

On the volumes of integral convex polytopes satisfying
certain conditions

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Let P be an integral convex polytope in $\mathbb{R}^n$, i.e., P is the convex hull of finite points of $\mathbb{Z}^n$. Assume that $\dim P = n$ and that $\text{Int}(P) \cap \mathbb{Z}^n = \{0\}$. Then the dual polytope $P^* := \{v \in \mathbb{R}^n \mid \langle v, u \rangle \geq -1 \text{ for all } u \in P\}$ is the convex hull of finite points of $\mathbb{Q}^n$. Let X_P be the union of $X_F := \text{Spec} \mathbb{C}[(\mathbb{R}_{\geq 0}^F)^* \cap \mathbb{Z}^n]$ for all faces F of P , where $(\mathbb{R}_{\geq 0}^F)^* = \{v \in \mathbb{R}^n \mid \langle v, u \rangle \geq 0 \text{ for all } u \in \mathbb{R}_{\geq 0}^F\}$. Then X_P is a compact toric variety whose anti-canonical divisor is ample and $(-K_{X_P})^n = n! \text{vol}(P^*)$. On the other hand, Hensley[3] showed that the volume of P has a upper-bound $K(n)$. However, $K(n)$ is much greater than the volume of the following example P_m first given by Zaks, Perles and Wills[4] and no examples are known whose volumes are greater than $\text{vol}(P_m)$.

Example. Assume that $n \geq 3$. Let $y_1 = 2$, $y_2 = 3$ and $y_k = y_1 y_2 \cdots y_{k-1} + 1$ for $k \geq 3$. Then $y_1^{-1} + \cdots + y_{n-1}^{-1} + (y_n - 1)^{-1} = 1$. Let P_m be the convex hull of the $n+1$ points ${}^t(y_1-1, -1, \dots, -1)$, ${}^t(-1, y_2-1, -1, \dots, -1)$, $\dots$, ${}^t(-1, \dots, -1, y_{n-1}-1, -1)$, ${}^t(-1, \dots, -1, 2(y_n-1)-1)$, ${}^t(-1, -1, \dots, -1)$.

Then $\text{Int}(P_m) \cap \mathbb{Z}^n = \{0\}$, P_m^* is integral and $\text{vol}(P_m) = \frac{1}{n!} y_1 \cdots y_{n-1}^2 (y_n - 1) = \frac{2}{n!} (y_n - 1)^2$.

In this note, we show that the above example has the maximal volume among n -dimensional integral convex polytopes satisfying certain additional conditions, if $n \geq 3$.

Theorem. If $n \geq 3$ and P is an n -dimensional integral simplex in $\mathbb{R}^n$ such that $\text{Int}(P) \cap \mathbb{Z}^n = \{0\}$ and that P^* is also integral, then $\text{vol}(P) \leq \text{vol}(P_m)$.

Remark. (1) If P is a simplex, then X_P is isomorphic to the quotient of $\mathbb{P}^n$ under the action of a finite subgroup of $(\mathbb{C}^\times)^n$. (2) X_P is $\mathbb{Q}$ -factorial, i.e., X_P is a $\mathbb{Q}$ -Fano variety, if and only if each $(n-1)$ -dimensional face of P is a simplex. In particular, if P is a simplex, then X_P is a $\mathbb{Q}$ -Fano variety.

Assume that $n \geq 3$. Let P be an n -dimensional integral simplex, i.e., P is the convex hull of $n + 1$ points $u_1, u_2, \dots$ and u_{n+1} of $\mathbb{Z}^n$ such that $u_1 - u_{n+1}, u_2 - u_{n+1}, \dots$ and $u_n - u_{n+1}$ are linearly independent. Assume that $\text{Int}(P) \cap \mathbb{Z}^n = \{0\}$. Then the dual polytope P^* is the convex hull of the $n + 1$ points $v_1, v_2, \dots, v_{n+1}$ of $\mathbb{Q}^n$ defined by $\langle v_j, u_k \rangle = -1$ if $j \neq k$. Also assume that P^* is integral,

i.e., $v_j \in \mathbf{Z}^n$. Hence $a_j := \langle v_j, u_j \rangle + 1$ are positive integers. Let $u_0 = a_1^{-1}u_1 + a_2^{-1}u_2 + \dots + a_{n+1}^{-1}u_{n+1}$. Then $\langle v_j, u_0 \rangle = -a_1^{-1} - \dots - a_{j-1}^{-1} + a_j^{-1}(a_j - 1) - a_{j+1}^{-1} - \dots - a_{n+1}^{-1} = 1 - \sum_{k=1}^{n+1} a_k^{-1}$ for all j . Since $0 \in P^* = \overline{v_1 v_2 \dots v_{n+1}}$, we have:

Proposition 1. $u_0 = 0$ and $a_1^{-1} + a_2^{-1} + \dots + a_{n+1}^{-1} = 1$.

Proposition 2. $\text{vol}(P) \leq \frac{1}{n!} a_1 a_2 \dots a_n$.

Proof. Let L be the sublattice of $\mathbf{Z}^n$ generated by $u_1 - u_{n+1}, u_2 - u_{n+1}, \dots$ and $u_n - u_{n+1}$. Then $n! \text{vol}(P) = [\mathbf{Z}^n : L]$. On the other hand, $\langle v_j, a_k^{-1}(u_k - u_{n+1}) \rangle = \delta_{jk}$ for $1 \leq j, k \leq n$. Hence the $\mathbf{Z}$ -module M generated by $a_1^{-1}(u_1 - u_{n+1}), a_2^{-1}(u_2 - u_{n+1}), \dots$ and $a_n^{-1}(u_n - u_{n+1})$ contains $\mathbf{Z}^n$. Therefore, $[\mathbf{Z}^n : L] \leq [M : L] = a_1 a_2 \dots a_n$. q.e.d.

Hence the theorem follows from the following proposition.

Proposition 3. Let $a_1, a_2, \dots$ and a_{n+1} be positive integers such that $a_1 \leq a_2 \leq \dots \leq a_{n+1}$ and that $a_1^{-1} + a_2^{-1} + \dots + a_{n+1}^{-1} = 1$. Then $a_1 a_2 \dots a_n \leq y_1 y_2 \dots y_{n-1} 2^{(y_n - 1)}$.

For the proof we need the following two lemmas.

Lemma 4. Let $a_1, a_2, \dots$ and a_{n+1} be those as in Proposition 3. Then there exist positive integers b_{n-1} and b_n satisfying the following three conditions.

$$(C1) \quad a_{n-1}, b_{n-1} \leq b_n.$$

$$(C2) \quad b_{n-1}^{-1} + b_n^{-1} \leq a_{n-1}^{-1} + a_n^{-1} + a_{n+1}^{-1} < b_{n-1}^{-1} + (b_{n-1})^{-1}.$$

$$(C3) \quad 2b_{n-1}b_n \geq a_{n-1}a_n.$$

Proof. Since $n \geq 3$ and $a_{n-2} \leq a_{n-1}$, we have $a_{n-1} \geq 3$. There exists a integer q greater than 1 such that $q^{-1} \leq a_n^{-1} + a_{n+1}^{-1} < (q-1)^{-1}$. Then $2q \geq a_n$, because $a_n \leq a_{n+1}$. Hence if $q \geq a_{n-1}$, then $b_{n-1} = a_{n-1}$ and $b_n = q$ satisfy C1, C2 and C3. Now assume that $q < a_{n-1}$. Then $a_{n-1}^{-1} + a_n^{-1} + a_{n+1}^{-1} > 2a_{n-1}^{-1}$.

(I) First, we consider the case that a_{n-1} is even, i.e., $a_{n-1} = 2r$ for a positive integer r . Then $r^{-1} < a_{n-1}^{-1} + a_n^{-1} + a_{n+1}^{-1} < 1$. Hence there exists a positive integer b_n such that $b_{n-1} = r$ and b_n satisfy C2. Since $r^{-1} + a_n^{-1} = 2a_{n-1}^{-1} + a_n^{-1} \geq a_{n-1}^{-1} + a_n^{-1} + a_{n+1}^{-1}$, we have $b_n \geq a_n$. Hence b_{n-1} and b_n satisfy C1 and C3.

(II) Next, we consider the case that a_{n-1} is odd, i.e., $a_{n-1} = 2r + 1$ for a positive integer r . Then $q \leq 2r$.

(i) Assume that $q < 2r$. Then $1 > a_{n-1}^{-1} + a_n^{-1} + a_{n+1}^{-1} \geq (2r+1)^{-1} + q^{-1} \geq (2r+1)^{-1} + (2r-1)^{-1} > r^{-1}$. Hence there exists a positive integer b_n such that $b_{n-1} = r$ and b_n satisfy C2. Since $(r^{-1} + (a_n + 1)^{-1}) - (a_{n-1}^{-1} + a_n^{-1} + a_{n+1}^{-1}) = ((2r+1)^{-1} - a_{n+1}^{-1}) + (r^{-1}(2r+1)^{-1} - a_n^{-1}(a_n + 1)^{-1}) > 0$, we have $b_n \geq a_n +$

2. Hence b_{n-1} and b_n satisfy C3, because $2rb_n - a_{n-1}a_n \geq 2r(a_n + 2) - (2r+1)a_n = 4r - a_n \geq 4r - 2q > 0$.

(ii) Assume that $q = 2r$. Suppose that $r = 1$. Then $q^{-1} \leq 1 - a_{n-2}^{-1} - a_{n-1}^{-1} \leq 1 - 2(2r+1)^{-1} = 3^{-1}$. Hence $r = \frac{q}{2} > 1$. (ii-i)

Assume that $r^{-1} < a_{n-1}^{-1} + a_n^{-1} + a_{n+1}^{-1}$. Then there exists a positive integer b_n such that $b_{n-1} = r$ and b_n satisfy C2. Since $a_{n-1}^{-1} + a_n^{-1} + a_{n+1}^{-1} < (2r+1)^{-1} + (2r-1)^{-1} = 4r(4r^2-1)^{-1} = r^{-1} + r^{-1}(4r^2-1)^{-1}$, we have $b_n > r(4r^2-1) > 2(2r+1)$. Hence

$2rb_n > (2r+1)4r \geq a_{n-1}a_n$. Therefore, b_{n-1} and b_n satisfy C1

and C3. (ii-ii) Assume that $a_{n-1}^{-1} + a_n^{-1} + a_{n+1}^{-1} \leq r^{-1}$. Since $(r+1)^{-1} < (2r+1)^{-1} + (2r)^{-1} = a_{n-1}^{-1} + q^{-1}$, there exists a

positive integer b_n such that $b_{n-1} = r+1$ and b_n satisfy C2. Then $b_n \geq r(r+1)$, because $r^{-1} = (r+1)^{-1} + r^{-1}(r+1)^{-1}$.

Hence b_{n-1} and b_n satisfy C1. Moreover, $2b_{n-1}b_n \geq 2r(r+1)^2 > (2r+1)4r \geq a_{n-1}a_n$, if $r \geq 3$. Finally, we consider the case

that $r = 2$. Then $a_n \leq 2q = 8$. If $a_n \leq 7$, then $2b_{n-1}b_n \geq 2 \cdot 3 \cdot 6 > 35 \geq a_{n-1}a_n$. If $a_n = 8$, then $a_{n-1}^{-1} + a_n^{-1} + a_{n+1}^{-1} \leq 5^{-1} + 8^{-1} + 8^{-1} = 9/20$. Hence $b_n \geq (9/20 - 1/3)^{-1} > 8$. Therefore,

$2b_{n-1}b_n \geq 2 \cdot 3 \cdot 9 > a_{n-1}a_n$. q.e.d.

Lemma 5. Let $a_1, a_2, \dots$ and a_n be positive integers such that $a_1 \leq a_2 \leq \dots \leq a_n$. If $a_1^{-1} + a_2^{-1} + \dots + a_n^{-1} < ($ resp. $\leq)$ $1 \leq ($ resp. $<)$ $a_1^{-1} + \dots + a_{n-1}^{-1} + (a_n - 1)^{-1}$, then $a_1a_2 \dots a_n \leq y_1y_2 \dots y_n$ (resp. $y_1 \dots y_{n-1}(y_n - 1)$).

Proof. In the case that $n = 3$, it is easy to verify the lemma. So assume that $n \geq 4$.

Sublemma. Let ε be a positive real number not greater than $1/2$, let a, b, c and d be positive integers with $c \leq d$. Assume that $a^{-1} + b^{-1} < (\text{resp. } \leq) \varepsilon \leq (a-1)^{-1}$ and that $c^{-1} + d^{-1} < (\text{resp. } \leq) \varepsilon \leq (\text{resp. } <) c^{-1} + (d-1)^{-1}$. Then $ab \geq cd$.

Proof. Since $a^{-1} + b^{-1} \leq (a-1)^{-1}$, we have $b \geq a(a-1)$. Clearly, $a \leq c$. If $a = c$, then $b \geq d$ and hence $ab \geq cd$. So assume that $c - a \geq 1$. Since $a^{-1} + b^{-1} < c^{-1} + (d-1)^{-1}$, we have $b > ac(d-1)(a(c+d-1) - c(d-1))^{-1}$. Hence $ab - cd > c(d((c-a)(d-a) - c) + a(d-a))(a(c+d-1) - c(d-1))^{-1}$. Here we note that $a(c+d-1) - c(d-1) > 0$, because $a^{-1} < c^{-1} + (d-1)^{-1}$. Hence if $E := (c-a)(d-a) - c > 0$, then $ab > cd$. Since $c - a \geq 1$, if $d \geq a + c$, then $ab > cd$. So we assume that $d < a + c$. Since $(a-1)^{-1} \geq c^{-1} + d^{-1} > c^{-1} + (2c)^{-1}$, we have $c - a > \frac{c}{3} - 1$.

(I) First, we consider the case that $c - a \geq 3$. Then $(c-a)^2 \geq 3(c-a) > c-3$. Hence $E > (c-a)(d-a) - (c-a)^2 - 3 = (c-a)(d-c) - 3 \geq (c-a)(d-c-1)$. Therefore, if $d > c$, then $ab > cd$. (i) Assume that $c = d$ is even, i.e., $c = d = 2r$ for a positive integer r . Since $r^{-1} = c^{-1} + d^{-1} \leq \varepsilon \leq c^{-1} + (d-1)^{-1} < (r-1)^{-1}$, $a = r$ or $r+1$. When

$a = r + 1$, $ab \geq a^2(a - 1) = r(r + 1)^2 \geq (2r)^2 = cd$. When $a = r$, $b^{-1} \leq \varepsilon - r^{-1} < (2r)^{-1} + (2r - 1)^{-1} - r^{-1} = (2r)^{-1}(2r - 1)^{-1}$. Hence $ab \geq 2r^2(2r - 1) > (2r)^2 = cd$, because $r = \frac{c}{2} \geq \frac{a+3}{2} > 2$. (ii) Assume that $c = d$ is odd, i.e., $c = d = 2r + 1$ for a positive integer r . Since $(r + 1)^{-1} < c^{-1} + d^{-1} \leq \varepsilon \leq c^{-1} + (d - 1)^{-1} < r^{-1}$, $a = r + 1$. Hence $b^{-1} \leq \varepsilon - a^{-1} \leq c^{-1} + (d - 1)^{-1} - a^{-1} = (3r + 1)(2r(2r + 1)(r + 1))^{-1}$. Therefore, $ab \geq 2r(r + 1)^2(2r + 1)(3r + 1)^{-1} > (2r + 1)^2 = cd$, because $r \geq 3$.

(II) Next, we consider the case that $c - a = 2$. Since $a \geq 3$ and $\frac{c}{3} - 1 < c - a = 2$, we have $5 \leq c \leq 8$. (i) When $c = 8$ (resp. 7), $d < a + c = 14$ (resp. 12) and $b \geq a(a - 1) = 30$ (resp. 20). Hence $ab \geq 6 \cdot 30 > 8 \cdot 14 > cd$ (resp. $ab \geq 5 \cdot 20 > 7 \cdot 12 > cd$). (ii) When $c = 6$, $ab \geq a^2(a - 1) = 48$. If $d < 8$, then $cd < ab$. If $d \geq 8$, then $E \geq 2(8 - 4) - 6 > 0$ and hence $ab > cd$. (iii) When $c = 5$, $a = 3$. If $d \geq 6$, then $E \geq 2(6 - 3) - 5 > 0$. If $d = 5$, then $b^{-1} \leq c^{-1} + (d - 1)^{-1} - a^{-1} = 7/60 < 1/8$ and hence $ab \geq 3 \cdot 9 > 5 \cdot 5 = cd$.

(III) Finally, we consider the case that $c - a = 1$. Since $a \geq 3$ and $\frac{c}{3} - 1 < 1$, we have $4 \leq c \leq 5$. (i) When $c = 5$, $b \geq a(a - 1) = 12$ and $d < a + c = 9$. Hence $ab \geq 4 \cdot 12 > 5 \cdot 9 > cd$. (ii) When $c = 4$, $b \geq 6$ and $4 \leq d < 7$. If $d = 4$, then $cd = 4 \cdot 4 < 3 \cdot 6 \leq ab$. If $d = 5$, then $b^{-1} < c^{-1} + (d - 1)^{-1} - a^{-1} = 1/6$ and hence $ab \geq 3 \cdot 7 > 4 \cdot 5 = cd$. If $d = 6$, then $b^{-1} < 7/60 < 1/8$ and hence $ab \geq 3 \cdot 9 > 4 \cdot 6 = cd$. q.e.d.

Proof of Lemma 5 continued. Suppose that $a_1^{-1} + a_2^{-1} + \dots + a_n^{-1} < (\text{resp. } \leq) 1 \leq (\text{resp. } <) a_1^{-1} + \dots + a_{n-1}^{-1} + (a_n - 1)^{-1}$ and that $a_1 a_2 \dots a_n > y_1 y_2 \dots y_n$ ($\text{resp. } y_1 \dots y_{n-1}(y_n - 1)$). Let $\varepsilon = 1 - a_1^{-1} - \dots - a_{n-2}^{-1}$. Then $a_{n-1}^{-1} + a_n^{-1} < (\text{resp. } \leq) \varepsilon \leq (\text{resp. } <) a_{n-1}^{-1} + (a_n - 1)^{-1}$ and $\varepsilon \leq \frac{1}{2}$, because $n \geq 4$. Hence we may assume that $a_1^{-1} + a_2^{-1} + \dots + a_{n-1}^{-1} < 1 \leq a_1^{-1} + \dots + a_{n-2}^{-1} + (a_{n-1} - 1)^{-1}$, by using the above sublemma repeatedly. On the other hand, $(a_n - 1)^{-1} \geq (\text{resp. } >) 1 - a_1^{-1} - a_2^{-1} - \dots - a_{n-1}^{-1} \geq 1 - y_1^{-1} - y_2^{-1} - \dots - y_{n-1}^{-1} = (y_n - 1)^{-1}$, by [1,2]. Hence $a_n \leq y_n$ ($\text{resp. } \leq y_n - 1$). Therefore, $a_1 a_2 \dots a_{n-1} > y_1 y_2 \dots y_{n-1}$. By the induction for n , we have $a_1 a_2 a_3 > y_1 y_2 y_3$, a contradiction.

q.e.d.

Proof of Proposition 3. By Lemma 4, there exist positive integers b_{n-1} and b_n such that $b_n = \max\{b_j \mid 1 \leq j \leq n\}$, that $b_1^{-1} + b_2^{-1} + \dots + b_n^{-1} \leq 1 < b_1^{-1} + \dots + b_{n-1}^{-1} + (b_n - 1)^{-1}$ and that $2b_{n-1}b_n \geq a_{n-1}a_n$, where $b_j = a_j$ for $j = 1$ through $n-2$. Hence $a_1 a_2 \dots a_n \leq 2b_1 b_2 \dots b_n \leq 2y_1 \dots y_{n-1}(y_n - 1)$, by Lemma 5.

q.e.d.

References

- [1] Curtis, On Kellogg's Diophantine Problem, Amer. Math. Monthly, 29(1922), 380-387.
- [2] P. Erdős, Az $\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} = \frac{a}{b}$ egyenlet egész számú megoldásairól, Math Lapok, (1950), 192-210.
- [3] D. Hensley, Lattice vertex polytopes with interior lattice points, Pacific J. of Math., Vol. 105 No. 1 (1983), 183-191.
- [4] J. Zaks, M. A. Perles and J. M. Wills, On the lattice polytopes having interior lattice points, Elemente der Math., 37 No. 2, (1982), 44-46.

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