

On the Wellposedness of the Cauchy Problem for Weakly Hyperbolic Operators

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§1. Introduction

We first consider the Cauchy problem on $[0, T] \times \mathbf{R}_x^n$

$$(P_1) \quad \begin{cases} D_t^m u = \sum_{j+|\alpha|=m} A_{j,\alpha}(t) D_t^j D_x^\alpha u + \sum_{j+|\alpha| \leq d} B_{j,\alpha}(t) D_t^j D_x^\alpha u + F(t, x) \\ D_t^j u(0, x) = u_j(x) \quad (j = 0, \dots, m-1), \end{cases}$$

where $D_t = -i\partial_t$, $D_x = -i(\partial_{x_1}, \dots, \partial_{x_n})$ and $0 \leq d \leq m-1$. The coefficients of the principal part depend only on t . We shall write for the principal part

$$p(t, \tau, \xi) = \prod_{k=1}^m (\tau - \lambda_k(t, \xi)) = \tau^m - \sum_{j+|\alpha|=m} A_{j,\alpha}(t) \tau^j \xi^\alpha,$$

and for the lower order terms

$$p_d(t, \tau, \xi) = \sum_{j+|\alpha| \leq d} B_{j,\alpha}(t) \tau^j \xi^\alpha.$$

We assume that the principal part p is hyperbolic with respect to τ , that is, the characteristic roots are all real-valued and named $\lambda_k(t, \xi)$ according to the rule

$$(1) \quad \lambda_1(t, \xi) \geq \lambda_2(t, \xi) \geq \dots \geq \lambda_m(t, \xi).$$

We recall that the functions $\lambda_k(t, \xi)$ are homogeneous of degree 1 in ξ .

In this paper we use the following notations:

$$\Omega_\sigma^k(\xi) = \{t \in [0, T] : |\lambda_k(t, \xi) - \lambda_{k+1}(t, \xi)| \leq \sigma\} \text{ for } 0 < \sigma < 1 \text{ and } 1 \leq k \leq m-1,$$

$$\Omega_\sigma^*(\xi) = \{t \in [0, T] : |\lambda_1(t, \xi) - \lambda_m(t, \xi)| \leq \sigma\}, \quad \Omega_\sigma(\xi) = \bigcup_{k=1}^{m-1} \Omega_\sigma^k(\xi) \text{ for } 0 < \sigma < 1$$

$\mu(S)$ is the Lebesgue measure in $\mathbf{R}_t$ of the set $S \subseteq [0, T]$.

$AC([0, T])$ is the space of absolutely continuous functions on $[0, T]$.

$G^s(\mathbf{R}^n)$ ($s \geq 1$) is the space of Gevrey functions $g(x)$ satisfying,

$$\sup_{x \in K} |D_x^\alpha g(x)| \leq C_K \rho_K^{|\alpha|} |\alpha|!^s \text{ for any compact set } K \subset \mathbf{R}^n \text{ and } \alpha \in \mathbf{N}^n.$$

Our previous result is the following:

THEOREM 0. ([DK]). Assume that $B_{j,\alpha}$ belong to $C^0([0, T])$ and $\lambda_1, \dots, \lambda_m$ belong to $AC([0, T])$ and that there exist $C > 0$, $a \geq 0$ and $b > 0$ such that for any $0 < \sigma < 1$, $|\xi| = 1$,

$$(2) \quad \mu(\Omega_\sigma(\xi)) \leq C\sigma^a,$$

$$(3) \quad \int_{[0, T] \setminus \Omega_\sigma^*(\xi)} \frac{|\lambda'_k(t, \xi)| + |\lambda'_{k+1}(t, \xi)|}{|\lambda_k(t, \xi) - \lambda_{k+1}(t, \xi)|} dt \leq C\sigma^{-b} \text{ for } 1 \leq k \leq m-1,$$

Then the Cauchy problem (P_1) is wellposed in G^s if

$$(4) \quad 1 \leq s < \begin{cases} 1 + \frac{a+1}{b} & \text{when } d \leq \frac{m(a+b)}{a+b+1}, \\ \frac{m}{d+a(d-m)} & \text{when } d > \frac{m(a+b)}{a+b+1}, \end{cases}$$

i.e., for any data $u_j \in G^s(\mathbf{R}^n)$ and $f \in C^0([0, T]; G^s(\mathbf{R}^n))$ the Cauchy problem (P_1) has a unique solution $u \in C^m([0, T]; G^s(\mathbf{R}^n))$.

Remark 1. If $\lambda_1, \dots, \lambda_m$ and $\psi_1, \dots, \psi_{m-1}$ are analytic in t and vanish at $t = 0$ and there exist $C > 0$, $c > 0$ and $0 < \alpha < \beta$ such that for any $t \in [0, T]$, $|\xi| = 1$

$$(5) \quad |\lambda_k(t, \xi)| \leq Ct^\alpha \text{ for } 1 \leq k \leq m,$$

$$(6) \quad |\lambda_{k+1}(t, \xi) - \lambda_k(t, \xi)| \geq ct^\beta \text{ for } 1 \leq k \leq m-1,$$

we can take $a = 1/\beta$ in (2) and $b = 1 - \alpha/\beta$ in (3). Then the Cauchy problem (P_1) when $B_{j,\alpha} \equiv 0$ is wellposed in G^s if

$$(7) \quad 1 \leq s < 1 + \frac{\beta+1}{\beta-\alpha}.$$

Remark 2. When $\lambda_1, \dots, \lambda_m$ vanish of infinite order, (2) can be dropped (one is forced to choose $a = 0$). Then the Cauchy problem (P_1) is wellposed in G^s if

$$1 \leq s < \min \left\{ 1 + \frac{1}{b}, \frac{m}{d} \right\}.$$

In case of constant multiplicity, we can take $a = 0$ in (2), but (3) does not make a sense, since $[0, T] \setminus \Omega_\sigma^k(\xi) = \phi$. From Remark 8 in §3, we may take $b = 1$. Then the Cauchy problem (P_1) is wellposed in G^s if $1 \leq s < \min\{2, m/d\}$.

Remark 3. When $\lambda_1, \dots, \lambda_m$ belong to $C^1([0, T])$, we can take $0 \leq a < 1$ and $0 < b \leq 1 - a$ (see Lemma 3 and Remark 7). Then the Cauchy problem (P_1) is wellposed in G^s if

$$1 \leq s < \min \left\{ \frac{2}{1-a}, \frac{m}{d+a(d-m)} \right\}.$$

We next consider the Cauchy problem on $[0, T] \times \mathbf{R}_x^n$

$$(P_2) \quad \begin{cases} D_t U = \sum_{|\alpha|=1} A_\alpha(t) D_x^\alpha U + B(t)U + F(t, x) \\ U(0, x) = U_0(x), \end{cases}$$

where $A_\alpha(t)$ and $B(t)$ are $m \times m$ matrices.

By the Cauchy-Kowalevski Theorem any type of systems can be solvable locally in the analytic class. Weakly hyperbolic systems are solvable globally in the Gevrey classes of order $1 \leq s < m/(m-1)$. In this paper the main result is the following:

THEOREM 1. Assume that A_α and B belong to $C^{m-1}([0, T])$ and $\lambda_1, \dots, \lambda_m$ belong to $AC([0, T])$ and satisfy (2) and (3). Then the Cauchy problem (P_2) is wellposed in G^s if

$$(8) \quad 1 \leq s < \min \left\{ 1 + \frac{a+1}{b}, \frac{m}{m-1-a} \right\},$$

i.e., for any data $U_0 \in G^s(\mathbf{R}^n)$ and $F \in C^{m-1}([0, T]; G^s(\mathbf{R}^n))$ the Cauchy problem (P_2) has a unique solution $U \in C^1([0, T]; G^s(\mathbf{R}^n))$.

Remark 4. When $\lambda_1, \dots, \lambda_m$ belong to $C^1([0, T])$, we can take $0 \leq a < 1$ and $0 \leq b \leq 1 - a$ (see Lemma 3 and Remark 7). Then the Cauchy problem (P_2) is wellposed in G^s if

$$1 \leq s < \min \left\{ \frac{2}{1-a}, \frac{m}{m-1-a} \right\} = \frac{m}{m-1-a}.$$

We also consider the special case when

$$\sum_{|\alpha|=1} A_\alpha(t) D_x^\alpha \equiv \begin{pmatrix} \lambda_1(t, D_x) & \psi_1(t, D_x) & 0 & \dots & 0 \\ 0 & \lambda_2(t, D_x) & \psi_2(t, D_x) & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \lambda_{m-1}(t, D_x) & \psi_{m-1}(t, D_x) \\ 0 & \dots & 0 & \lambda_m(t, D_x) \end{pmatrix},$$

where $\lambda_k(t, \xi)$ ($1 \leq k \leq m$) and $\psi_k(t, \xi)$ ($1 \leq k \leq m-1$) are homogeneous of degree 1 in ξ .

THEOREM 2. Assume that B belongs to $C^{m-1}([0, T])$ and $\lambda_1, \dots, \lambda_m$ and $\psi_1, \dots, \psi_{m-1}$ belong to $C^{m-1}([0, T])$ and satisfy (2) and that there exists $C > 0$ and $b > 0$ such that for any $0 < \sigma < 1$, $|\xi| = 1$

$$(9) \quad \int_{[0, T] \setminus \Omega_\sigma^*(\xi)} \frac{|\lambda'_k(t, \xi)|}{|\lambda_1(t, \xi) - \lambda_m(t, \xi)|} dt \leq C\sigma^{-b} \text{ for } 1 \leq k \leq m,$$

$$(10) \quad \int_{[0, T] \setminus \Omega_\sigma^*(\xi)} \frac{|\psi'_k(t, \xi)|}{|\lambda_1(t, \xi) - \lambda_m(t, \xi)|} dt \leq C\sigma^{-b} \text{ for } 1 \leq k \leq m-1.$$

Then the Cauchy problem (P_2) is wellposed in G^s if

$$(11) \quad 1 \leq s < 1 + \frac{a+1}{b+m-2}.$$

Remark 5. (9) is a more relaxed condition than (3) (when $m = 2$, (9) is the same as (3)). Since $\lambda_1, \dots, \lambda_m$ belong to $C^{m-1}([0, T]) \subset C^1([0, T])$, we find that $b \leq 1 - a$ and by Theorem 1 we get the wellposedness in G^s if

$$1 \leq s < \frac{m}{m-1-a} \left(= 1 + \frac{a+1}{1-a+m-2} \right) \leq 1 + \frac{a+1}{b+m-2}$$

(see Remark 4). The Gevrey index (11) for this special case is larger than (8).

Example A. Theorem 2 can be also applied when the principal part has the form of the Jordan matrix:

$$\sum_{|\alpha|=1} A_\alpha(t) D_x^\alpha \equiv \sum_{|\alpha|=1} \begin{pmatrix} \lambda_1(t) & 1 & 0 & \dots & 0 \\ 0 & \lambda_2(t) & 1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \lambda_{m-1}(t) & 1 \\ 0 & \dots & & 0 & \lambda_m(t) \end{pmatrix} D_x^\alpha.$$

Since (10) is valid for any $0 < b (\leq 1 - a)$, with only (9) we obtain (11).

Example B. Yamahara [Y] studied 4 by 4 systems with the principal part

$$\sum_{|\alpha|=1} A_\alpha(t) D_x^\alpha \equiv \begin{pmatrix} \lambda t^\beta & 1 & 0 & 0 \\ 0 & \lambda t^\beta & \psi t^\alpha & 0 \\ 0 & 0 & \mu t^\beta & 1 \\ 0 & 0 & 0 & \mu t^\beta \end{pmatrix} D_x, \text{ where } \alpha, \beta > 0, \lambda \neq \mu \text{ and } \psi \neq 0.$$

and proved that the Cauchy problem (P_2) is wellposed in G^s if

$$(12) \quad 1 \leq s < \begin{cases} 2 & \text{for } \alpha \geq 2\beta, \\ 1 + \frac{\beta}{3\beta - \alpha} & \text{for } 0 < \alpha < 2\beta, \end{cases}$$

and the Cauchy problem (P_2) is not wellposed in G^s otherwise. In particular, let us consider the case when $0 < \alpha < \beta$. Taking $m = 4$, $a = 0$ and $b = 1 - \alpha/\beta$ which is determined by (10), by Theorem 2 we can also get the optimal Gevrey index (12) if $0 < \alpha < \beta$.

§2. Sketch of Proof of Theorem 0

When $s = 1$, the Cauchy problem (P_1) is wellposed in the class of real analytic functions. Therefore we can suppose that $s > 1$ for the proof. By Fourier transform with respect to x , the Cauchy problem (P_1) turns into

$$\begin{cases} p(t, D_t, \xi) \hat{u} = \hat{f}(t, \xi) + p_d(t, D_t, \xi) \hat{u} \\ D_t^j \hat{u}(0, \xi) = \hat{u}_j(\xi) \quad (j = 0, \dots, m-1). \end{cases}$$

Let $0 < \sigma < 1$ and $\varphi(r)$ be a non-negative function such that $\varphi \in C_0^\infty(\mathbf{R})$, $\varphi(r) \equiv 0$ for $|r| \geq 2$ and $\varphi(r) \equiv 1$ for $|r| \leq 1$. We define

$$\begin{aligned} \omega(t, \xi) &= \sigma |\xi| \sum_{l=1}^{m-1} \varphi \left(\sigma^{-1} \left\{ \lambda_l \left(t, \frac{\xi}{|\xi|} \right) - \lambda_{l+1} \left(t, \frac{\xi}{|\xi|} \right) \right\} \right), \\ \mu_k(t, \xi) &= \lambda_k(t, \xi) + ik\omega(t, \xi) \quad \text{for } k = 1, \dots, m. \end{aligned}$$

Moreover we denote by $q(t, \tau, \xi)$ the polynomial of degree m in τ

$$q(t, \tau, \xi) = \prod_{k=1}^m (\tau - \mu_k(t, \xi)).$$

Now we set the energy density

$$E(t, \xi) = \frac{1}{2} \sum_{l=1}^m |q_l(t, D_t, \xi) \hat{u}|^2,$$

where $q_l(t, \tau, \xi)$ is the polynomial of degree $m-1$ in τ defined by

$$q_l(t, \tau, \xi) = \frac{q(t, \tau, \xi)}{\tau - \mu_l(t, \xi)} \left(= \prod_{k=1, k \neq l}^m (\tau - \mu_k(t, \xi)) \right).$$

Hence we can derive the energy estimate

$$\sqrt{E(t, \xi)} \leq \exp \left\{ C \int_0^T \left(\max_{1 \leq k \leq m-1} \frac{|\lambda'_k| + |\lambda'_{k+1}| + |\omega'|}{|\lambda_k - \lambda_{k+1}| + \omega} + \omega + \frac{|\xi|^d}{\prod_{k=1}^{m-1} |\lambda_k - \lambda_{k+1}| + \omega^m} \right) \right. \\ \left. \times \left\{ \sqrt{E(0, \xi)} + \int_0^T |\hat{f}(t, \xi)| dt \right\} \right\}.$$

By (2)-(4), there exist $C > 0$ and $\rho > 0$ such that for any $(t, \xi) \in [0, T] \times \mathbf{R}_\xi^n \setminus 0$

$$\sqrt{E(t, \xi)} \leq C \exp \{ \rho |\xi|^{\frac{1}{2}} \} \left\{ \sqrt{E(0, \xi)} + \int_0^T |\hat{f}(t, \xi)| dt \right\}.$$

§3. Sketch of Proof of Theorem 1

In Theorem 1 the coefficients A_α , B and F belong to $C^{m-1}([0, T])$ and the assumptions for the characteristic roots of the principal part are the same as Theorem 0. So, we shall use the result of Theorem 0. Let $C(t, \tau, \xi)$ be the cofactor matrix of $\tau I - \sum_{|\alpha|=1} A_\alpha(t) \xi^\alpha$, i.e.,

$$\begin{cases} C(t, \tau, \xi) \cdot \left\{ \tau I - \sum_{|\alpha|=1} A_\alpha(t) \xi^\alpha \right\} = p(t, \tau, \xi) I, \\ \left\{ \tau I - \sum_{|\alpha|=1} A_\alpha(t) \xi^\alpha \right\} \cdot C(t, \tau, \xi) = p(t, \tau, \xi) I, \end{cases}$$

where I is the $m \times m$ identity matrix and the polynomial p has degree m :

$$p(t, \tau, \xi) = \det \left\{ \tau I - \sum_{|\alpha|=1} A_\alpha(t) \xi^\alpha \right\}.$$

Multiplying $\left\{ D_t - \sum_{|\alpha|=1} A_\alpha(t) D_x^\alpha \right\}$ by $C(t, D_t, D_x)$, we have the following operator:

$$(13) \quad p(t, D_t, D_x) I + G(t, D_t, D_x),$$

where G contains all the coefficients A_α , B and their derivatives up to order $m-1$ in each row. Applying Theorem 0 into (13), we get the existence and the uniqueness.

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