

Legendrian dualities and their applications

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Dedicated to Professor Donghe Pei on the occasion of his 60th birthday

and Professor Weizhi Sun on the occasion of his 70th birthday

Abstract

This paper presents a review of Legendrian dualities. These dualities were initially developed by S. Izumiya and subsequently generalized by S. Izumiya and the author. They serve as fundamental tools for researching the extrinsic differential geometry of submanifolds in non-flat spaces.

Keywords: Legendrian duality; non-flat space; front; caustic; semi-Euclidean space.

1 Introduction

S. Izumiya demonstrated a theorem named Legendrian dualities for pseudo spheres in Minkowski space in [7, 8]. This theorem has since become a fundamental tool for studying the extrinsic differential geometry of submanifolds immersed in these pseudo spheres from the perspective of singularity theory (cf., [3, 4, 5, 9, 10, 11, 12]). Subsequently, S. Izumiya and the author of this paper developed similar Legendrian dualities between pseudo spheres in general semi-Euclidean spaces [2]. In this paper, we will provide a review of these dualities and present some aspects of their applications, though this review may be somewhat limited.

We first prepare basic notions on semi-Euclidean space. Let $\mathbb{R}^{n+1} = \{(x_1, \dots, x_{n+1}) \mid x_i \in \mathbb{R}, i = 1, \dots, n+1\}$ be an $(n+1)$ -dimensional vector space. For any vectors $\mathbf{x} = (x_1, \dots, x_{n+1})$, $\mathbf{y} = (y_1, \dots, y_{n+1})$ in $\mathbb{R}^{n+1}$, the *pseudo scalar product* of $\mathbf{x}$ and $\mathbf{y}$ is defined by $\langle \mathbf{x}, \mathbf{y} \rangle = -\sum_{i=1}^r x_i y_i + \sum_{i=r+1}^{n+1} x_i y_i$. The space $(\mathbb{R}^{n+1}, \langle, \rangle)$ is called *semi-Euclidean $(n+1)$ -space with index r* and denoted by $\mathbb{R}_r^{n+1}$. We say that a vector $\mathbf{x}$ in $\mathbb{R}_r^{n+1} \setminus \{\mathbf{0}\}$ is *spacelike*, *null* or *timelike* if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$, $= 0$ or < 0 respectively. The norm of the vector $\mathbf{x} \in \mathbb{R}_r^{n+1}$ is defined by $\|\mathbf{x}\| = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$. We have the following three kinds of pseudo-spheres in $\mathbb{R}_r^{n+1}$: The *pseudohyperbolic n -space with index $r-1$* is defined by

$$H_{r-1}^n = \{\mathbf{x} \in \mathbb{R}_r^{n+1} \mid \langle \mathbf{x}, \mathbf{x} \rangle = -1\},$$

the *pseudo n -sphere with index r* by

$$S_r^n = \{\mathbf{x} \in \mathbb{R}_r^{n+1} \mid \langle \mathbf{x}, \mathbf{x} \rangle = 1\}$$

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and the (*open*) *nullcone* by

$$\Lambda^n = \{\mathbf{x} \in \mathbb{R}_r^{n+1} \setminus \{\mathbf{0}\} \mid \langle \mathbf{x}, \mathbf{x} \rangle = 0\}.$$

In relativity theory $\mathbb{R}_1^{n+1}$ is called *Minkowski* $(n+1)$ -space, S_1^n *de Sitter* n -space and H_1^n is *Anti de Sitter* n -space which is denoted by AdS^n . These are the Lorentzian space forms. Moreover, H_0^n is called *hyperbolic* n -space and S_0^n is the Euclidean unit sphere which are the Riemannian space forms.

2 Legendrian dualities

In [7], S. Izumiya has shown the basic duality theorem which is the fundamental tool for the study of spacelike hypersurfaces in Minkowski pseudo-spheres and then we generalize the similar dualities in semi-Euclidean space in [2]. We now introduce these dualities as follows:

- (1) (a) $H_{r-1}^n \times S_r^n \supset \Delta_1 = \{(\mathbf{v}, \mathbf{w}) \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0\}$,
 (b) $\pi_{11} : \Delta_1 \longrightarrow H_{r-1}^n, \pi_{12} : \Delta_1 \longrightarrow S_r^n$,
 (c) $\theta_{11} = \langle d\mathbf{v}, \mathbf{w} \rangle|_{\Delta_1}$, $\theta_{12} = \langle \mathbf{v}, d\mathbf{w} \rangle|_{\Delta_1}$.
- (2) (a) $H_{r-1}^n \times \Lambda^n \supset \Delta_2 = \{(\mathbf{v}, \mathbf{w}) \mid \langle \mathbf{v}, \mathbf{w} \rangle = -1\}$,
 (b) $\pi_{21} : \Delta_2 \longrightarrow H_{r-1}^n, \pi_{22} : \Delta_2 \longrightarrow \Lambda^n$,
 (c) $\theta_{21} = \langle d\mathbf{v}, \mathbf{w} \rangle|_{\Delta_2}$, $\theta_{22} = \langle \mathbf{v}, d\mathbf{w} \rangle|_{\Delta_2}$.
- (3) (a) $\Lambda^n \times S_r^n \supset \Delta_3 = \{(\mathbf{v}, \mathbf{w}) \mid \langle \mathbf{v}, \mathbf{w} \rangle = 1\}$,
 (b) $\pi_{31} : \Delta_3 \longrightarrow \Lambda^n, \pi_{32} : \Delta_3 \longrightarrow S_r^n$,
 (c) $\theta_{31} = \langle d\mathbf{v}, \mathbf{w} \rangle|_{\Delta_3}$, $\theta_{32} = \langle \mathbf{v}, d\mathbf{w} \rangle|_{\Delta_3}$.
- (4) (a) $\Lambda^n \times \Lambda^n \supset \Delta_4 = \{(\mathbf{v}, \mathbf{w}) \mid \langle \mathbf{v}, \mathbf{w} \rangle = -2\}$,
 (b) $\pi_{41} : \Delta_4 \longrightarrow \Lambda^n, \pi_{42} : \Delta_4 \longrightarrow \Lambda^n$,
 (c) $\theta_{41} = \langle d\mathbf{v}, \mathbf{w} \rangle|_{\Delta_4}$, $\theta_{42} = \langle \mathbf{v}, d\mathbf{w} \rangle|_{\Delta_4}$.

Here, $\pi_{i1}(\mathbf{v}, \mathbf{w}) = \mathbf{v}$, $\pi_{i2}(\mathbf{v}, \mathbf{w}) = \mathbf{w}$, $\langle d\mathbf{v}, \mathbf{w} \rangle = -w_0 dv_0 + \sum_{i=1}^n w_i dv_i$ and $\langle \mathbf{v}, d\mathbf{w} \rangle = -v_0 dw_0 + \sum_{i=1}^n v_i dw_i$ are one-forms on $\mathbb{R}_r^{n+1} \times \mathbb{R}_r^{n+1}$.

We remark that $\theta_{i1}^{-1}(0)$ and $\theta_{i2}^{-1}(0)$ define the same tangent hyperplane field over Δ_i which is denoted by K_i . The basic duality theorem is the following theorem:

Theorem 2.1 *Under the same notations as the previous paragraph, each (Δ_i, K_i) ($i = 1, 2, 3, 4$) is a contact manifold and both of π_{ij} ($j = 1, 2$) are Legendrian fibrations. Moreover those contact manifolds are contact diffeomorphic each other.*

We now explain the situation by a “mandala of Legendrian dualities” as the following commutative diagram:

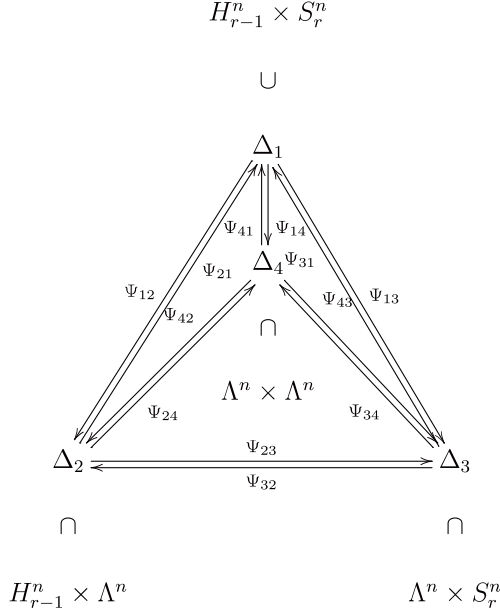


Figure 1. The Mandala of Legendrian Dualities

We can also consider the following two extra double fibrations:

- (5) (a) $S_r^n \times S_r^n \supset \Delta_5 = \{(\mathbf{v}, \mathbf{w}) \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0\}$,
 (b) $\pi_{51} : \Delta_5 \rightarrow S_r^n, \pi_{52} : \Delta_5 \rightarrow S_r^n$,
 (c) $\theta_{51} = \langle d\mathbf{v}, \mathbf{w} \rangle|_{\Delta_5}, \theta_{52} = \langle \mathbf{v}, d\mathbf{w} \rangle|_{\Delta_5}$.
- (6) (a) $H_{r-1}^n \times H_{r-1}^n \supset \Delta_6 = \{(\mathbf{v}, \mathbf{w}) \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0\}$,
 (b) $\pi_{61} : \Delta_6 \rightarrow H_{r-1}^n, \pi_{62} : \Delta_6 \rightarrow H_{r-1}^n$,
 (c) $\theta_{61} = \langle d\mathbf{v}, \mathbf{w} \rangle|_{\Delta_6}, \theta_{62} = \langle \mathbf{v}, d\mathbf{w} \rangle|_{\Delta_6}$.

We have the following theorem.

Theorem 2.2 *Under the same notations as the above, each (Δ_i, K_i) ($i = 5, 6$) is a contact manifold and both of π_{ij} ($j = 1, 2$) are Legendrian fibrations.*

We can show that (Δ_5, K_5) (respectively, (Δ_6, K_6)) is locally diffeomorphic to the projective cotangent bundle $\pi : PT^*H_{r-1}^n \rightarrow H_{r-1}^n$ (respectively, $\pi : PT^*S_r^n \rightarrow S_r^n$) which sends K_i to the canonical contact structure. We remark that these contact manifolds (Δ_j, K_j) ($j = 5, 6$) are not canonically contact diffeomorphic to (Δ_i, K_i) ($i = 1, 2, 3, 4$). Therefore we cannot add these contact manifolds to the mandala of Legendrian dualities. By definition, S_0^n is a unit sphere in Euclidean space $\mathbb{R}_0^{n+1}$, so that (Δ_5, K_5) is the well known classical spherical duality in this case. Finally we remark that $\Delta_6 = \emptyset$ in $H_0^n \times H_0^n$.

3 Applications

3.1 Linear Weingarten surfaces

A surface M immersed in different ambient 3-space is called a *Weingarten surfaces*, if the principal curvatures κ_1, κ_2 of M satisfy $f(\kappa_1, \kappa_2) = 0$ or the Gauss curvature K and the mean curvature H satisfy $g(K, H) = 0$, where f and g are smooth functions. Moreover, if there exist constants a, b and c , such that the smooth functions f and g are linear, namely $a\kappa_1 + b\kappa_2 = c$ or $aK + bH = c$, we term M a *linear Weingarten surfaces*. Furthermore, a linear Weingarten surface is also called a *Bryant type* provided that $a + b \neq 0$, briefly *BLW-surface*. We would like to point out that when a linear Weingarten surface of Bryant type is immersed in the de Sitter 3-space, the resulting surface is termed a *linear Weingarten surface of Bianchi type*, briefly, *BLW-surface*. We note that the theory of linear Weingarten surfaces has a long standing research history. Nevertheless, from our point of view, we only reference the research results achieved by Gálvez, Martínez and Milán in [6] and those by Aledo and Espinar in [1] as follows.

Theorem 3.1 ([6], Theorem 2 ii) *Let V be a noncompact, simply connected domain. Fix a meromorphic map $A : V \rightarrow SL(2, \mathbb{C})$ satisfying*

$$A^{-1}dA = \begin{pmatrix} 0 & \omega \\ dh & 0 \end{pmatrix},$$

where h is a meromorphic function and ω a holomorphic one-form. If

$$\sigma = (a + b) \left((1 + \varepsilon \|h\|^2)^2 \|\omega\|^2 - \frac{(1 - \varepsilon)^2 \|dh\|^2}{(1 + \varepsilon \|h\|^2)^2} \right)$$

is positive definite then $f = A(\Omega_+)A^$ is a linear Weingarten surface. Where,*

$$\Omega_{\pm} = \begin{pmatrix} \frac{(1 \pm \varepsilon^2 \|h\|^2)}{1 + \varepsilon \|h\|^2} & \mp \varepsilon \bar{h} \\ \mp \varepsilon h & \pm (1 + \varepsilon \|h\|^2) \end{pmatrix}, \quad \varepsilon = \frac{a}{a + b}, \quad 1 + \varepsilon \|h\|^2 > 0.$$

We remark that the hyperbolic Gauss map g of f is given by $g = A(\Omega_-)A^*$. Aledo and Espinar in [1] showed that g is a BLW-surface in S_1^3 . Moreover, from the Legendrian duality viewpoint, we can show that f and g are Δ_1 dual to each other. Consequently, we arrive at the following assertion proposed by Izumiya and Saji.

Theorem 3.2 ([12], Theorem 5.2) *Let $\mathcal{L}_1 : U \rightarrow \Delta_1$ be a Legendrian immersion. Suppose that both of $\pi_{11} \circ \mathcal{L}_1 : U \rightarrow H_0^3$ and $\pi_{12} \circ \mathcal{L}_1 : U \rightarrow S_1^3$ are immersions. Then $\pi_{11} \circ \mathcal{L}_1 = f$ is a linear Weingarten surface of Bryant type if and only if $\pi_{12} \circ \mathcal{L}_1 = g$ is a linear Weingarten surface of Bianchi type.*

This implies that the dual surface of a linear Weingarten surface in H_0^3 is also a linear Weingarten surface in S_1^3 and vice versa.

3.2 The evolutes of fronts in hyperbolic 2-space and de Sitter 2-space

In this subsection, we review the differential geometry of smooth curves in hyperbolic 2-space and de Sitter 2-space. For details, see [5].

3.2.1 The evolutes of fronts in hyperbolic 2-space

We firstly introduce the differential geometry of curves in H_0^2 . Let $\gamma_h : I \rightarrow H_0^2$ be a smooth curve. We call γ_h the *spacelike frontal* in H_0^2 , if there exists a smooth mapping $\gamma_h^d : I \rightarrow S_1^2$, such that the pair $(\gamma_h, \gamma_h^d) : I \rightarrow \Delta_1$ satisfies $(\gamma_h(t), \gamma_h^d(t))^* \theta = 0$ for all $t \in I$. Here θ is a canonical contact 1-form on Δ_1 . The condition $(\gamma_h(t), \gamma_h^d(t))^* \theta = 0$ is equivalent to $\langle \dot{\gamma}_h(t), \gamma_h^d(t) \rangle = 0$, for all $t \in I$. We call (γ_h, γ_h^d) the *spacelike Legendrian curve* in Δ_1 . Moreover, if (γ_h, γ_h^d) is an immersion, we call γ_h the *spacelike front* in H_0^2 and (γ_h, γ_h^d) the *spacelike Legendrian immersion* in Δ_1 . Let $\gamma_h^s(t) = \gamma_h(t) \wedge \gamma_h^d(t) \in S_1^2$. We have a moving frame $\{\gamma_h, \gamma_h^d, \gamma_h^s\}$ which called the *hyperbolic Legendrian Frenet frame* of $\mathbb{R}_1^3$ along γ_h . By the standard arguments, we have the following *hyperbolic Legendrian Frenet-Serret type formula*:

$$\begin{pmatrix} \dot{\gamma}_h(t) \\ \dot{\gamma}_h^d(t) \\ \dot{\gamma}_h^s(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & m_h(t) \\ 0 & 0 & n_h(t) \\ m_h(t) & -n_h(t) & 0 \end{pmatrix} \begin{pmatrix} \gamma_h(t) \\ \gamma_h^d(t) \\ \gamma_h^s(t) \end{pmatrix},$$

where $m_h(t) = \langle \dot{\gamma}_h(t), \gamma_h^s(t) \rangle$ and $n_h(t) = \langle \dot{\gamma}_h^d(t), \gamma_h^s(t) \rangle$. We call the pair (m_h, n_h) the *spacelike hyperbolic Legendrian curvature* of spacelike Legendrian curve (γ_h, γ_h^d) .

Example 3.3 Let γ_h be a regular curve in H_0^2 with the hyperbolic geodesic curvature κ_h . If we take $\gamma_h^d = \mathbf{e}_h$, then (γ_h, γ_h^d) is a spacelike Legendrian curve with the spacelike hyperbolic Legendrian curvature $(-||\dot{\gamma}_h||, ||\dot{\gamma}_h||\kappa_h)$. In fact, it is a spacelike Legendrian immersion. Moreover, by a straightforward calculation, we have $n_h(t) = |m_h(t)|\kappa_h(t)$ for all $t \in I$. In this case, we have $n_h(t) = 0$ if and only if $\kappa_h(t) = 0$.

Example 3.4 Let $\gamma_h : I \rightarrow H_0^2$ be $\gamma_h(t) = (\sqrt{1+t^4+t^6}, t^2, t^3)$. It is obviously that the origin is the singular point of γ_h . We assume that

$$\gamma_h^d(t) = \frac{1}{\sqrt{4+9t^2+t^6}} \left(t^3\sqrt{1+t^4+t^6}, t^5+3t, t^6-2 \right).$$

By a straightforward calculation, we have

$$\gamma_h^s(t) = \frac{\sqrt{1+t^4+t^6}}{\sqrt{t^6+9t^2+4}} \left(\frac{3t^4+2t^2}{\sqrt{1+t^4+t^6}}, 2, 3t \right)$$

and (γ_h, γ_h^d) is a spacelike Legendrian immersion with the spacelike hyperbolic Legendrian curvature (m_h, n_h) , where

$$\begin{aligned} m_h(t) &= \frac{t\sqrt{4+9t^2+t^6}}{\sqrt{1+t^4+t^6}}, \\ n_h(t) &= \frac{t^{10}+15t^6+10t^4+6}{(4+9t^2+t^6)\sqrt{1+t^4+t^6}}. \end{aligned}$$

We call γ_h the hyperbolic 3/2-cusp, see Figure 2.

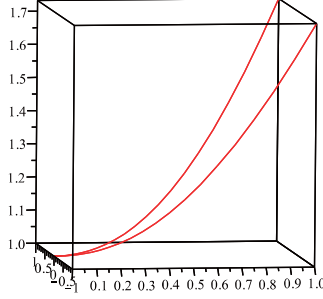


Figure 2. hyperbolic 3/2-cusp

We now consider the geometric meanings of evolutes of spacelike fronts in H_0^2 . Let $(\gamma_h, \gamma_h^d) : I \rightarrow \Delta_1$ be a spacelike Legendrian immersion with the spacelike hyperbolic Legendrian curvature (m_h, n_h) which satisfies $n_h^2(t) \neq m_h^2(t)$ for all $t \in I$. We define a mapping $\mathcal{E}_v(\gamma_h) : I \rightarrow \mathbb{R}_1^3$ by

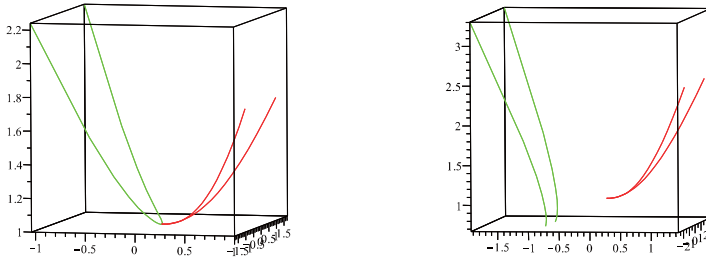
$$\mathcal{E}_v(\gamma_h)(t) = \pm \frac{1}{\sqrt{|n_h^2(t) - m_h^2(t)|}} (n_h(t)\gamma_h(t) - m_h(t)\gamma_h^d(t))$$

and call it the *totally evolute* of γ_h in $\mathbb{R}_1^3$. We remark that if $n_h^2(t) > m_h^2(t)$, then $\mathcal{E}_v(\gamma_h)(t) \in H_0^2$. In this case, we denote it by $\mathcal{E}_v^h(\gamma_h)$ and call it the *hyperbolic evolute* of γ_h . Moreover, if $n_h^2(t) < m_h^2(t)$, then $\mathcal{E}_v(\gamma_h)(t) \in S_1^2$. We rewrite it as $\mathcal{E}_v^d(\gamma_h)$ and call it the *de Sitter evolute* of γ_h .

Example 3.5 Let $\gamma_h : I \rightarrow H^2(-1)$ be $\gamma_h(t) = (\sqrt{1+t^4+t^6}, t^2, t^3)$ be the hyperbolic 3/2-cusp defined in the Example 3.4. By a straightforward calculation, we have

$$\mathcal{E}_v(\gamma_h)(t) = \pm \frac{(6(1+t^4+t^6)^{3/2}, 3t^8+6t^6-27t^4-6t^2, 6t^9+8t^7+24t^3+8t)}{\sqrt{|(t^{10}+15t^6+10t^4+6)^2-t^2(4+9t^2+t^6)|}},$$

see Figure 3.



hyperbolic evolute of the hyperbolic 3/2-cusp de Sitter evolute of the hyperbolic 3/2-cusp

Figure 3

3.2.2 The evolutes of fronts in de Sitter 2-space

We now consider the differential geometry of spacelike curves in S_1^2 . Suppose $\gamma_d : I \rightarrow S_1^2$ is a spacelike curve at any regular points $t \in I$, namely, $\dot{\gamma}_d(t)$ is a spacelike vector at the regular points. We call γ_d the *spacelike frontal* in S_1^2 if there exists a smooth mapping $\gamma_d^h : I \rightarrow H_0^2$ such that the pair $(\gamma_d^h, \gamma_d) : I \rightarrow \Delta_1$ satisfies $(\dot{\gamma}_d^h(t), \gamma_d(t))^* \theta = 0$ for all $t \in I$. The condition $(\gamma_d^h(t), \gamma_d(t))^* \theta = 0$ is equivalent to $\langle \dot{\gamma}_d(t), \gamma_d^h(t) \rangle = 0$, for all $t \in I$. We call (γ_d, γ_d^h) the *spacelike Legendrian curve* in Δ_1 . Moreover, if (γ_d, γ_d^h) is an immersion, we call γ_d the *spacelike front* in S_1^2 and (γ_d, γ_d^h) the *spacelike Legendrian immersion* in Δ_1 . Let $\gamma_d^s(t) = \gamma_d(t) \wedge \gamma_d^h(t) \in S_1^2$. We have a moving frame $\{\gamma_d, \gamma_d^h, \gamma_d^s\}$ which called the *spacelike de Sitter Legendrian Frenet frame* of $\mathbb{R}_1^3$ along γ_d . By the standard arguments, we have the following *spacelike de Sitter Legendrian Frenet-Serret type formula*:

$$\begin{pmatrix} \dot{\gamma}_d(t) \\ \dot{\gamma}_d^h(t) \\ \dot{\gamma}_d^s(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & m_d(t) \\ 0 & 0 & n_d(t) \\ -m_d(t) & n_d(t) & 0 \end{pmatrix} \begin{pmatrix} \gamma_d(t) \\ \gamma_d^h(t) \\ \gamma_d^s(t) \end{pmatrix},$$

where $m_d(t) = \langle \dot{\gamma}_d(t), \gamma_d^s(t) \rangle$ and $n_d(t) = \langle \dot{\gamma}_d^h(t), \gamma_d^s(t) \rangle$. We call the pair (m_d, n_d) the *spacelike de Sitter Legendrian curvature* of the spacelike Legendrian curve (γ_d, γ_d^h) . Under the assumption $n_d^2(t) \neq m_d^2(t)$ for all $t \in I$, we define a mapping $\mathcal{E}_v(\gamma_d) : I \rightarrow \mathbb{R}_1^3$ by

$$\mathcal{E}_v(\gamma_d)(t) = \pm \frac{1}{\sqrt{|n_d^2(t) - m_d^2(t)|}} (n_d(t)\gamma_d(t) - m_d(t)\gamma_d^h(t))$$

and call it the *totally evolute* of γ_d in $\mathbb{R}_1^3$. We remark that if $n_d^2(t) < m_d^2(t)$, then $\mathcal{E}_v(\gamma_d)(t) \in H_0^2$. In this case, we denote it by $\mathcal{E}_v^h(\gamma_d)$ and call it the *hyperbolic evolute* of γ_d . Moreover, if $n_d^2(t) > m_d^2(t)$, then $\mathcal{E}_v(\gamma_d)(t) \in S_1^2$. We rewrite it as $\mathcal{E}_v^d(\gamma_d)$ and call it the *de Sitter evolute* of γ_d .

We now show the relationships between the totally evolute $\mathcal{E}_v(\gamma_h)$ of a spacelike front γ_h in H_0^2 and the totally evolute $\mathcal{E}_v(\gamma_d)$ of a spacelike front γ_d in S_1^2 as follows.

Theorem 3.6 ([5], **Theorem 5.1**) *Suppose that $(\gamma_h, \gamma_d) : I \rightarrow \Delta_1$ is a spacelike Legendrian immersion with the spacelike hyperbolic Legendrian curvature (m_h, n_h) which satisfies $m_h^2(t) \neq n_h^2(t)$ for all $t \in I$. Then we have:*

- (i) *If $m_h^2(t) < n_h^2(t)$, then $\mathcal{E}_v^h(\gamma_h)(t) = \mathcal{E}_v^h(\gamma_d)(t)$.*
- (ii) *If $m_h^2(t) > n_h^2(t)$, then $\mathcal{E}_v^d(\gamma_h)(t) = \mathcal{E}_v^d(\gamma_d)(t)$.*

This assertion implies that when γ_h and γ_d are Δ_1 -dual to each other, they share identical evolutes.

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