

On the number of independent orders

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Abstract

In this note, we present and prove some lemmas that are useful when studying the number of independent orders. We can show $\kappa_{srd}^m(T) = \infty \Rightarrow \kappa_{srd}^1(T) = \infty$, using these lemmas. Its proof will be given in a forthcoming paper. (The details are not given in this note.)

We fix a complete theory T , and we work in a very saturated model of T . Letters $x, y, \dots$ are used to denote finite tuples of variables. X is a set of x -tuples and Y is a set of y -tuples. In many cases, they have the form

$$X = (x_\eta)_{\eta \in \omega^n} \text{ and } Y = (y_\nu)_{\nu \in n \times \omega},$$

where $n \in \omega$. For sets Z, W of finite tuples of variables and a set $\Gamma(Z, W)$ of formulas, the set of all formulas $\exists z_0 \dots \exists z_{m-1} (\gamma_0(z, w) \wedge \dots \gamma_{m-1}(z, w))$, where $m \in \omega$, $\gamma_i(z_i, w_i) \in \Gamma$, $z_i \subset Z$, $w_i \subset W$, is denoted by $\exists Z \Gamma(Z, W)$.

Definition 1. Let $n \in \omega$.

1. Let $X = (x_\eta : \eta \in \omega^n)$ be a set of variables. Let $\Delta(X)$ be a set of formulas whose free variables are in X . We say that Δ has the subarray property if there is a set $A = (a_{i_0, \dots, i_{n-1}} : \langle i_0, \dots, i_{n-1} \rangle \in \omega^n)$ such that for any strictly increasing functions $f_i : \omega \rightarrow \omega$ ($i < n$), $A_{f_0, \dots, f_{n-1}} = (a_{f_0(i_0), \dots, f_{n-1}(i_{n-1})} : \langle i_0, \dots, i_{n-1} \rangle \in \omega^n)$ realizes Δ .
2. Let $Y = (y_\nu)_{\nu \in n \times \omega}$. Let $\mathcal{E}(Y)$ be a set of formulas whose free variables are in Y . We say that $\mathcal{E}$ has the (n -dimensional) subsequence property if there is a set $B = (b_{i,j})_{\langle i,j \rangle \in n \times \omega}$ such that for any strictly increasing functions $f_i : \omega \rightarrow \omega$ ($i < n$), $B_{f_0, \dots, f_{n-1}} = (b_{i, f_i(j)})_{\langle i,j \rangle \in n \times \omega}$ realizes $\mathcal{E}(Y)$.

Lemma 2. Suppose that $\Delta(X)$, where $X = (x_\eta : \eta \in \omega^n)$, has the sub-array property. Then a realization $A = (a_\eta : \eta \in \omega^n)$ of Δ can be chosen as an indiscernible array in the following sense:

(*) For finite subsets F, F' of ω^n , if F and F' are isomorphic as $\{\leq_0, \dots, \leq_{n-1}\}$ -structures then a_F and $a_{F'}$ have the same L -type.

Proof. For simplicity, we assume $n = 2$. We write X as $X = (X_0, X_1, \dots)$, where $X_i = (x_{i,j})_{j \in \omega}$. For each i , let $X_i = (x_{i,j})_{j \in \omega}$ be the i -th row vector of X . Then

$$\Delta = \Delta((X_i)_{i \in \omega}) = \Delta(X_0, X_1, \dots)$$

has the subsequence property. So, for $A = (A_i)_{i \in \omega}$ realizing Δ , we can assume the A_i 's form an indiscernible sequence. Similarly, we can also assume $(A'_j)_{j \in \omega}$, where $A'_j = (a_{i,j})_{i \in \omega}$, is an indiscernible sequence. So A is an indiscernible array. $\square$

For $A = (a_\eta)_{\eta \in \omega^n}$ and a subset F of ω^2 , a_F will denote the set $(a_\eta)_{\eta \in F}$.

Lemma 3. Suppose that $\Delta(X)$ is realized by an indiscernible array $A = (a_\eta : \eta \in \omega^n)$. Let $X^* = (x_\eta)_{\eta \in I^n}$, where I is an arbitrary ordered set. We define $\Delta^*(X^*)$ by: For all φ and $F^* \subset_{fin} I^n$,

$$\varphi(x_{F^*}) \in \Delta^* \iff \varphi(x_F) \in \Delta, \text{ for some } F \subset \omega^n \text{ with } F \cong_{\leq_0, \dots, \leq_{n-1}} F^*.$$

Then Δ^* is consistent and is realized by an indiscernible array.

Proof. It is sufficient to show the consistency, since the indiscernibility condition can be added to Δ^* . Let $\varphi_i(x_{F_i^*}) \in \Delta^*$ ($i < m$). Choose $F_i \subset \omega^n$ ($i < m$) witnessing the definition of Δ^* . Then $\varphi_i(a_{F_i})$ holds for all $i < m$. We can also choose $F'_i \subset \omega^n$ such that $F_0^* \dots F_{n-1}^* \cong F'_0 \dots F'_{n-1}$. By the indiscernibility, $\varphi_i(a_{F'_i})$ holds for all $i < m$. This shows that $\bigwedge \varphi_i(x_{F_i^*})$ is satisfiable. $\square$

Lemma 4. Suppose that $\mathcal{E}(Y)$, where $Y = (y_{\langle i,j \rangle} : \langle i,j \rangle \in n \times \omega)$, has the n -dimensional subsequence property. Then $\mathcal{E}(Y)$ is realized by $B = (b_{\langle i,j \rangle} : \langle i,j \rangle \in n \times \omega)$ with the following property:

(**) By letting $B_i = (b_{i,j})_{j \in \omega}$ ($i < n$), B_i is an indiscernible sequence over $\bigcup_{k \neq i} B_k$.

Proof. Easy. $\square$

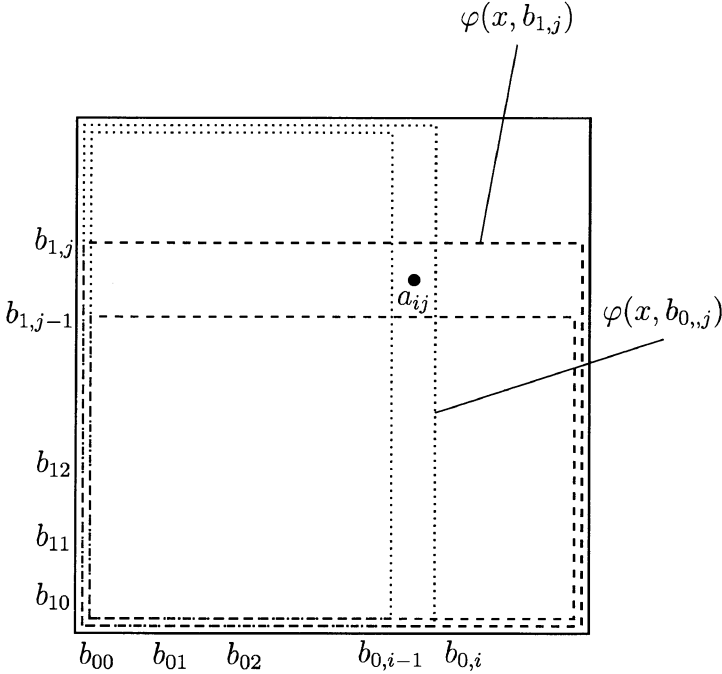
Example 5. Let $\varphi(x, y)$ be a formula. We say that T has n independent orders uniformly defined by φ if there are $A = (a_\eta : \eta \in \omega^n)$ and $B = (b_{i,j})_{\langle i,j \rangle \in n \times \omega}$ such that, for all $\eta \in \omega^n$ and $\langle i, j \rangle \in n \times \omega$,

$$\varphi(a_\eta, b_{ij}) \text{ holds iff } j \geq \eta(i).$$

Let

$$\Gamma(X, Y) := \{\varphi(x_\eta, y_{i,j}) \text{ if } j \geq \eta(i) : \eta \in \omega^n, \langle i, j \rangle \in n \times \omega\}.$$

Then T has n independent orders iff $\Gamma(X, Y)$ is consistent (with T). The set $\Delta(X) := \exists Y \Gamma(X, Y)$ has the subarray property and $\mathcal{E}(Y) := \exists X \Gamma(X, Y)$ has the n -dimensional subsequence property. (Notice that Δ and $\mathcal{E}$ are sets of first-order formulas.)



2-dimensional case

From now on, $\Gamma_{\varphi,n,\omega}(X, Y)$ denotes the set described by the above example. By Lemma 3 (or by a direct argument), $\Gamma_{\varphi,n,\mathbb{Q}}$ is naturally defined. In particular, if T has n independent orders defined by φ , then $\Gamma_{\varphi,n,\mathbb{Q}}(X, Y)$ is consistent, and $\Delta(X) := \exists Y \Gamma_{\varphi,n,\mathbb{Q}}(X, Y)$ has the subarray property. We simply write $\Gamma_{\varphi,n}$ if we are not interested in the ordered set (ω or $\mathbb{Q}$).

Definition 6 (The Number of Independent Orders). Let $m, n \in \omega$. We write

1. $\kappa_{ird}^m(T) \geq n$ if $\Gamma_{\varphi(x,y),n}$ is consistent for some $\varphi(x,y)$ with $|x| = m$.
2. $\kappa_{ird}^m(T) = n$ if $\kappa_{ird}^m(T) \geq n$ and $\kappa_{ird}^m(T) \not\geq n+1$.
3. $\kappa_{ird}^m(T) = \infty$ if $\kappa_{ird}^m(T) \geq n$ ($\forall n$).

Definition 7 (The Number of Independent Strict Orders). Let $\Gamma_{\varphi(x,y),n}^s(X,Y)$ be the set:

$$\Gamma_{\varphi(x,y),n}(X,Y) \cup \bigcup_{j < n} \{\forall x(\varphi(x, y_{i,j}) \rightarrow \varphi(x, y_{i+1,j})) : i \in \omega\}.$$

We write

1. $\kappa_{srd}^m(T) \geq n$ if $\Gamma_{\varphi(x,y),n}^s$ is consistent for some $\varphi(x,y)$ with $|x| = m$.
2. $\kappa_{srd}^m(T) = n$ if $\kappa_{srd}^m(T) \geq n$ and $\kappa_{srd}^m(T) \not\geq n+1$.
3. $\kappa_{srd}^m(T) = \infty$ if $\kappa_{srd}^m(T) \geq n$ ($\forall n$).

The definition of above invariants are due to Shelah, but with a slight modification.

- Remark 8.** 1. Suppose that T has the independence property. Then $\kappa_{ird}^1(T) = \infty$: Since T has the independence property, there is a formula $\varphi(x,y)$ with $|x| = 1$ and $I = (b_i)_{i \in \omega}$ such that $\{\varphi(x, b_i)^{\text{if } i \in F} : i \in \omega\}$ is consistent for any $F \subset \omega$. Choose an indiscernible sequence $I^* = (b_i)_{i \in \omega^2}$ extending I . Then I^* realizes $\exists X \Delta_{\varphi,\omega}(X,Y)$. By compactness, this shows $\kappa_{ird}^1(T) = \infty$.
2. Let T_{rg} be the theory of random graphs. Then $\kappa_{ird}^1(T_{rg}) = \infty$ and $\kappa_{srd}^m(T) = 1$.
3. If T has the order property, then $\kappa_{ird}^m(T) \geq m+1$. If T has the strict order property, then $\kappa_{srd}^m(T) \geq m+1$: Both can be proven similarly. For the case of strict order property, choose $\psi(x,y)$ with $|x| = 1$ and $I = (b_i)$ witnessing the property. For $u = u_0, \dots, u_{m-1}$, let $\varphi_i(u,y) := \psi(u_i, y)$ ($i < m$). Then $\{\varphi_i(u, b_j)^{\text{if } j \geq \eta(i)} : i < m, j \in \omega\}$ is consistent, for any $\eta \in \omega^m$. This shows $\kappa_{srd}^m(T) \geq m+1$, since there is a formula with additional variables such that each φ_i is a specialization of the formula.

References

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