

On an exterior problem for fully non-linear wave equation
(非線形波動方程式の外部問題について)

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§1. Introduction. Klainerman [1], [2] proved that one and only one C^∞ solution exists for Cauchy problem of a fully nonlinear wave equation of the form: $\square u + F(t, x, u', u'') = 0$, with sufficiently small and smooth initial data under the following assumptions:

- (1) $F = F_{u'} = F_{u''} = 0$ for $u' = u'' = 0$ if $n \geq 6$,
- (2) $F = F_{u'} = F_{u''} = F_{u'u'} = F_{u'u''} = F_{u''u''} = 0$ for $u' = u'' = 0$ if $3 \leq n \leq 5$.

Here n is the space dimension and $\square$ denotes the d'Alembertian, u' represents the vector of first derivatives, u'' that of second derivatives with respect to the $x = (x_1, \dots, x_n)$ and t . See also Klainerman-Ponce [3] and Shatah [4]. The case $n=3$ is, of course, of special importance for applications. Under the assumption (1) Klainerman's theorem cannot be extended to $n=3$, which was showed by John [5].

According to John's result, for example we have that every non-trivial C^2 -solution of the equation: $\square u = 2u_t u_{tt}$ for which $u(0, x), u_t(0, x)$ are of compact support and $\int_{\mathbb{R}^3} [u_t(0, x) - u_t^2(0, x)] dx \geq 0$ blows up in finite time. Therefore, when $n=3$, the assumption (2) is needed to get a global existence theorem. For example, a classical non-linear wave operator: $u_{tt} - \Delta u (1 + \sum_{j=1}^n (u_{x_j})^2)^{-1/2}$, satisfies the assumption (2).

In this note, our purpose is to extend Klainerman's theorems to an exterior problem. Our results stated here will be published elsewhere (see Shibata and Tsutsumi [6] and [7]). Let Ω be an unbounded

in R^n , its boundary $\partial\Omega$ being C^∞ and compact. We denote a time variable by t or x_0 and a space variable by $x = (x_1, \dots, x_n)$, respectively.

We abbreviate $\partial/\partial t$, $\partial/\partial x_j$ and $(\partial/\partial x_1)^{\alpha_1} \dots (\partial/\partial x_n)^{\alpha_n}$ to ∂_t or ∂_0 , ∂_j and ∂_x^α , respectively, where α is a multi-index with $|\alpha| = \alpha_1 + \dots + \alpha_n$, and $j = 1, \dots, n$. We consider the following problem:

$$\square u + F(t, x, u', u'') = f(t, x) \text{ in } [0, T) \times \Omega,$$

$$(M.P) \quad u = 0 \quad \text{on } [0, T) \times \partial\Omega,$$

$$u(0, x) = u_0(x), \quad (\partial_t u)(0, x) = u_1(x) \text{ in } \Omega.$$

Roughly speaking, under the assumptions (1) and (2) we can establish unique existence theorem of time global C^∞ solutions for (M.P) ($T = \infty$) if data u_0 , u_1 and f are sufficiently small and smooth. In §2, we consider the local existence theorem of C^∞ solutions for (M.P). In §3, we consider the global existence theorem of C^∞ solutions for (M.P) under the additional assumption: Ω is non-trapping.

To conclude we list notations. For p with $1 \leq p \leq \infty$ we denote the standard L^p space defined on Ω and its norm by $L^p(\Omega)$ and $\|\cdot\|_p$, respectively. For a vector valued function $h = (h_1, \dots, h_s)$, we put $\|h\|_p = \sum_{j=1}^s \|h_j\|_p$. For a positive integer N we put $\|f\|_{p,N} = \sum_{|\alpha| \leq N} \|\partial_x^\alpha f\|_p$ and $\|h\|_{p,N} = \sum_{j=1}^s \sum_{|\alpha| \leq N} \|\partial_x^\alpha h_j\|_p$. We set $H_p^N(\Omega) = \{f \in L^p(\Omega); \|f\|_{p,N} < \infty\}$. By $\overset{\circ}{H}_p^N(\Omega)$ we denote the completion of $C_0^\infty(\Omega)$ in $H_p^N(\Omega)$. $H_p^\infty(\Omega)$ denotes $\bigcap_{j=0}^\infty H_p^j(\Omega)$. For $-\infty \leq a < b \leq \infty$, a non-negative integer k and a Banach space E , $C^k([a, b]; E)$ denotes the set of all E -valued functions having all derivatives of order $\leq k$ continuous in $[a, b]$.

For $u \in \bigcap_{j=0}^L C^j([a, b]; H_p^{L-j}(\Omega))$, $v \in \bigcap_{j=0}^L C^j([0, \infty); H_p^{L-j}(\Omega))$, we put

$$|u|_{L, [a, b], p} = \sup_{a \leq t \leq b} \sum_{j+|\alpha| \leq L} \|\partial_t^j \partial_x^\alpha u(t, \cdot)\|_p, \text{ and } |u|_{L, k, p} =$$

$\sup_{t \geq 0} (1+t)^k \sum_{j+|\alpha| \leq L} \|\partial_t^j \partial_x^\alpha u(t, \cdot)\|_p$, respectively.

§2. Local existence theorem. In this section, for finite $T > 0$ we consider the problem (M.P) under the following assumption.

Assumption 2.1. (1) $F(t, x, u', u'')$ is a real-valued function defined on $[0, T] \times \bar{\Omega} \times \{(u', u'') \in R^{(n+1)(n+2)}; |u'| + |u''| \leq 3\lambda_0\}$ such that $F = F_{u'} = F_{u''} = 0$ for $u' = u'' = 0$.

(2) Let us define functions F_2^j, F_2^{ij}, F_1^j and F_0 by the formula:

$$(dF/d\theta)(t, x, u' + \theta u'')|_{\theta=0} = \sum_{j=0}^n F_2^j(t, x, u', u'') \partial_t \partial_j v - \sum_{ij=1}^n F_2^{ij}(t, x, u', u'') \partial_i \partial_j v + \sum_{j=0}^n F_1^j(t, x, u', u'') \partial_j v + F_0(t, x, u', u'') v.$$

Then there exists a positive number d such that

$$\sum_{ij=1}^n [\delta_{ij} + F_2^{ij}(t, x, u', u'')] \xi_i \xi_j \geq d |\xi|^2, \quad 1 + F_2^0(t, x, u', u'') \geq d$$

for all $(t, x) \in [0, T] \times \bar{\Omega}$, $|u'| + |u''| \leq 3\lambda_0$ and $\xi \in R^n$. Here $\delta_{ij} = 1$ if $i = j$ and $= 0$ if $i \neq j$. //

Before stating main results in this section, we define a certain class of data and compatibility condition.

Definition 2.2. We shall say that a pair of functions $(u_0(x), u_1(x), f(t, x))$ with $u_0 \in B^2(\bar{\Omega})$, $u_1 \in B^1(\bar{\Omega})$ and $f(0, x) \in B^0(\bar{\Omega})$ belongs to $\mathcal{D}$ if there exists a $u_2(x) \in B^0(\bar{\Omega})$ such that $\|u_0\|_{\infty, 2} + \|u_1\|_{\infty, 1} + \|u_2\|_{\infty} \leq \lambda_0$ and $u_2(x) - \Delta u_0(x) + F(0, x, D_x^2 u_0(x), D_x^1 u_1(x), u_2(x)) = f(0, x)$ in Ω , where $D_x^2 u_0 = (\partial_1 u_0, \dots, \partial_n u_0, \partial_i \partial_j u_0; i, j=1, \dots, n)$ and $D_x^1 u_1 = (\partial_1 u_1, \dots, \partial_n u_1)$. //

Of course, it follows from Assumption 2.1 that u_2 is unique if exists. If u_0, u_1 and f are sufficiently small, u_2 exists near 0 by implicit function theorem. To discuss the compatibility condition,

we have to introduce some notations as follows. For a smooth function

$v(t, x)$, we put $(\partial_t^p v)(0, x) = v_p(x)$. We define functions G_{p-2} as follows:

$$\begin{aligned} \partial_t^{p-2} F(t, x, v, v'') \Big|_{t=0} &= F_2^0(0, x, \bar{D}_x^2 v_0, \bar{D}_x^1 v_1, v_2) v_p + \\ &+ G_{p-2}(x, \bar{D}_x^1 v_{p-1}, \bar{D}_x^2 v_{p-2}, \dots, \bar{D}_x^p v_0) \end{aligned}$$

where $\bar{D}_x^j v = (\partial_x^\alpha v, |\alpha| \leq j)$. If $(u_0, u_1, f) \in \mathcal{D}$, it follows from assumption 2.1 that $1 + F_2^0(0, x, \bar{D}_x^2 u_0, \bar{D}_x^1 u_1, u_2) \geq d > 0$. Therefore, we can define

$u_p, p \geq 3$, successively as follows:

$$\begin{aligned} u_p(x) &= [1 + F_2^0(0, x, \bar{D}_x^2 u_0, \bar{D}_x^1 u_1, u_2)]^{-1} [\Delta u_{p-2} - G_{p-2}(x, \bar{D}_x^1 u_{p-1}, \dots, \bar{D}_x^p u_0) + \\ &+ (\partial_t^{p-2} f)(0, x)]. \end{aligned}$$

Now, we define the compatibility condition in terms of u_p .

Definition 2.3. We assume that the assumption 2.1 holds.

Let L be an integer ≥ 2 . We shall say that data $(u_0, u_1, f) \in \mathcal{D}$ satisfy the compatibility condition of order L if $u_j \in \dot{H}_2^1(\Omega)$, $0 \leq j \leq L$. //

We state the local existence theorem.

Main Theorem I. We assume that Assumption 2.1 holds. Let Ω be a domain in R^n , its boundary $\partial\Omega$ being C^∞ and compact. If data u_0, u_1, f belong to class $\mathcal{D}$ and satisfy the compatibility condition of order infinity and $u_0, u_1 \in H_2^\infty(\Omega)$ and $f \in C^\infty([0, T]; H_2^\infty(\Omega))$, then there exists one and only one solution $u \in C^\infty([0, T']; H_2^\infty(\Omega))$ of the problem (M.P) for some positive $T' \leq T$. Here T' depends essentially only on n, Ω, F and the bound for $\|u_0\|_{2L+2}, \|u_1\|_{2L+1}, \|f\|_{2L, [0, T], 2}$ and $L = \max(2[\frac{n}{2}] + 4, [\frac{n}{2}] + 7)$. //

§3. Global existence theorem. We consider the global existence theorem of (M.P) with $T = \infty$ under the assumptions (1) and (2) and additional assumption: Ω is non-trapping. More precisely, we introduce the following assumption.

Assumption 3.1. (1) The spatial dimension $n \geq 3$.

(2) The nonlinear mapping F is a real-valued function belonging to $B^\infty([0, \infty) \times \overline{\Omega} \times \{(u', u'') \in R^{(n+1)(n+2)}; |u'| + |u''| \leq 1\})$.

(3) Put $\lambda = (u', u'')$.

$$F(t, x, \lambda) = \begin{cases} O(|\lambda|^2) & \text{near } \lambda = 0 \text{ if } n \geq 6, \\ O(|\lambda|^3) & \text{near } \lambda = 0 \text{ if } 3 \leq n \leq 5. \end{cases}$$

(4) The exterior domain Ω is "non-trapping" in the following sense:

Let $G(t, x, y)$ be the Green function for the following problem:

$$\square G = 0 \text{ in } [0, \infty), \quad \lim_{t \rightarrow 0+} (\partial/\partial t)^j G = \begin{cases} 0 & \text{if } j=0, \\ \delta(x-y) & \text{if } j=1, \end{cases} \quad G|_{x \in \partial\Omega} = 0.$$

where y is an arbitrary point in Ω and $\square$ is the d'Alembertian with

respect to the x and t . Let a and b be arbitrary positive constants

such that $b \geq a \geq r_0$ with $\partial\Omega \subset \{x \in R^n; |x| \leq r_0\}$. For any $v \in$

$L^2(\Omega)$ with $\text{supp } v \subset \{x \in R^n; |x| \leq a\}$ we put

$$(\mathbb{G}v)(t, x) = \int_{\Omega} G(t, x, y) v(y) dy.$$

Then there exists a $T_0 > 0$ such that $(\mathbb{G}v)(t, x) \in C^\infty([T_0, \infty) \times \overline{\Omega}_b)$ for

any $v \in L^2(\Omega)$ with $\text{supp } v \subset \{x \in R^n; |x| \leq a\}$, where T_0 depends

only on n , a , b and Ω , and Ω_b denotes the set $\Omega_b = \{x \in \Omega; |x| \leq b\}$.

Remark 3.2. (1) It is well known that if the complement of Ω

is convex, then Assumption 3.1 (4) is satisfied. (see e.g. Melrose

[8]). (2) It is well known that under the Assumption 3.1. (4) the

local energy of solutions of wave equations of the form: $\square u = 0$

decays exponentially if $n \geq 3$ and n is odd (see e.g. Lax-Phillips [9]). Melrose [8] proved that if $n \geq 4$ and n is even, the order of local energy of solutions of wave equations is $-n$. But, as byproduct of our proof in Shibata and Tsutsumi [6] we proved that its order is $-2n+2$. This is a sharper result than Melrose [8]. Furthermore, comparing with the order of local energy in the whole space R^n , our result is best.//

Now, we shall state our global existence theorem.

Main Theorem II. Let m be an arbitrary integer $m \geq 0$. Let Assumption 1.1 be all satisfied. (1) Put $m^0 = 2\max(4[\frac{n}{2}]+7, m+1)+4[\frac{n}{2}]+8$. If $n \geq 6$, then there exist positive constants a and δ_0 having the following properties: If $\phi_0 \in B^{2m^0+[\frac{n}{2}]+3}(\bar{\Omega})$, $\phi_1 \in B^{2m^0+[\frac{n}{2}]+2}(\bar{\Omega})$ and $f \in B^{2m^0+[\frac{n}{2}]+1}([0, \infty) \times \bar{\Omega})$ satisfy for some δ with $0 < \delta \leq \delta_0$

$$\|\phi_0\|_{4/3, 2m^0} + \|\phi_1\|_{4/3, 2m^0-1} + \|f\|_{4/3, (n-1)/4, 2m^0-2} \leq a\delta,$$

$$\|\phi_0\|_{4, 2m^0+2} + \|\phi_1\|_{4, 2m^0+1} + \|f\|_{4, 0, 2m^0} \leq a\delta,$$

$$\|\phi_0\|_{\infty, 2m^0+2} + \|\phi_1\|_{\infty, 2m^0+1} + \|f\|_{\infty, 0, 2m^0} \leq a\delta$$

and the compatibility condition of order m^0 , then Problem (M.P) has a solution $u \in C^{m+2}([0, \infty) \times \bar{\Omega})$ satisfying

$$|(u', u'')|_{2, 0, m} + |(u', u'')|_{4, (n-1)/4, m} \leq \delta.$$

(2) Put $m^0 = 2\max(3[\frac{n}{2}]+6, m-1) + 3[\frac{n}{2}]+7$. If $4 \leq n \leq 5$, then there exist positive constants a and δ_0 having the following properties: If $\phi_0 \in B^{2m^0+2}(\bar{\Omega})$, $\phi_1 \in B^{2m^0+1}(\bar{\Omega})$ and $f \in B^{2m^0}([0, \infty) \times \bar{\Omega})$ satisfy for some δ with $0 < \delta \leq \delta_0$

$$\|\phi_0\|_{1, 2m^0} + \|\phi_1\|_{1, 2m^0-1} + \|f\|_{1, (n-1)/2, 2m^0-2} \leq a\delta,$$

$$\|\phi_0\|_{2, 2m^0+2} + \|\phi_1\|_{2, 2m^0+1} + \|f\|_{2, (n-1)/2, 2m^0} \leq a\delta,$$

$$\|\phi_0\|_{\infty, 2m^0+2} + \|\phi_1\|_{\infty, 2m^0+1} + \|f\|_{\infty, 0, 2m^0} \leq a\delta$$

and the compatibility condition of order m^0 , then Problem (M.P) has a solution $u \in C^{m+2}([0, \infty) \times \bar{\Omega})$ satisfying

$$|(u', u'')|_{2, 0, m} + |(u', u'')|_{\infty, (n-1)/2, m} < \delta.$$

(3) Let σ be a positive constant with $0 < \sigma \leq 1/(7m+18)$, and m^0 an integer with $m^0 \geq \frac{7}{\sigma}[\frac{3}{2} + (3m+7)\sigma] + 3[\frac{n}{2}] + 6$. If $n = 3$, then there exist positive constants a and δ_0 having the following properties: If $\phi_0 \in B^{2m^0+2}(\bar{\Omega})$, $\phi_1 \in B^{2m^0+1}(\bar{\Omega})$ and $f \in B^{2m^0}([0, \infty) \times \bar{\Omega})$ satisfy for some δ with $0 < \delta \leq \delta_0$

$$\|\phi_0\|_{1, 2m^0} + \|\phi_1\|_{1, 2m^0-1} + \|f\|_{1, 1+\sigma, 2m^0-2} \leq a\delta,$$

$$\|\phi_0\|_{2, 2m^0+2} + \|\phi_1\|_{2, 2m^0+1} + \|f\|_{2, 1+\sigma, 2m^0} \leq a\delta,$$

$$\|\phi_0\|_{\infty, 2m^0+2} + \|\phi_1\|_{\infty, 2m^0+1} + \|f\|_{\infty, 0, 2m^0} \leq a\delta$$

and the compatibility condition of order m^0 , the Problem (M.P) has a solution $u \in C^{m+2}([0, \infty) \times \bar{\Omega})$ satisfying

$$|(u', u'')|_{2, 0, m} + |(u', u'')|_{\infty, \frac{1}{2}+\sigma, m} \leq \delta.$$

Furthermore, there exists a small constant $\delta_1 > 0$ such that if $u, v \in C^3([0, \infty) \times \bar{\Omega})$ are two solutions of Problem (M.P) for the same data with $|(u', u'')|_{\infty, 0, 0} \leq \delta_1$ and $|(v', v'')|_{\infty, 0, 0} \leq 1$, then $u = v$.

Combining Main Theorems I and II, we have

Main Theorem III. The same assumption and conditions as in Main Theorem II are satisfied by data ϕ_0, ϕ_1 and f , nonlinear term F and the domain Ω . Furthermore, assume that $\phi_0, \phi_1 \in H_2^\infty(\bar{\Omega})$ and $f \in C^\infty([0, \infty); H_2^\infty(\bar{\Omega}))$ and that ϕ_0, ϕ_1 and f satisfy the compatibility condition of infinite order. Then there exists one and only one solution of Problem (M.P) belonging to $C^\infty([0, \infty); H_2^\infty(\bar{\Omega}))$.

Concluding Remarks.

(1) The local existence theorem can be extended to more general fully nonlinear 2nd order hyperbolic operators (see Shibata and Tsutsumi [7]).

(2) We can also obtain the analogous results for the mixed problems of the nonlinear Klein-Gordon equation and the nonlinear Schrödinger equation.//

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