

Highlights

A kernel of resonance: perturbative finite-time responses in spin-lock magnetic resonance

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- Spin-lock detection is cast as a finite-time observation operator.
- Dyson expansion yields kernels for arbitrary weak target waveforms.
- Unmodulated spin-lock and SPARX share one perturbative framework.
- Bloch simulations verify accuracy and breakdown regimes.
- The model links pulse design to weak-field waveform reconstruction.

A kernel of resonance: perturbative finite-time responses in spin-lock magnetic resonance

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ABSTRACT

Spin-lock preparations have been proposed as a route toward magnetic resonance detection of weak low-frequency magnetic fields, including neuronal magnetic fields. Existing analytic treatments are typically restricted to sinusoidal target fields or to long-time effective descriptions, which limits their direct use for reconstructing arbitrary temporal waveforms. Here we formulate spin-lock detection as a finite-time perturbative observation problem. Starting from the Bloch equation in the rotating frame, the weak target field is separated from the reference spin-lock dynamics and treated using the first-order Dyson expansion. For an unmodulated spin-lock preparation, the resulting longitudinal signal perturbation is written as a linear functional of the target waveform, or equivalently as a convolution-like response with a finite-time kernel. The same construction is extended to Spin-lock Phase-modulated Active Resonance eXcitation (SPARX), where the modulation-induced pseudo-magnetic field is included in the reference dynamics and the weak target field enters only as a perturbation. Numerical Bloch simulations verify the reference propagator, the first-order endpoint response, and the kernel representation for sinusoidal, square, and band-limited target waveforms. Amplitude sweeps identify the perturbative regime and show where higher-order terms become visible. For SPARX, an adjoint numerical kernel is compared with a finite damped-trigonometric basis, and frequency-response calculations show how pseudo-field modulation shifts the sensitive bands relative to unmodulated spin-lock. This framework provides a forward model that bridges spin-lock pulse design and time-series reconstruction of weak magnetic-field waveforms.

1. Introduction

Direct detection of weak neuronal magnetic fields by magnetic resonance remains an important but difficult target for functional neuroimaging. Spin-lock preparations are attractive in this context because magnetization locked by a radio-frequency field can be sensitive to low-frequency field perturbations in the rotating frame. The relevant target fields, however, are not generally pure sinusoids. Physiological or stimulus-locked magnetic fields are time-dependent waveforms, and the experimental question is ultimately how a finite set of spin-lock observations can constrain the underlying temporal signal.

The motivation for this direction is the gap between the spatial encoding strength of MRI and the temporal mechanism of conventional functional MRI. Blood-oxygenation-level-dependent MRI has become a central neuroimaging tool, but its contrast reflects vascular and metabolic consequences of neural activity rather than electrophysiological currents themselves [1, 2]. Spin-lock-based methods were proposed as a nonhemodynamic alternative because an oscillatory magnetic field can interact resonantly with magnetization in the rotating frame. Stimulus-induced rotary saturation (SIRS) established the basic concept [3], and subsequent work demonstrated current or oscillatory-field detection in phantoms and biological settings [4, 5, 6]. More recent

developments include in vivo spin-lock imaging of neuro-electromagnetic oscillations [7], steady-state and ultra-low-field implementations [8], robustness-oriented spin-lock preparations [9, 10], validation strategies based on rotary excitation [11], and extensions to non-sinusoidal pulsed fields [12]. These studies define a technical map in which spin-lock MRI has moved from a resonance-based sensing concept to a family of pulse designs for field detection, robustness, and experimental validation.

Within this map, analytic studies have mainly clarified magnetization dynamics for controlled oscillatory targets. Closed-form and approximate solutions of the Bloch equation have been used to interpret SIRS-type dynamics, to account for relaxation, and to reduce reliance on repeated numerical integration [13, 14]. Low-field phantom studies further showed that the theoretical response can be connected to experimentally detectable tiny oscillatory fields [15]. In parallel, phase-modulated spin-lock designs such as SPARX introduce a controllable pseudo-magnetic field in the rotating frame to shift the effective resonance condition and improve sensitivity to ultra-weak oscillatory fields [16]. However, the current analytic endpoint is still largely organized around sinusoidal or otherwise prescribed target classes. This is a limitation for reconstruction-oriented applications, where the desired object is not only the response to a single frequency but a forward operator mapping an arbitrary weak waveform during the preparation to the measured magnetization after the preparation.

The purpose of this work is to derive and numerically verify such a finite-time response model. We separate the Bloch dynamics into a reference spin-lock system and a weak

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target-field perturbation. The first-order Dyson expansion then expresses the measured longitudinal magnetization perturbation as an integral of the target waveform against a kernel determined by the reference trajectory and propagator. This construction is first developed for an unmodulated spin-lock preparation, where the reference system has constant coefficients and the kernel has a transparent form. It is then extended to SPARX, where the modulation-generated pseudo-magnetic field is intentionally large and must be included in the reference system rather than treated as a perturbation.

The contribution of the paper is therefore not another single-waveform Bloch solution and not a replacement for numerical integration of the Bloch equation. Direct integration can always predict a specified experiment. Instead, the contribution is an interpretable linear forward model that connects pulse design, kernel shape, and endpoint contrast. The model can be evaluated numerically, compared across unmodulated and phase-modulated spin-lock preparations, and assembled into an observation matrix for time-series reconstruction. The numerical part of the paper tests the validity of this model against full Bloch simulations, quantifies where the first-order approximation breaks down, and identifies response metrics that are useful for subsequent inverse-problem studies.

2. Theory

This section formulates spin-lock detection as a finite-time perturbative observation problem. We first write the Bloch equation as an affine linear system and separate the dynamics into a reference system and a weak target-field perturbation. The reference trajectory and its homogeneous propagator determine the response kernel. We then derive the kernel explicitly for an unmodulated spin-lock preparation and finally extend the same construction to the phase-modulated SPARX sequence.

2.1. Bloch equation and reference system

Let

$$\mathbf{M}(t) = [M_x(t), M_y(t), M_z(t)]^\top$$

denote the magnetization vector in the rotating frame. The Bloch equation is written as

$$\frac{d}{dt}\mathbf{M}(t) = \mathbf{A}(t)\mathbf{M}(t) + \mathbf{c}, \quad \mathbf{c} = [0, 0, R_1 M_\infty]^\top,$$

where

$$\mathbf{A}(t) = \mathbf{\Gamma} - \omega_x(t)\mathbf{L}_x - \omega_y(t)\mathbf{L}_y - \omega_z(t)\mathbf{L}_z,$$

and

$$\mathbf{\Gamma} = \begin{bmatrix} -R_2 & 0 & 0 \\ 0 & -R_2 & 0 \\ 0 & 0 & -R_1 \end{bmatrix}.$$

The rotation generators are defined as

$$\mathbf{L}_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{L}_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},$$

$$\mathbf{L}_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

With this sign convention, these matrices generate active rotations about the coordinate axes:

$$\exp(\theta \mathbf{L}_i) = \mathbf{R}_i(\theta), \quad [\mathbf{L}_i, \mathbf{L}_j] = \epsilon_{ijk} \mathbf{L}_k, \quad i, j, k \in \{x, y, z\}.$$

Here, $\mathbf{R}_i(\theta)$ denotes the rotation matrix about the i -axis by angle θ , and ϵ_{ijk} denotes the Levi-Civita symbol.

We separate the system matrix into a reference part and a weak perturbation,

$$\mathbf{A}(t) = \mathbf{A}_0(t) + \epsilon \mathbf{A}_1(t), \quad (1)$$

where ϵ represents the scale of the target field. The reference magnetization $\mathbf{M}_0(t)$ satisfies

$$\frac{d}{dt}\mathbf{M}_0(t) = \mathbf{A}_0(t)\mathbf{M}_0(t) + \mathbf{c}.$$

The corresponding homogeneous propagator $\Phi_0(t, \tau)$ is defined by

$$\frac{\partial}{\partial t}\Phi_0(t, \tau) = \mathbf{A}_0(t)\Phi_0(t, \tau), \quad \Phi_0(\tau, \tau) = \mathbf{I}.$$

The reference solution is then

$$\mathbf{M}_0(t) = \Phi_0(t, 0)\mathbf{M}(0) + \int_0^t \Phi_0(t, \tau)\mathbf{c} \, d\tau.$$

If $\mathbf{A}_0$ is time invariant, this expression reduces to

$$\mathbf{M}_0(t) = \Phi_0(t, 0)[\mathbf{M}(0) + \mathbf{A}_0^{-1}\mathbf{c}] - \mathbf{A}_0^{-1}\mathbf{c},$$

$$\Phi_0(t, \tau) = \exp[\mathbf{A}_0(t - \tau)].$$

2.2. First-order Dyson response

We expand the magnetization as

$$\mathbf{M}(t) = \mathbf{M}_0(t) + \delta \mathbf{M}^{(1)}(t) + O(\epsilon^2).$$

Substitution into Eq. (1) and retention of first-order terms gives

$$\frac{d}{dt}\delta \mathbf{M}^{(1)}(t) = \mathbf{A}_0(t)\delta \mathbf{M}^{(1)}(t) + \epsilon \mathbf{A}_1(t)\mathbf{M}_0(t),$$

$$\delta \mathbf{M}^{(1)}(0) = \mathbf{0}.$$

Variation of constants yields the first-order Dyson response

$$\delta \mathbf{M}^{(1)}(t) = \int_0^t \Phi_0(t, \tau)\epsilon \mathbf{A}_1(\tau)\mathbf{M}_0(\tau) \, d\tau.$$

Thus, once the reference trajectory and propagator are known, the endpoint perturbation is a linear functional of the weak target waveform. This point is important for reconstruction-oriented applications: the pulse sequence defines a finite-time observation kernel rather than an instantaneous sample of the field.

2.3. Unmodulated spin-lock response

We first apply the above construction to an unmodulated spin-lock preparation. In the rotating frame, the applied field is

$$\mathbf{B}(t) = [B_1, 0, B_{\text{tar}}(t) + \Delta B_0]^\top.$$

Here, B_1 is the spin-lock field, $B_{\text{tar}}(t)$ is the weak target field, and ΔB_0 is a static off-resonance term. We express the target field in angular-frequency units as

$$\omega_{\text{tar}}(t) = \gamma B_{\text{tar}}(t), \quad \omega_1 = \gamma B_1, \quad \Delta_0 = \gamma \Delta B_0.$$

For clarity of the closed-form expressions, we first consider the on-resonance reference case $\Delta_0 = 0$. The Bloch equation becomes

$$\frac{d}{dt}\mathbf{M}(t) = [\mathbf{A}_0 - \omega_{\text{tar}}(t)\mathbf{L}_z]\mathbf{M}(t) + \mathbf{c}, \quad (2)$$

where

$$\mathbf{A}_0 = \begin{bmatrix} -R_2 & 0 & 0 \\ 0 & -R_2 & \omega_1 \\ 0 & -\omega_1 & -R_1 \end{bmatrix}.$$

A nonzero static offset can also be included in $\mathbf{A}_0$. In that case, the propagator is still a closed-form matrix exponential, but the simple y - z block expression below is replaced by the general spectral or Cayley–Hamilton form (See Appendix A).

Because $\mathbf{A}_0$ is time invariant, the reference propagator is

$$\Phi_0(t, \tau) = \exp[\mathbf{A}_0(t - \tau)]. \quad (3)$$

The x component is decoupled from the y - z subspace. Define

$$\alpha := \frac{R_1 + R_2}{2}, \quad d := \frac{R_1 - R_2}{2}, \quad \Omega_\rho := \sqrt{\omega_1^2 - d^2}.$$

For the underdamped case $\Omega_\rho \in \mathbb{R}$, and with $s = t - \tau$, the matrix exponential is

$$\begin{aligned} \Phi_0(t, \tau) &= \begin{bmatrix} e^{-R_2 s} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &+ e^{-\alpha s} \frac{\omega_1}{\Omega_\rho} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \cos(\Omega_\rho s - \theta_\rho) & \sin(\Omega_\rho s) \\ 0 & -\sin(\Omega_\rho s) & \cos(\Omega_\rho s + \theta_\rho) \end{bmatrix}, \\ \theta_\rho &:= \arctan\left(\frac{d}{\Omega_\rho}\right). \end{aligned}$$

This expression shows explicitly that the reference dynamics consists of ordinary transverse relaxation in the x component and damped rotation in the y - z subspace. The affine shift is

$$\mathbf{A}_0^{-1}\mathbf{c} = \frac{R_1 M_\infty}{R_1 R_2 + \omega_1^2} \begin{bmatrix} 0 \\ \omega_1 \\ -R_2 \end{bmatrix},$$

and the reference trajectory is therefore

$$\mathbf{M}_0(t) = \Phi_0(t, 0)[\mathbf{M}(0) + \mathbf{A}_0^{-1}\mathbf{c}] - \mathbf{A}_0^{-1}\mathbf{c}. \quad (4)$$

For the unmodulated spin-lock sequence, the perturbation matrix is $-\omega_{\text{tar}}(t)\mathbf{L}_z$. The first-order response is therefore

$$\delta\mathbf{M}^{(1)}(t) = -\int_0^t \Phi_0(t, \tau)\mathbf{L}_z\mathbf{M}_0(\tau)\omega_{\text{tar}}(\tau) d\tau. \quad (5)$$

For the observed component $j \in \{x, y, z\}$, define

$$K_j(t, \tau) := -e_j^\top \Phi_0(t, \tau)\mathbf{L}_z\mathbf{M}_0(\tau).$$

Then

$$\delta M_j^{(1)}(t) = \int_0^t K_j(t, \tau)\omega_{\text{tar}}(\tau) d\tau. \quad (6)$$

Using Eq. (4), the kernel can also be written as

$$K_j(t, \tau) = -e_j^\top \Phi_0(t, \tau)\mathbf{L}_z \{ \Phi_0(\tau, 0)[\mathbf{M}(0) + \mathbf{A}_0^{-1}\mathbf{c}] - \mathbf{A}_0^{-1}\mathbf{c} \}.$$

This is the finite-time observation kernel generated by the reference spin-lock dynamics.

The structure of the kernel is most transparent for an initial spin-locked magnetization $\mathbf{M}(0) = [M_x(0), 0, 0]^\top$. In this case, the part of the longitudinal response generated by the homogeneous reference trajectory is

$$\begin{aligned} \delta M_z^{(1)}(t) &= \frac{\omega_1}{\Omega_\rho} M_x(0) e^{-R_2 t} \\ &\int_0^t e^{-d(t-\tau)} \sin[\Omega_\rho(t - \tau)] \omega_{\text{tar}}(\tau) d\tau. \end{aligned}$$

Equivalently, with

$$\begin{aligned} k_{\rho,z}(t) &:= e^{-dt} \sin(\Omega_\rho t), \\ (k_{\rho,z} * f)(t) &:= \int_0^t k_{\rho,z}(t - \tau) f(\tau) d\tau, \end{aligned}$$

we obtain

$$\delta M_z^{(1)}(t) = \frac{\omega_1}{\Omega_\rho} M_x(0) e^{-R_2 t} (k_{\rho,z} * \omega_{\text{tar}})(t).$$

Thus, the unmodulated spin-lock preparation does not measure the target field instantaneously. It measures a finite-time convolution of the target waveform with a damped sinusoidal kernel. The corresponding quadrature kernel for the y component is

$$k_{\rho,y}(t) := e^{-dt} \cos(\Omega_\rho t - \theta_\rho).$$

These two kernels provide the main physical interpretation of the unmodulated sequence: the sequence-dependent response is a damped oscillatory filter whose frequency and decay are set by ω_1 , R_1 , and R_2 .

2.4. Error scale of the first-order approximation

The first-order model neglects the second and higher Dyson terms. A useful way to quantify the expected error is to express the exact propagator in the interaction picture. Let

$\Phi_{\text{ex}}(t, \tau)$ be the homogeneous propagator of the full perturbed system,

$$\frac{\partial}{\partial t} \Phi_{\text{ex}}(t, \tau) = [\mathbf{A}_0 - \omega_{\text{tar}}(t) \mathbf{L}_z] \Phi_{\text{ex}}(t, \tau).$$

Writing

$$\Phi_{\text{ex}}(t, \tau) = \Phi_0(t, \tau) \Psi(t, \tau),$$

we obtain

$$\frac{\partial}{\partial t} \Psi(t, \tau) = \mathbf{V}_1(t; \tau) \Psi(t, \tau),$$

where

$$\mathbf{V}_1(t; \tau) := \Phi_0^{-1}(t, \tau) [-\omega_{\text{tar}}(t) \mathbf{L}_z] \Phi_0(t, \tau).$$

The size of the neglected terms is controlled by the integrated spectral norm

$$\epsilon_1(t) := \int_0^t \|\mathbf{V}_1(s; 0)\|_2 ds.$$

For the on-resonance underdamped spin-lock case, the derivation in Appendix B gives the coarse bound

$$\epsilon_1(t) \lesssim \frac{\omega_1}{\Omega_\rho} \sqrt{1 + \sin \theta_\rho} \int_0^t |\omega_{\text{tar}}(\tau)| e^{-d\tau} d\tau. \quad (7)$$

This estimate is not intended as a sharp error bound for all parameter regimes. Its practical role is to indicate when the first-order treatment is meaningful. A deliberately large modulation-induced pseudo-field, as used in SPARX, should therefore be placed in the reference system rather than in the perturbation.

2.5. Phase-modulated SPARX response

We next consider a phase-modulated spin-lock preparation. The RF field in the rotating frame is written as

$$\mathbf{B}_1(t) = [B_1 \cos \phi(t), -B_1 \sin \phi(t), 0]^T.$$

We transform to the phase-following frame by

$$\mathbf{M}(t) = \mathbf{R}_z[-\phi(t)] \mathbf{M}^{(r)}(t).$$

In this frame, the transverse RF field is locked along the x -axis, and the phase derivative appears as a pseudo-field along the z -axis. We define

$$\omega_p(t) := \frac{d\phi(t)}{dt}.$$

The weak target field remains a perturbation, whereas the pseudo-field is part of the reference system:

$$\frac{d}{dt} \mathbf{M}^{(r)}(t) = [\mathbf{A}_0^{(r)}(t) - \omega_{\text{tar}}(t) \mathbf{L}_z] \mathbf{M}^{(r)}(t) + \mathbf{c}, \quad (8)$$

with

$$\mathbf{A}_0^{(r)}(t) = \begin{bmatrix} -R_2 & \omega_p(t) & 0 \\ -\omega_p(t) & -R_2 & \omega_1 \\ 0 & -\omega_1 & -R_1 \end{bmatrix}. \quad (9)$$

For SPARX, the pseudo-field is chosen as a resonant sinusoidal modulation,

$$\omega_p(t) = \Omega_p \sin(\omega_1 t),$$

where Ω_p is the pseudo-field amplitude. This amplitude is not assumed to be small.

To obtain an interpretable reference propagator, we introduce a second rotating frame about the spin-lock axis,

$$\mathbf{M}_0^{(r)}(t) = \mathbf{R}_x[-\omega_1 t] \mathbf{M}_0^{(dr)}(t).$$

Under this transformation, the homogeneous reference dynamics can be decomposed into a dc part and rapidly oscillating terms,

$$\frac{d}{dt} \mathbf{M}_0^{(dr)}(t) = [\mathbf{A}_{0,\text{DC}}^{(dr)} + \mathbf{A}_{0,\text{AC}}^{(dr)}(t)] \mathbf{M}_0^{(dr)}(t).$$

The dc matrix retained by the rotating-wave approximation is

$$\mathbf{A}_{0,\text{DC}}^{(dr)} = \begin{bmatrix} -R_{1\rho} & 0 & \omega_\rho \\ 0 & -R_{2\rho} & 0 \\ -\omega_\rho & 0 & -R_{2\rho} \end{bmatrix},$$

where

$$R_{1\rho} := R_2, \quad R_{2\rho} := \frac{R_1 + R_2}{2}, \quad \omega_\rho := \frac{\Omega_p}{2}.$$

The factor $1/2$ in ω_ρ is the resonant component retained by the rotating-wave approximation. The dc propagator is again available in closed form. Define

$$\alpha_{\text{dr}} := \frac{R_1 + 3R_2}{4}, \quad \beta_{\text{dr}} := 2\sqrt{\omega_\rho^2 - d^2},$$

$$\theta_{\text{dr}} := \arctan\left(\frac{2d}{\beta_{\text{dr}}}\right).$$

Then, with $s = t - \tau$,

$$\begin{aligned} \Psi_{\text{DC}}(s) &:= \exp[\mathbf{A}_{0,\text{DC}}^{(dr)} s] \\ &= \frac{2\omega_\rho}{\beta_{\text{dr}}} e^{-\alpha_{\text{dr}} s} \begin{bmatrix} \cos\left(\frac{\beta_{\text{dr}} s}{2} - \theta_{\text{dr}}\right) & 0 & \sin\left(\frac{\beta_{\text{dr}} s}{2}\right) \\ 0 & 0 & 0 \\ -\sin\left(\frac{\beta_{\text{dr}} s}{2}\right) & 0 & \cos\left(\frac{\beta_{\text{dr}} s}{2} + \theta_{\text{dr}}\right) \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & 0 & 0 \\ 0 & e^{-R_{2\rho} s} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

The RWA reference propagator in the phase-following frame is therefore

$$U_0(t, \tau) := \mathbf{R}_x[-\omega_1 t] \Psi_{\text{DC}}(t - \tau) \mathbf{R}_x[\omega_1 \tau].$$

This expression is the SPARX counterpart of Eq. (3). The difference is that the designed pseudo-field is now part of the reference propagator.

The affine reference trajectory in the phase-following frame is

$$\mathbf{M}_0^{(r)}(t) = U_0(t, 0)\mathbf{M}^{(r)}(0) + R_1 \mathbf{M}_\infty \int_0^t U_0(t, \tau) \mathbf{e}_z d\tau.$$

The first-order target-field response is then

$$\delta \mathbf{M}^{(r,1)}(t) = - \int_0^t U_0(t, \tau) \mathbf{L}_z \mathbf{M}_0^{(r)}(\tau) \omega_{\text{tar}}(\tau) d\tau.$$

Therefore, the SPARX observation kernel is

$$K_j^{\text{SP}}(t, \tau) := -\mathbf{e}_j^T U_0(t, \tau) \mathbf{L}_z \mathbf{M}_0^{(r)}(\tau), \quad (10)$$

which gives

$$\delta M_j^{(r,1)}(t) = \int_0^t K_j^{\text{SP}}(t, \tau) \omega_{\text{tar}}(\tau) d\tau.$$

Unlike the unmodulated spin-lock kernel, the SPARX kernel does not generally reduce to a single damped oscillatory response. This behavior is expected because the phase modulation modifies both the reference trajectory and the reference propagator through the pseudo-field dynamics. Consequently, the resulting first-order kernel contains multiple relaxation modes and modulation-induced sideband frequencies.

The component-wise derivation of Eq. (10) is lengthy and is therefore given in Appendix C. For the present discussion, it is sufficient to note that, in the high-power-drive regime

$$\beta_{\text{dr}} \simeq 2\omega_\rho, \quad \theta_{\text{dr}} \simeq 0,$$

each kernel component can be written as a finite sum of damped trigonometric basis functions,

$$K_j^{\text{SP}}(t, \tau) = \sum_{\lambda \in \mathcal{R}} \sum_{v \in \mathcal{F}_\lambda} [a_{j,\lambda,v}(t) \cos(v\tau) + b_{j,\lambda,v}(t) \sin(v\tau)] e^{-\lambda\tau},$$

where $j \in \{x, y, z\}$.

The complete relaxation-rate set is

$$\mathcal{R} = \{0, \alpha_{\text{dr}}, R_{2\rho}, \alpha_{\text{dr}} - R_{2\rho}, -\alpha_{\text{dr}} + R_{2\rho}\}.$$

The corresponding complete frequency sets $\mathcal{F}_\lambda$ contain the spin-lock frequency, the pseudo-field frequency, and their modulation sidebands. They are

$$\begin{aligned} \mathcal{F}_0 &= \{\omega_1, \omega_\rho \pm \omega_1\}, \\ \mathcal{F}_{\alpha_{\text{dr}}} &= \{\omega_1, \omega_1 \pm 2\omega_\rho, 3\omega_1 \pm 2\omega_\rho, 3\omega_1\}, \\ \mathcal{F}_{R_{2\rho}} &= \{0, 2\omega_1\}, \\ \mathcal{F}_{\alpha_{\text{dr}} - R_{2\rho}} &= \{\omega_1, \omega_1 \pm \omega_\rho, 3\omega_1 \pm 2\omega_\rho\}, \\ \mathcal{F}_{-\alpha_{\text{dr}} + R_{2\rho}} &= \{\omega_1 \pm \omega_\rho, \omega_1 \pm 3\omega_\rho\}. \end{aligned}$$

Appendix C gives the derivation of these relaxation-rate and frequency sets from the homogeneous and inhomogeneous SPARX reference contributions. Therefore, unlike the unmodulated spin-lock sequence, whose response is described by a single damped resonance kernel, SPARX generates a structured family of kernels composed of multiple relaxation modes and sideband frequencies.

3. Methods

All numerical calculations were performed with verification code that is made available in the public repository described in the Data and code availability statement. The code solves the Bloch equation either for the unmodulated spin-lock system in Eq. (2) or for the SPARX system in Eq. (8). Direct numerical integration was used as the reference calculation. Unless otherwise stated, the ODEs were integrated with a high-order adaptive Runge–Kutta method using stringent tolerances, and the same time grid was used for target-field evaluation, kernel construction, and numerical quadrature.

The default relaxation and preparation parameters were $R_1 = 1.67 \text{ s}^{-1}$, $R_2 = 10.0 \text{ s}^{-1}$, $t_{\text{SL}} = 80 \text{ ms}$, and $M_\infty = 1$. For the unmodulated and SPARX model-validation calculations, $\omega_1/2\pi = 80 \text{ Hz}$. The weak target-field amplitude was $\omega_a = 0.25 \text{ rad s}^{-1}$, corresponding to 0.935 nT for the proton gyromagnetic ratio. The target waveforms were a sinusoid, a square wave, and a band-limited multitone waveform. These three cases were chosen to test whether the first-order response is tied to sinusoidal driving or remains valid for arbitrary finite-time waveforms.

For the unmodulated sequence, the time-invariant reference solution in Eq. (4) was compared with direct numerical integration in the absence of a target field. This target-free calculation was used as an implementation check for the reference matrix and propagator before testing the perturbative response. The first-order Dyson response in Eq. (5) and the kernel observation formula in Eq. (6) were then evaluated for each target waveform. Endpoint accuracy was quantified by

$$E_{\text{abs}} = \left| M_{z,\text{full}}(t_{\text{SL}}) - [M_{z,\text{ref}}(t_{\text{SL}}) + \delta M_z^{(1)}(t_{\text{SL}})] \right|,$$

and

$$E_{\text{rel}} = \frac{E_{\text{abs}}}{\left| M_{z,\text{full}}(t_{\text{SL}}) - M_{z,\text{ref}}(t_{\text{SL}}) \right| + \eta},$$

where η is a small numerical floor. The target-field amplitude was further swept on a logarithmic grid to identify the range over which the first-order model remains accurate and the point at which higher-order terms become visible.

For SPARX, the phase-modulation-induced pseudo-field is included in the reference system rather than in the perturbation. The numerical reference trajectory was therefore obtained by integrating the target-free SPARX Bloch equation with the time-dependent matrix in Eq. (9). The numerical SPARX kernel was constructed from the first-order adjoint relation corresponding to Eq. (10). Specifically, if $q_z(\tau)$ denotes the row vector that propagates a perturbation at time τ to the final z -magnetization, the kernel was evaluated as $K_z^{\text{SP}}(t_{\text{SL}}, \tau) = -q_z(\tau) \mathbf{L}_z \mathbf{M}_0(\tau)$. The resulting nonparametric numerical kernel was compared with the damped-trigonometric finite basis derived in Appendix C.

The finite-basis fit was evaluated over the sampled kernel values. With K_i denoting the numerical kernel and $\hat{K}_i$ the

basis prediction, the primary fit metric was

$$\text{MSE} = \frac{1}{N} \sum_{i=1}^N (K_i - \hat{K}_i)^2.$$

The root mean squared error,

$$\text{RMSE} = \sqrt{\text{MSE}},$$

was also reported because it has the same units as the kernel samples. Endpoint errors for sinusoidal, square, and band-limited target waveforms were then used to test whether the fitted kernel predicts physically relevant observations, rather than only pointwise kernel values.

To compare frequency selectivity, the target field was taken to be a single-frequency waveform,

$$\omega_{\text{tar}}(t) = \omega_a \sin(\omega_s t),$$

and ω_s was swept from zero to $3\omega_1$. To remove the dependence on the initial phase of the sinusoidal target, both sine and cosine quadratures were evaluated and combined as

$$G_z(\omega_s) = \frac{1}{\omega_a} \sqrt{\delta M_{z,\sin}^2(t_{\text{SL}}; \omega_s) + \delta M_{z,\cos}^2(t_{\text{SL}}; \omega_s)}.$$

The plotted response is $\omega_a G_z$, i.e., the phase-optimized endpoint perturbation $|\delta M_z|$. Finally, the pseudo-field amplitude Ω_p was swept at fixed $\omega_1/2\pi = 40$ Hz to test how SPARX shifts the sensitive frequency bands. This sweep was interpreted as a frequency-shaping analysis at fixed spin-lock amplitude.

4. Results

Fig. 1 verifies the unmodulated spin-lock response model. In the absence of a target field, the matrix-exponential reference trajectory agreed with direct Bloch integration with a maximum norm error of 4.96×10^{-12} . For a sinusoidal target field, the full Bloch endpoint perturbation was $\Delta M_z = 3.28268 \times 10^{-4}$, compared with 3.28274×10^{-4} from the first-order Dyson equation and 3.28264×10^{-4} from direct kernel quadrature. The square and band-limited waveforms gave relative endpoint errors of 1.14×10^{-4} and 4.75×10^{-6} , respectively. Thus the finite-time kernel representation reproduced non-sinusoidal target waveforms without changing the model form.

The amplitude sweep in Fig. 1 and Fig. 2 shows the expected perturbative behavior. For small target amplitudes, the absolute endpoint error increases approximately quadratically with amplitude; the fitted small-amplitude error slopes were 2.00 for the sinusoidal target, 2.00 for the square target, and 2.06 for the band-limited target. The sinusoidal case also showed a sharp local decrease in both absolute and relative error at $B_{\text{tar}} = 15.46$ nT, where the absolute endpoint error was 2.74×10^{-9} and the relative error was 5.04×10^{-7} . At this point the perturbation amplitude was $\omega_a/\omega_1 = 8.23 \times 10^{-3}$, and the signed endpoint residual crossed zero at $B_{\text{tar}} = 15.49$ nT. At larger amplitudes the curves depart from simple

quadratic scaling, indicating the onset of higher-order effects. This behavior is consistent with the interpretation that the first-order kernel is a weak-field observation model, not a replacement for the full nonlinear Bloch dynamics at arbitrary target amplitude.

Fig. 3 shows the corresponding SPARX kernel calculation. The numerical kernel was obtained from the phase-modulated reference trajectory and the adjoint endpoint sensitivity, whereas the finite-basis model used the damped-trigonometric functions suggested by the analytic structure in Appendix C. The finite-basis fit had $\text{MSE} = 5.93 \times 10^{-7}$, $\text{RMSE} = 7.70 \times 10^{-4}$, numerical rank 40, and condition number 1.96×10^6 . For weak target fields, the first-order SPARX model reproduced the full Bloch endpoint response with relative differences of approximately 2.2×10^{-3} , 2.0×10^{-3} , and 1.3×10^{-3} for sinusoidal, square, and band-limited targets, respectively. This endpoint test corresponds to the physical observation made by the sequence, namely the final longitudinal magnetization, and complements the pointwise kernel RMSE. The direct numerical kernel and the finite-basis approximation gave comparable endpoint predictions.

The single-frequency response spectra in Fig. 4 compare unmodulated spin-lock and SPARX for $\omega_1/2\pi = 80$ Hz. The unmodulated response peaked near 79.7 Hz, close to the spin-lock frequency, with $|\delta M_z| = 5.41 \times 10^{-3}$. With $\Omega_p/2\pi = 40$ Hz, the SPARX response had a peak near 81.0 Hz with $|\delta M_z| = 3.35 \times 10^{-3}$ and a broader sideband structure. The purpose of this comparison is not to maximize the response for a single frequency, but to show that phase modulation reshapes the finite-time response spectrum.

Fig. 5 examines this spectral reshaping at $\omega_1/2\pi = 40$ Hz. The SPARX response follows the expected sideband structure, with sensitive bands near $\omega_1/2\pi \pm \Omega_p/4\pi$. The lower sideband gives the largest global peak in the present parameter set, whereas the upper sideband remains visible and is relevant when the target application prioritizes higher-frequency sensitivity. These results show that the pseudo-field amplitude can move the sensitive band away from the nominal spin-lock frequency while preserving a finite response over a broader band.

5. Discussion

The numerical results support the central claim that spin-lock detection can be formulated as a finite-time linear response model for weak magnetic-field waveforms. In the unmodulated case, the kernel integral is not merely a reformulation of a sinusoidal steady-state response. It accurately predicts endpoint magnetization for square and band-limited inputs as long as the target field remains within the perturbative regime. This is the essential step for using spin-lock preparations as temporal encoding modules rather than as single-frequency detectors.

The amplitude-dependence results also clarify the limitation of the model. The first-order kernel is valid when the target-field term is small compared with the reference dynamics over the preparation interval. Once the target

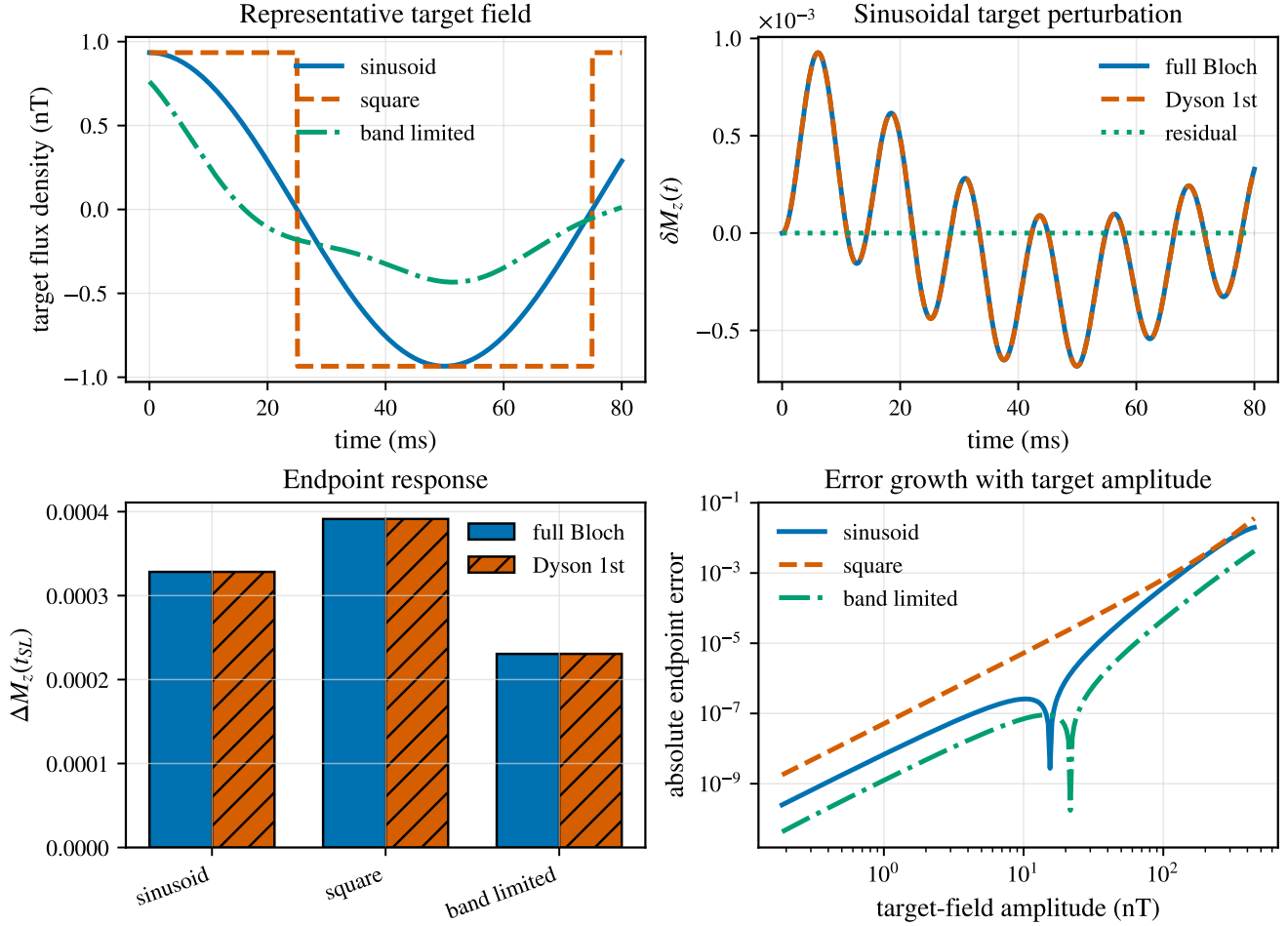


Figure 1: Numerical verification of the unmodulated spin-lock perturbation model. The target waveforms are expressed as magnetic flux density, the time-domain response compares the full Bloch solution with the first-order Dyson solution for a sinusoidal target, and the endpoint and amplitude-sweep panels test arbitrary-waveform behavior and perturbative breakdown.

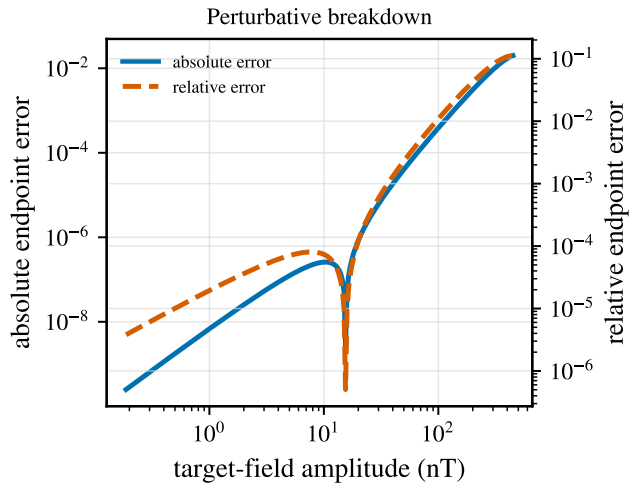


Figure 2: Amplitude-dependent breakdown of the first-order unmodulated spin-lock approximation. Absolute and relative endpoint errors are shown on the same amplitude grid.

amplitude grows, the neglected higher-order terms become visible and the endpoint error no longer follows the small-amplitude quadratic scaling. The sharp error minimum in the sinusoidal sweep should therefore not be over-interpreted as an extended validity region. Because the endpoint error is defined by the difference between the full Bloch endpoint and the first-order prediction, its numerator can become accidentally close to zero at isolated amplitudes. This is what occurs here: the signed residual changes sign near $B_{\text{tar}} = 15.49$ nT. By contrast, $B_{\text{tar}} = 15.46$ nT gives $(1/2)\gamma B_{\text{tar}} t_{\text{SL}} = 0.165$ rad $= 0.0527\pi$, while a simple π -phase condition would require $B_{\text{tar}} = 293.6$ nT. The observed dip is therefore best interpreted as a finite-time cancellation in the endpoint residual, not as a true singularity of the Bloch dynamics or as a universal condition set only by the relaxation-dependent phase θ_ρ . In particular, the cancellation condition depends on the preparation time, waveform, and higher-order response terms. This breakdown analysis is useful experimentally because it provides a quantitative range in which linear reconstruction algorithms are meaningful.

For SPARX, the phase modulation should be interpreted as a control mechanism that changes the reference dynamics

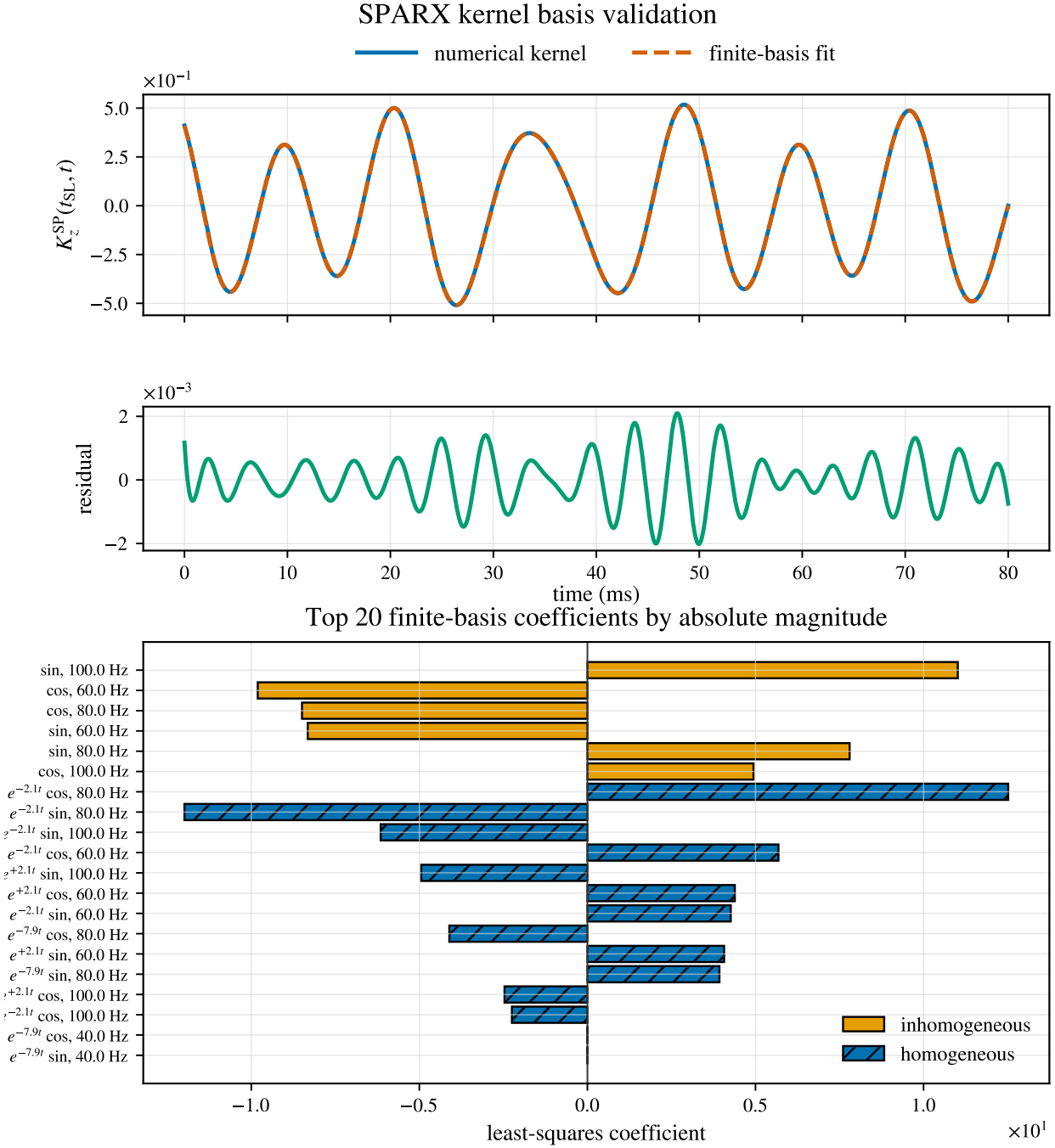


Figure 3: SPARX kernel-model validation. The numerical kernel is computed from the target-free phase-modulated reference trajectory and its adjoint sensitivity. A finite damped-trigonometric basis is fit to this numerical kernel; the largest coefficients indicate the dominant basis components. The kernel-fit errors are $\text{MSE} = 5.93 \times 10^{-7}$ and $\text{RMSE} = 7.70 \times 10^{-4}$.

and the response kernel. It is not part of the weak target field. This distinction is important because it allows the pseudo-field amplitude to be used as a design parameter. The pseudo-field sweep shows that the sensitive band can be shifted away from the nominal spin-lock frequency according to the sideband structure of the modulated reference system. The

present calculations show how phase modulation reshapes the frequency response at fixed ω_1 .

The SPARX finite-basis fit should be interpreted cautiously. The basis captures the dominant damped trigonometric structure of the numerical kernel, but its condition number indicates that individual coefficients are not necessarily robust physical observables. The more reliable object is the kernel

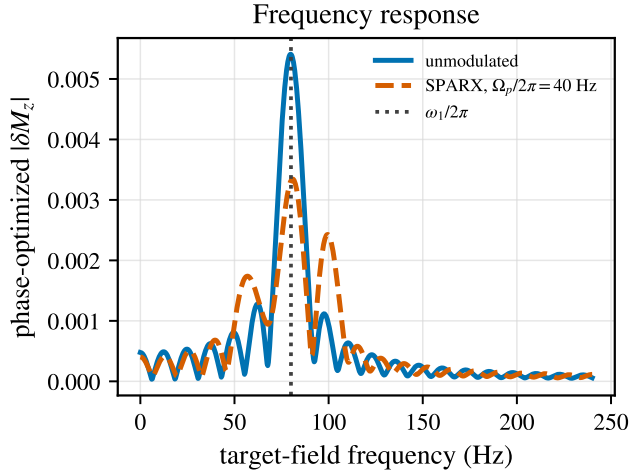


Figure 4: Single-frequency endpoint response of unmodulated spin-lock and SPARX. The SPARX calculation uses $\Omega_p/2\pi = 40$ Hz. The plotted quantity is the phase-optimized endpoint perturbation $|\delta M_z|$.

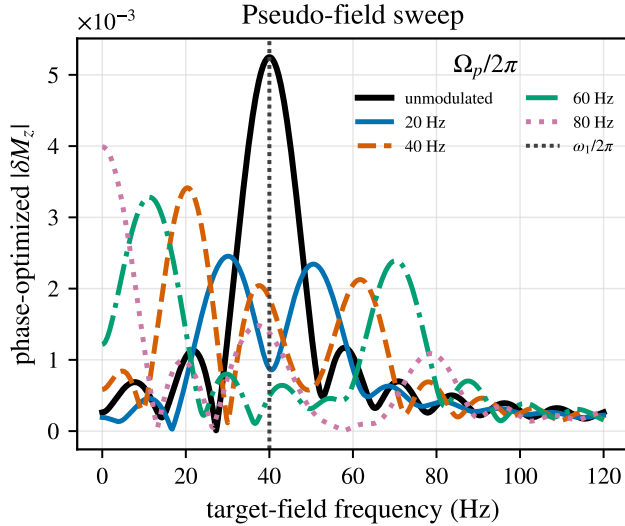


Figure 5: Pseudo-field amplitude dependence of the SPARX response at fixed $\omega_1/2\pi = 40$ Hz. The unmodulated response is shown as a reference, and SPARX responses are shown for $\Omega_p/2\pi = 20, 40, 60$, and 80 Hz.

itself, or predictions obtained by integrating the kernel against target waveforms. For this reason, the finite-basis expansion is most useful as a compact model for response design and interpretation, whereas the adjoint numerical kernel remains the reference for quantitative validation.

A direct implication of the kernel formulation is that waveform reconstruction can be posed as a linear inverse problem. Each spin-lock or SPARX preparation defines one temporal observation kernel, and multiple preparations with different ω_1 , Ω_p , or t_{SL} can provide complementary rows of a sensing matrix. The present work establishes the forward model and its perturbative validity. Future work should quantify identifiability, noise amplification, regularization,

and Cramer–Rao bounds for reconstructing weak neuronal magnetic-field time series.

6. Conclusion

We derived and numerically verified a finite-time perturbative response model for spin-lock detection of weak magnetic-field waveforms. For an unmodulated spin-lock preparation, the first-order Dyson expansion yields a kernel integral that maps arbitrary weak target waveforms to the endpoint longitudinal magnetization. The same framework extends to SPARX by including the phase-modulation-induced pseudo-field in the reference dynamics. The resulting model provides a bridge from spin-lock dynamics to inverse reconstruction of weak temporal magnetic fields.

CRediT authorship contribution statement

Hiroyuki Ueda: Conceptualization, Methodology, Software, Formal analysis, Writing - original draft.

Data and code availability

The Python scripts used to generate the numerical Bloch simulations, kernel calculations, and figures in this study are available at [repository DOI or URL]. The numerical data underlying the figures are included in the same repository.

Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

During the preparation of this work, the author used OpenAI ChatGPT and Codex to support language editing, LaTeX drafting, and code assistance. After using these tools, the author reviewed and edited the content as needed and takes full responsibility for the content of the manuscript.

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A. General reference propagator including ΔB_0

A nonzero static offset can be included in the reference system by using

$$\mathbf{A}_0(\Delta_0) = \begin{bmatrix} -R_2 & -\Delta_0 & 0 \\ \Delta_0 & -R_2 & \omega_1 \\ 0 & -\omega_1 & -R_1 \end{bmatrix}.$$

In this case, the x component is no longer decoupled from the y - z subspace. The eigenvalues λ_k of $\mathbf{A}_0(\Delta_0)$ are the roots of the cubic equation

$$\det[\lambda \mathbf{I} - \mathbf{A}_0(\Delta_0)] = 0, \\ (\lambda + R_1)[(\lambda + R_2)^2 + \omega_1^2] + \omega_1^2(\lambda + R_2) = 0.$$

Therefore, although the propagator remains available in closed form as

$$\Phi_0(t, \tau) = \sum_{k=1}^3 e^{\lambda_k(t-\tau)} \prod_{\substack{j=1 \\ j \neq k}}^3 \frac{\mathbf{A}_0(\Delta_0) - \lambda_j \mathbf{I}}{\lambda_k - \lambda_j},$$

this expression is much less transparent than the on-resonance block form. Equivalently, by the Cayley–Hamilton theorem,

$$\Phi_0(t, \tau) = a_0(s)\mathbf{I} + a_1(s)\mathbf{A}_0(\Delta_0) + a_2(s)\mathbf{A}_0^2(\Delta_0), \\ s = t - \tau,$$

where the scalar coefficients are determined by

$$e^{\lambda_k s} = a_0(s) + a_1(s)\lambda_k + a_2(s)\lambda_k^2, \quad k = 1, 2, 3.$$

These formulas are useful for numerical evaluation and symbolic consistency checks, but the resulting kernels involve all three coupled eigenmodes. For this reason, the following derivation focuses on the on-resonance case $\Delta_0 = 0$, where the response kernel can be written in an interpretable damped-oscillatory form.

B. Derivation of the interaction-picture error bound

This appendix derives the error indicator used in Eq. (7). We consider the on-resonance unmodulated spin-lock system,

$$\frac{d\Phi_{\text{ex}}(t, \tau)}{dt} = [\mathbf{A}_0 - \omega_{\text{tar}}(t)\mathbf{L}_z]\Phi_{\text{ex}}(t, \tau), \quad \Phi_{\text{ex}}(\tau, \tau) = \mathbf{I}.$$

The exact propagator is factorized as

$$\Phi_{\text{ex}}(t, \tau) = \Phi_0(t, \tau)\Psi(t, \tau),$$

where $\Phi_0(t, \tau) = \exp[\mathbf{A}_0(t - \tau)]$. Then $\Psi(t, \tau)$ satisfies

$$\frac{d\Psi(t, \tau)}{dt} = \mathbf{V}_1(t; \tau)\Psi(t, \tau),$$

with the interaction-picture perturbation

$$\mathbf{V}_1(t; \tau) = \Phi_0^{-1}(t, \tau)(-\omega_{\text{tar}}(t)\mathbf{L}_z)\Phi_0(t, \tau).$$

The first-order Dyson approximation is accurate when the time-integrated size of this interaction-picture perturbation is small. We therefore define

$$\epsilon_1(t) := \int_0^t \|\mathbf{V}_1(\tau; 0)\|_2 d\tau,$$

where the spectral norm is

$$\|\mathbf{V}\|_2 = \sqrt{\lambda_{\max}(\mathbf{V}^T \mathbf{V})}.$$

Let $s = t - \tau$ and $\theta_\rho := \arctan(d/\Omega_\rho)$. For the on-resonance underdamped case, direct substitution of the

closed-form propagator gives

$$\mathbf{V}_I = -\frac{\omega_1 \omega_{\text{tar}}(\tau)}{\Omega_\rho} \times \left(e^{(\alpha-R_2)s} \begin{bmatrix} 0 & -\cos(\Omega_\rho s + \theta_\rho) & -\sin(\Omega_\rho s) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e^{-(\alpha-R_2)s} \begin{bmatrix} 0 & 0 & 0 \\ \cos(\Omega_\rho s - \theta_\rho) & 0 & 0 \\ \sin(\Omega_\rho s) & 0 & 0 \end{bmatrix} \right).$$

Therefore,

$$\begin{aligned} \mathbf{V}_I^T \mathbf{V}_I &= \left(\frac{\omega_1 \omega_{\text{tar}}(\tau)}{\Omega_\rho} \right)^2 \begin{bmatrix} W_{11} & \mathbf{0}^T \\ \mathbf{0} & W_{22} \end{bmatrix} \\ W_{11} &= e^{-2(\alpha-R_2)s} [\cos^2(\Omega_\rho s - \theta_\rho) + \sin^2(\Omega_\rho s)] \\ W_{22} &= e^{2(\alpha-R_2)s} \\ &\begin{bmatrix} \cos^2(\Omega_\rho s + \theta_\rho) & \sin(\Omega_\rho s) \cos(\Omega_\rho s + \theta_\rho) \\ \sin(\Omega_\rho s) \cos(\Omega_\rho s + \theta_\rho) & \sin^2(\Omega_\rho s) \end{bmatrix} \end{aligned}$$

The nonzero eigenvalues of the matrix part are

$$\begin{aligned} \lambda_1(s) &= e^{-2(\alpha-R_2)s} [1 - \sin \theta_\rho \sin(2\Omega_\rho s - \theta_\rho)], \\ \lambda_2(s) &= e^{2(\alpha-R_2)s} [1 + \sin \theta_\rho \sin(2\Omega_\rho s + \theta_\rho)], \\ \lambda_3(s) &= 0. \end{aligned}$$

Using

$$|\sin(2\Omega_\rho s \pm \theta_\rho)| \leq 1,$$

we obtain the upper bound

$$\lambda_{\max} \leq (1 + \sin \theta_\rho) \max\{e^{2(\alpha-R_2)s}, e^{-2(\alpha-R_2)s}\}.$$

Since $\alpha - R_2 = (R_1 - R_2)/2 = d$, this gives

$$\|\mathbf{V}_I(\tau; 0)\|_2 \leq \frac{\omega_1}{\Omega_\rho} |\omega_{\text{tar}}(\tau)| \sqrt{1 + \sin \theta_\rho} e^{-d\tau},$$

for the usual relaxation ordering $R_2 \geq R_1$, where $d \leq 0$ and the dominant exponential factor can be written as $e^{-d\tau}$. Integrating this inequality over the spin-lock duration yields

$$\epsilon_I(t) \lesssim \frac{\omega_1}{\Omega_\rho} \sqrt{1 + \sin \theta_\rho} \int_0^t |\omega_{\text{tar}}(\tau)| e^{-d\tau} d\tau.$$

This is the bound used in the main text. It should be interpreted as a practical smallness indicator rather than a sharp estimate: when $\epsilon_I(t) \ll 1$, the neglected higher-order Dyson terms are expected to remain small, whereas large values indicate that the target field should be incorporated into the reference dynamics or treated by a higher-order approximation.

C. Derivation of the SPARX kernel basis

This appendix summarizes the component structure of the SPARX response kernel. The purpose is not to give a compact

physical interpretation, as in the unmodulated spin-lock case, but to show that the kernel can be constructed reproducibly from a finite set of damped trigonometric basis functions.

We consider the high-power-drive approximation of the SPARX reference propagator, where

$$\beta_{\text{dr}} \simeq 2\omega_\rho, \quad \theta_{\text{dr}} \simeq 0.$$

Under this approximation, the transformed-frame propagator $U_0(t, \tau)$ is composed of products of exponential factors, $\cos(\omega_1 t)$, $\sin(\omega_1 t)$, $\cos(\omega_1 \tau)$, $\sin(\omega_1 \tau)$, and the pseudo-field oscillation $\cos(\omega_\rho(t - \tau))$ or $\sin(\omega_\rho(t - \tau))$.

The reference trajectory consists of homogeneous and inhomogeneous parts:

$$\mathbf{M}_0^{(r)}(t) = \mathbf{M}_{0,\text{hom}}^{(r)}(t) + \mathbf{M}_{0,\text{inh}}^{(r)}(t).$$

Accordingly, the SPARX kernel is decomposed as

$$K_j^{\text{SP}}(t, \tau) = K_{j,\text{hom}}^{\text{SP}}(t, \tau) + K_{j,\text{inh}}^{\text{SP}}(t, \tau), \quad j \in \{x, y, z\},$$

where

$$K_{j,\text{hom}}^{\text{SP}}(t, \tau) = -e_j^T U_0(t, \tau) \mathbf{L}_z \mathbf{M}_{0,\text{hom}}^{(r)}(\tau),$$

and

$$K_{j,\text{inh}}^{\text{SP}}(t, \tau) = -e_j^T U_0(t, \tau) \mathbf{L}_z \mathbf{M}_{0,\text{inh}}^{(r)}(\tau). \quad (\text{C.1})$$

To express the inhomogeneous reference solution, we first define the elementary integrals

$$C_\lambda(v; t) := \int_0^t e^{-\lambda u} \cos(vu) du,$$

$$S_\lambda(v; t) := \int_0^t e^{-\lambda u} \sin(vu) du.$$

They are evaluated as

$$C_\lambda(v; t) = \frac{\lambda - \lambda e^{-\lambda t} \cos(vt) + v e^{-\lambda t} \sin(vt)}{\lambda^2 + v^2},$$

$$S_\lambda(v; t) = \frac{v - \lambda e^{-\lambda t} \sin(vt) - v e^{-\lambda t} \cos(vt)}{\lambda^2 + v^2}.$$

Using these auxiliary functions, the inhomogeneous reference trajectory can be written as

$$\mathbf{M}_{0,\text{inh}}^{(r)}(t) = R_1 \mathbf{M}_\infty \begin{bmatrix} B(t) \\ -\cos(\omega_1 t) Q(t) + \sin(\omega_1 t) E(t) \\ \sin(\omega_1 t) Q(t) + \cos(\omega_1 t) E(t) \end{bmatrix}. \quad (\text{C.2})$$

Here,

$$\begin{aligned} B(t) &:= \int_0^t e^{-\alpha_{\text{dr}}(t-\tau)} \sin[\omega_\rho(t-\tau)] \cos(\omega_1 \tau) d\tau \\ &= \frac{\cos(\omega_1 t)}{2} [S_{\alpha_{\text{dr}}}(\omega_\rho + \omega_1; t) + S_{\alpha_{\text{dr}}}(\omega_\rho - \omega_1; t)] \\ &\quad + \frac{\sin(\omega_1 t)}{2} [C_{\alpha_{\text{dr}}}(\omega_\rho - \omega_1; t) - C_{\alpha_{\text{dr}}}(\omega_\rho + \omega_1; t)], \end{aligned}$$

$$\begin{aligned}
 Q(t) &:= \int_0^t e^{-R_{2\rho}(t-\tau)} \sin(\omega_1 \tau) d\tau \\
 &= \sin(\omega_1 t) C_{R_{2\rho}}(\omega_1; t) - \cos(\omega_1 t) S_{R_{2\rho}}(\omega_1; t), \\
 E(t) &:= \int_0^t e^{-\alpha_{\text{dr}}(t-\tau)} \cos[\omega_\rho(t-\tau)] \cos(\omega_1 \tau) d\tau \\
 &= \frac{\cos(\omega_1 t)}{2} \left[C_{\alpha_{\text{dr}}}(\omega_\rho - \omega_1; t) + C_{\alpha_{\text{dr}}}(\omega_\rho + \omega_1; t) \right] \\
 &\quad + \frac{\sin(\omega_1 t)}{2} \left[S_{\alpha_{\text{dr}}}(\omega_\rho + \omega_1; t) - S_{\alpha_{\text{dr}}}(\omega_\rho - \omega_1; t) \right]. \quad (3)
 \end{aligned}$$

Equations (C.2)– (C.3) show that the inhomogeneous reference trajectory is a finite sum of exponentially damped trigonometric functions. Substituting this trajectory into Eq. (C.1) and expanding the products of trigonometric functions gives a kernel of the form

$$\begin{aligned}
 K_{j,\text{inh}}^{\text{SP}}(t, \tau) &= \\
 &\sum_{\lambda \in \mathcal{R}_{\text{inh}}} \sum_{v \in \mathcal{F}_{\lambda}^{\text{inh}}} \left[a_{j,\lambda,v}^{\text{inh}}(t) \cos(v\tau) + b_{j,\lambda,v}^{\text{inh}}(t) \sin(v\tau) \right] e^{\lambda\tau}.
 \end{aligned}$$

The relaxation-rate set for the inhomogeneous kernel is

$$\mathcal{R}_{\text{inh}} = \{0, \alpha_{\text{dr}}, R_{2\rho}, \alpha_{\text{dr}} - R_{2\rho}, -\alpha_{\text{dr}} + R_{2\rho}\}.$$

The corresponding frequency sets are

$$\begin{aligned}
 \mathcal{F}_0^{\text{inh}} &= \{\omega_1, \omega_\rho \pm \omega_1\}, \\
 \mathcal{F}_{\alpha_{\text{dr}}}^{\text{inh}} &= \{\omega_1, \omega_1 \pm 2\omega_\rho, 3\omega_1 \pm 2\omega_\rho, 3\omega_1\}, \\
 \mathcal{F}_{R_{2\rho}}^{\text{inh}} &= \{0, 2\omega_1\}, \\
 \mathcal{F}_{\alpha_{\text{dr}}-R_{2\rho}}^{\text{inh}} &= \{\omega_1 \pm \omega_\rho, 3\omega_1 \pm 2\omega_\rho\}, \\
 \mathcal{F}_{-\alpha_{\text{dr}}+R_{2\rho}}^{\text{inh}} &= \{\omega_1 \pm \omega_\rho, \omega_1 \pm 3\omega_\rho\}.
 \end{aligned}$$

The homogeneous reference trajectory gives a simpler contribution. For the initial condition

$$\mathbf{M}^{(r)}(0) = \begin{bmatrix} M_\infty \\ 0 \\ 0 \end{bmatrix},$$

only the first column of $U_0(t, \tau) \mathbf{L}_z U_0(\tau, 0)$ is needed. The homogeneous kernel can therefore be written as

$$\begin{aligned}
 K_{j,\text{hom}}^{\text{SP}}(t, \tau) &= \\
 &\sum_{\lambda \in \mathcal{R}_{\text{hom}}} \sum_{v \in \mathcal{F}_{\lambda}^{\text{hom}}} \left[a_{j,\lambda,v}^{\text{hom}}(t) \cos(v\tau) + b_{j,\lambda,v}^{\text{hom}}(t) \sin(v\tau) \right] e^{\lambda\tau}.
 \end{aligned}$$

The relaxation-rate set is

$$\mathcal{R}_{\text{hom}} = \{0, \alpha_{\text{dr}} - R_{2\rho}, -\alpha_{\text{dr}} + R_{2\rho}\},$$

with frequency sets

$$\begin{aligned}
 \mathcal{F}_0^{\text{hom}} &= \{\omega_1\}, \\
 \mathcal{F}_{\alpha_{\text{dr}}-R_{2\rho}}^{\text{hom}} &= \{\omega_1\}, \\
 \mathcal{F}_{-\alpha_{\text{dr}}+R_{2\rho}}^{\text{hom}} &= \{\omega_1 \pm \omega_\rho\}.
 \end{aligned}$$

Combining the homogeneous and inhomogeneous contributions yields the full SPARX kernel basis

$$\begin{aligned}
 K_j^{\text{SP}}(t, \tau) &= \\
 &\sum_{\lambda \in \mathcal{R}} \sum_{v \in \mathcal{F}_{\lambda}} \left[a_{j,\lambda,v}(t) \cos(v\tau) + b_{j,\lambda,v}(t) \sin(v\tau) \right] e^{\lambda\tau},
 \end{aligned}$$

where

$$\mathcal{R} = \{0, \alpha_{\text{dr}}, R_{2\rho}, \alpha_{\text{dr}} - R_{2\rho}, -\alpha_{\text{dr}} + R_{2\rho}\}.$$

The combined frequency sets are

$$\begin{aligned}
 \mathcal{F}_0 &= \{\omega_1, \omega_\rho \pm \omega_1\}, \\
 \mathcal{F}_{\alpha_{\text{dr}}} &= \{\omega_1, \omega_1 \pm 2\omega_\rho, 3\omega_1 \pm 2\omega_\rho, 3\omega_1\}, \\
 \mathcal{F}_{R_{2\rho}} &= \{0, 2\omega_1\}, \\
 \mathcal{F}_{\alpha_{\text{dr}}-R_{2\rho}} &= \{\omega_1, \omega_1 \pm \omega_\rho, 3\omega_1 \pm 2\omega_\rho\}, \\
 \mathcal{F}_{-\alpha_{\text{dr}}+R_{2\rho}} &= \{\omega_1 \pm \omega_\rho, \omega_1 \pm 3\omega_\rho\}. \quad (C.4)
 \end{aligned}$$

The coefficients $a_{j,\lambda,v}(t)$ and $b_{j,\lambda,v}(t)$ are determined by the endpoint time t , the reference propagator, and the relaxation parameters. Their explicit component-wise forms are algebraically long, but Eqs. (C.2)– (C.4) specify all damping rates and angular frequencies required to reconstruct the SPARX kernel. Thus, SPARX is represented as a finite-dimensional damped-trigonometric observation model for the weak target waveform.