv 0$ on $K = \{z_0 = 0\}$ provided $b_\alpha(z) \not\equiv 0$. Let us assume in this paper that $L(z, \partial)$ satisfies the following conditions (A) and (B),

$$\begin{aligned} (A) \quad & a_k(0) \neq 0, \\ (B) \quad & j_\alpha - \alpha_0 > 0 \quad \text{for all } \alpha. \end{aligned}$$

We define an important constant γ by

$$(2.2) \quad \gamma := \begin{cases} \min\left\{\frac{j_\alpha - \alpha_0}{|\alpha| - k}; |\alpha| > k\right\} & \text{if } k < m, \\ +\infty & \text{if } k = m. \end{cases}$$

and a polynomial $\chi(\lambda, z')$ by

$$(2.3) \quad \chi(\lambda, z') = \sum_{i=0}^k a_i(z') \lambda^i.$$

Let us give examples, which show that the class of operators considered in this paper contains useful examples.

(1). Let

$$(2.4) \quad P(z, \partial_z) = \partial_{z_0}^k + \sum_{\substack{|\alpha| \leq m \\ \alpha_0 < k}} a_\alpha(z) \partial_z^\alpha \quad (m > k).$$

$P(z, \partial_z)$ is a linear partial differential operator with order m and is of the normal form with respect to ∂_{z_0} . By multiplying $P(z, \partial_z)$ by z_0^k , consider $z_0^k P(z, \partial_z)$. Then $z_0^k P(z, \partial_z)$ satisfies (A) and (B), by setting $A(z_0, \partial_{z_0}) = z_0^k \partial_{z_0}^k$ and $B(z, \partial_z) = \sum_{|\alpha| \leq m, \alpha_0 < k} z_0^k a_\alpha(z) \partial_z^\alpha$.

(2). Let $P(z, \partial_z)$ be an m -th operator of Fuchsian type weight $(m - h)$ in the sense of Baouendi-Goulaouic [1]. Then $z_0^{m-h} P(z, \partial_z)$ belongs to the class we

consider and $\gamma = +\infty$.

(3). We give a concrete example. Let $z = (z_0, z_1) \in \mathbb{C}^2$ and

$$(2.5) \quad L(z, \partial_z) = z_0 \partial_{z_0} - a(z) + z_0^j c(z) \partial_{z_1}^m,$$

where $j \geq 1$ and $c(0, z_1) \not\equiv 0$. Then $\chi(\lambda, z_1) = \lambda - a(0, z_1)$ and $\gamma = j/(m - 1)$ ($m > 1$), $\gamma = +\infty$ ($m = 1$).

Let us introduce function spaces on the sectorial region $U(\theta)$ for our aim.

Definition 2.1. $\mathcal{O}_{(\kappa)}(U(\theta))$ is the set of all $u(z) \in \mathcal{O}(U(\theta))$ such that for any $\varepsilon > 0$ and any θ' with $0 < \theta' < \theta$

$$(2.6) \quad |u(z)| \leq M \exp(\varepsilon |z_0|^{-\kappa}) \quad \text{for } z \in U(\theta')$$

holds for some constant $M = M(\varepsilon, \theta')$. We put $\mathcal{O}_{(+\infty)}(U(\theta)) = \mathcal{O}(U(\theta))$.

Definition 2.2. $\mathcal{O}_{temp,c}(U(\theta))$ is the set of all $u(z) \in \mathcal{O}(U(\theta))$ such that for any θ' with $0 < \theta' < \theta$

$$(2.7) \quad |u(z)| \leq M |z_0|^c \quad \text{for } z \in U(\theta')$$

holds for some constant $M = M(\theta')$.

Set $\mathcal{O}_{temp}(U(\theta)) = \cup_{c \in \mathbb{R}} \mathcal{O}_{temp,c}(U(\theta))$, which is the set of all holomorphic functions on $U(\theta)$ having singularities on $z_0 = 0$ with fractional order. We also say that $u(z) \in \mathcal{O}(U(\theta))$ is tempered singular on $(U(\theta))$, provided $u(z) \in \mathcal{O}_{temp}(U(\theta))$.

§3 Behaviors of singular solutions

Now let us return to the equation $L(z, \partial_z)u(z) = 0$, $u(z) \in \mathcal{O}(U(\theta))$. In order to study the behaviors of solutions more concretely we restrict the growth properties of singularities, that is, we assume $u(z) \in \mathcal{O}_{(\gamma)}(U(\theta))$ in this paper, where γ is defined by (2.2). Firstly we show that it follows from this assumption that the singularities of solutions are less irregular.

As for the zeros of $\chi(\lambda, z')$ it follows from the condition (A), that there are constants $r' > 0$, a_0 , a_1 and b such that $\chi(\lambda, z') = 0$ has k roots for $z' \in V' = \{|z'| \leq r'\}$ and

$$(3.1) \quad \{\lambda; \chi(\lambda, z') = 0\} \subset \{\lambda; a_0 \leq \Re \lambda \leq a_1, |\Im \lambda| \leq b\}.$$

holds. Then we have, by using the constant a_0 ,

Theorem 3.1. ([7]). *Let $u(z) \in \mathcal{O}_{(\gamma)}(U(\theta))$ be a solution of $L(z, \partial_z)u(z) = f(z) \in \mathcal{O}_{temp,c}(U(\theta))$. Then there is a polydisk V centered at $z = 0$ such that $u(z) \in \mathcal{O}_{temp,c'}(V(\theta))$ for any $c' < \min\{c, a_0\}$.*

We show Theorem 3.1 by constructing a parametrix and refer the details of the proof to Ōuchi [7]. It follows from Theorem 3.1 that singularities of homogeneous solutions of $L(z, \partial_z)$ are of fractional order, provided they are in $\mathcal{O}_{\{\gamma\}}(U(\theta))$. So we assume $u(z) \in \mathcal{O}_{temp,c}U(\theta)$ in the following of this paper.

In order to analyze singularities, we make use of the Mellin transform with respect to z_0

$$(3.2) \quad \hat{u}(\lambda, z') = \int_0^T t^{\lambda-1} u(t, z') dt,$$

where T is a small positive constant. The transform (3.2) is Mellin transform on $\arg z_0 = 0$, however, the Mellin transform on $\arg z_0 = \theta$ is also available to get the main result.

By the assumption $\hat{u}(\lambda, z')$ is holomorphic in $\{\lambda; \Re \lambda > -c\}$. It is the first aim to show it is meromorphically extensible to a larger region. Put $\Phi(\lambda, z') := \chi(-\lambda, z')$. We have

Theorem 3.2. *$\hat{u}(\lambda, z')$ ($z' \in V'$) is meromorphically extensible in λ to the whole λ -plain. Its poles are contained in $\cup_{n=0}^{\infty} \{\lambda; \Phi(z', \lambda + n) = 0\}$.*

Outline of the proof. $u(z)$ satisfies $A(z, \partial_{z_0})u(z) + B(z, \partial_z)u(z) = 0$, from which we have a partial differential difference equation $\hat{u}(\lambda, z')$ satisfies, that is, for any $N \in \mathbb{N}$

$$(3.3) \quad \begin{aligned} \Phi(\lambda, z') \hat{u}(\lambda, z') + \sum_{h=1}^N \mathcal{L}_h(\lambda, z', \partial') \hat{u}(\lambda + h, z') \\ + \hat{u}_N(\lambda, z') + T^\lambda H_N(\lambda, z') = 0, \end{aligned}$$

where $\mathcal{L}_h(\lambda, z', \partial')$ is a partial differential operator, whose coefficients are polynomial in λ . $H_N(\lambda, z')$ is a polynomial of λ and $\hat{u}_N(\lambda, z')$ is holomorphic in $\{\lambda; \Re \lambda > -N - c\}$. Equation (3.3) is obtained by the Mellin transform of the equation, integrations by parts and Taylor expansion of the coefficients. The order of Taylor expansion of the coefficients depends on N . We have easily the meromorphic extension by the relation (3.3).

Let us calculate the inverse Mellin transform and reconstruct $u(z)$. So

the second aim is to obtain estimates of $\hat{u}(\lambda, z')$ outside of poles. Set

$$(3.4) \quad \begin{aligned} Z(r) &= \bigcup_{|z'| \leq r} \bigcup_{n=0}^{\infty} \{\Phi(\lambda + n, z') = 0\} \\ Z(r, \delta) &= \{\lambda; d(\lambda, Z(r)) \leq \varepsilon_0\}, \end{aligned}$$

where $d(\lambda, A)$ means the distance of λ and set A . We choose $r > 0$ and $\delta > 0$ so small, if necessary. For $N \in \mathbb{N}$ set

$$(3.5) \quad \Lambda(N) = \{\lambda \notin Z(r', \varepsilon_0); -N + 1/2 - c \leq \Re \lambda \leq -N + 3/2 - c\}.$$

We have an estimate of $\hat{u}(\lambda, z')$ in $\Lambda(N)$

Proposition 3.3. *There are constants A, B and a polydisk V' such that for $z' \in V'$ and $\lambda \in \Lambda(N)$*

$$(3.6) \quad |\hat{u}(\lambda, z')| \leq AB^N T^{\Re \lambda} \frac{\prod_{s=1}^N (|\lambda + N| + s)^m}{N!^m} \Gamma\left(\frac{N}{\gamma} + 1\right).$$

Let $\{\sigma_N\}_{N \in \mathbb{N}}$ be a sequence of real numbers such that the vertical line $\Re \lambda = -\sigma_N$ lies in $\Lambda(N)$. Define

$$(3.7) \quad u_N(z) = \frac{1}{2\pi i} \int_{C_N} z_0^{-\lambda} \hat{u}(\lambda, z') d\lambda,$$

where C_N is a contour which encloses all the poles of $\hat{u}(\lambda, z')$ in $\Re \lambda > -\sigma_N$. $u_N(z)$ gives asymptotic behavior of $u(z)$. We have

Theorem 3.4. *Let $u(z) \in \mathcal{O}_{temp}(U(\theta))$ be a solution of $L(z, \partial_z)u(z) = 0$ and $u_N(z)$ be the function defined by (3.7). Then there is a polydisk V centered at $z = 0$ such that for any θ' with $0 < \theta' < \theta$ and any $N \in \mathbb{N}$*

$$(3.8) \quad |u(z) - u_N(z)| \leq AB^N |z_0|^{\sigma_N} \Gamma\left(\frac{N}{\gamma} + 1\right) \quad \text{in } V(\theta')$$

holds for some constants A and B depending on θ' .

To show the remainder estimate (3.8) consider

$$(3.9) \quad u_N^R(t, z') = \frac{1}{2\pi i} \int_{\Re \lambda = -\sigma_N} t^{-\lambda} \hat{u}(\lambda, z') d\lambda \quad \text{for } t > 0.$$

Then formally $u_N^R(t, z') = u(t, z') - u_N(t, z')$ holds for $t > 0$. However the convergence of the integral of (3.9) is vague, because the estimate of $\hat{u}(\lambda, z')$

in $\Lambda(N)$ obtained in Proposition 3.3 is of polynomial growth in $\Im\lambda$. So we do not calculate directly it. However by the assumption that $u(z)$ is holomorphic on the sectorial region $U(\theta)$ we can modify (3.9), estimate the difference $u(t, z') - u_N(t, z')$ by another method and get Theorem 3.4.

If $L(z, \partial_z)$ is an operator of Fuchsian type (see example 2), then $\gamma = \infty$, so it follows from (3.8) that $u(z) = \lim_{N \rightarrow \infty} u_N(z)$ in $V(\theta')$ for small z , which is a generalization of the result of Mandai and Tahara concerning the structure of homogeneous solutions of operator of Fuchsian type.

Corollary 3.5. *Let $u(z) \in \mathcal{O}_{temp}(U(\theta))$ be a solution of $L(z, \partial_z)u(z) = 0$ satisfying $|u(z)| \leq A|z_0|^a$ in $U(\theta)$ for some $a > a_1$, a_1 being the constant in (3.1). Then there is a polydisk V centered at $z = 0$ such that for any θ' with $0 < \theta' < \theta$*

$$(3.10) \quad |u(z)| \leq C \exp(-c|z_0|^{-\gamma}) \quad \text{in } V(\theta')$$

holds for some positive constants C and c .

We have Corollary 3.5 by showing that $\hat{u}(\lambda, z')$ has no poles.

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