

The existence of positive solutions to the semilinear elliptic equation involving the Sobolev and the Sobolev-Hardy critical terms

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Abstract

Let $n \geq 3$ and Ω be a C^2 -bounded domain in $\mathbb{R}^n$ with $0 \in \partial\Omega$. We consider

$$\begin{cases} \Delta u + \lambda u^p + \frac{u^{2^*-1}}{|x|^s} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (0.1)$$

where $0 < s < 2$, $2^* = \frac{2(n-s)}{n-2}$,

$$\begin{cases} \lambda > 0 & \text{if } 1 < p \leq \frac{n+2}{n-2}, \\ 0 < \lambda < \lambda_1(\Omega) & \text{if } p = 1, \end{cases}$$

and $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ with the Dirichlet boundary condition.

In this paper, we shall prove the equation (0.1) has a positive solution provided that $\frac{n}{n-2} < p < \frac{n+2}{n-2}$. If the mean curvature H at 0 is negative, then for each of the following cases, the equation (0.1) has a positive solution.

1. $1 \leq p < \frac{n+2}{n-2}$.

2. $p = \frac{n+2}{n-2}$ and

$$\begin{cases} 0 < s < 1 & \text{if } n = 3, \\ 0 < s < 2 & \text{if } n \geq 4. \end{cases}$$

If $H(0) < 0$, then we also prove the existence of a positive solution for the following equation:

$$\begin{cases} \Delta u - \lambda u^p + \frac{u^{2^*-1}}{|x|^s} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$\lambda > 0 \quad \text{and} \quad 1 \leq p < \min \left\{ \frac{n}{n-2}, 2^* - 1 \right\}.$$

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1 Introduction

In this paper, we consider the existence of positive solutions of the following nonlinear elliptic equations:

$$\begin{cases} \Delta u + \lambda u^p + \frac{u^{2^*-1}}{|x|^s} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where Ω is a bounded domain in $\mathbb{R}^n$ with $n \geq 3$, $0 \in \partial\Omega$, $0 < s < 2$, $2^* = 2^*(s) = \frac{2(n-s)}{n-2}$, $1 \leq p \leq \frac{n+2}{n-2}$ and λ is a real parameter. For the case $p = 1$, this problem was considered by Ghoussoub-Robert [3, 4], and they proved the following result.

Theorem A. *Suppose that Ω is a bounded smooth domain in $\mathbb{R}^n$ with $0 \in \partial\Omega$ and $\lambda < \lambda_1(\Omega)$, where $\lambda_1(\Omega) > 0$ is the first eigenvalue of $-\Delta$ with the Dirichlet boundary condition. Then the equation*

$$\begin{cases} \Delta u + \lambda u + \frac{u^{2^*-1}}{|x|^s} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a positive solution provided that the mean curvature H of $\partial\Omega$ at 0 is negative.

In an earlier paper by Brézis-Nirenberg [1], among other results, they showed:

Theorem B. *Suppose that Ω is a bounded smooth domain in $\mathbb{R}^n$ with $n \geq 4$. Then for every $\lambda \in (0, \lambda_1(\Omega))$, there exists a positive solution for the equation*

$$\begin{cases} \Delta u + \lambda u + u^{\frac{n+2}{n-2}} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In this article, we study more general cases. Namely, we obtain the following theorems.

Theorem 1.1. *Suppose that Ω is a bounded domain in $\mathbb{R}^n$ with $0 \in \partial\Omega$. Then the equation*

$$\begin{cases} \Delta u + \lambda u^p + \frac{u^{2^*-1}}{|x|^s} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1.2)$$

has a positive solution if one of the following conditions holds:

(i) $\partial\Omega$ is C^3 at 0 , $H(0) < 0$ and

$$\begin{cases} \lambda > 0 & \text{if } 1 < p < \frac{n+2}{n-2}, \\ 0 < \lambda < \lambda_1(\Omega) & \text{if } p = 1. \end{cases}$$

(ii) $\partial\Omega$ is C^3 at 0 , $H(0) = 0$, $\lambda > 0$ and $1 < p < \frac{n+2}{n-2}$.

(iii) $\partial\Omega$ is C^2 at 0 (no restriction for the mean curvature), $\lambda > 0$ and $\frac{n}{n-2} < p < \frac{n+2}{n-2}$.

(iv) $\partial\Omega$ is C^3 at 0 , $H(0) < 0$, $\lambda > 0$, $p = \frac{n+2}{n-2}$ and

$$\begin{cases} 0 < s < 1 & \text{if } n = 3, \\ 0 < s < 2 & \text{if } n \geq 4. \end{cases}$$

Our proof for Theorem 1.1 is inspired by the idea in Brézis-Nirenberg [1]. However, there is a major difference between our work and [1]. As it is well-known, the Sobolev best constant S_n is actually independent of Ω . This important fact was used in [1] implicitly. For our problem, $\mu_s(\Omega)$ defined in

Lemma 2.1 below does depend on the domain Ω . For the proof of the assertion (iv), besides using the fact that the Sobolev best constant S_n is independent of Ω , we also take advantage of that the energy level c^* in the mountain pass lemma (see the remark of Theorem C) is independent of the choice of v .

Furthermore, we prove the existence theorem of the following:

Theorem 1.2. *Suppose that Ω is a bounded domain in $\mathbb{R}^n$ with $0 \in \partial\Omega$, $\lambda > 0$, $\partial\Omega \in C^3$ at 0, $H(0) < 0$ and $1 \leq p < \min\{\frac{n}{n-2}, 2^* - 1\}$. Then the equation*

$$\begin{cases} \Delta u - \lambda u^p + \frac{u^{2^*-1}}{|x|^s} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1.3)$$

has a positive solution.

For $\Omega = \mathbb{R}_+^n$, we prove the following theorem by the Pohozaev identity together with the blow-up argument used in the proof of Theorem 1.1.

Theorem 1.3. *For $n \geq 3$ and $\lambda > 0$, there exists a positive solution of the equation*

$$\begin{cases} \Delta u + \lambda u^{\frac{n+2}{n-2}} + \frac{u^{2^*-1}}{|x|^s} = 0 & \text{in } \mathbb{R}_+^n, \\ u = 0 & \text{on } \partial\mathbb{R}_+^n \end{cases} \quad (1.4)$$

provided that

$$\begin{cases} 0 < s < 1 & \text{if } n = 3, \\ 0 < s < 2 & \text{if } n \geq 4. \end{cases}$$

This paper is organized as follows. Couples of auxiliary lemmas are collected in Section 2. The proof of theorems occupies Section 3-Section 5.

2 Preliminaries

In this section, we prove two lemmas for the proof of main theorems.

Lemma 2.1. *Suppose that Ω is a bounded domain with $0 \in \partial\Omega$, $\lambda > 0$ and assume one of the followings:*

- (i) $\partial\Omega$ is C^3 at 0, $H(0) < 0$ and $1 \leq p < \frac{n+2}{n-2}$.
- (ii) $\partial\Omega$ is C^3 at 0, $H(0) = 0$ and $1 < p < \frac{n+2}{n-2}$.
- (iii) $\partial\Omega$ is C^2 at 0 (no restriction for the mean curvature) and $\frac{n}{n-2} < p < \frac{n+2}{n-2}$.

Then there exists a nonnegative function $v_0 \in H_0^1(\Omega) \setminus \{0\}$ such that

$$\max_{t \geq 0} \Phi_{s,p}(tv_0) < \left(\frac{1}{2} - \frac{1}{2^*}\right) \mu_s(\Omega)^{\frac{2^*}{2^*-2}}, \quad (2.1)$$

where

$$\Phi_{s,p}(u) := \int_{\Omega} \left(\frac{1}{2} |\nabla u|^2 - \frac{\lambda}{p+1} (u^+)^{p+1} - \frac{1}{2^*} \frac{(u^+)^{2^*}}{|x|^s} \right) dx \quad \text{for } u \in H_0^1(\Omega),$$

and

$$\mu_s(\Omega) := \inf \left\{ \int_{\Omega} |\nabla u|^2 dx \mid u \in H_0^1(\Omega) \text{ and } \int_{\Omega} \frac{|u|^{2^*}}{|x|^s} dx = 1 \right\}.$$

For the proof of Lemma 2.1, we apply the following lemma:

Lemma 2.2. *Let $u \in H_0^1(\mathbb{R}_+^n)$ be an entire positive solution of*

$$\begin{cases} \Delta u + \mu_s(\mathbb{R}_+^n) \frac{u^{2^*-1}}{|y|^s} = 0 & \text{in } \mathbb{R}_+^n, \\ u = 0 & \text{on } \partial\mathbb{R}_+^n \quad \text{with } \int_{\mathbb{R}_+^n} \frac{u^{2^*}}{|y|^s} dy = 1, \end{cases} \quad (2.2)$$

or

$$\begin{cases} \Delta u + \lambda u^{\frac{n+2}{n-2}} + \frac{u^{2^*-1}}{|y|^s} = 0 & \text{in } \mathbb{R}_+^n, \\ u = 0 & \text{on } \partial\mathbb{R}_+^n. \end{cases} \quad (2.3)$$

Then the followings hold:

(i)

$$\begin{cases} u \in C^2(\overline{\mathbb{R}_+^n}) & \text{if } s < 1 + \frac{2}{n}, \\ u \in C^{1,\beta}(\overline{\mathbb{R}_+^n}) & \text{for all } 0 < \beta < 1 \quad \text{if } s = 1 + \frac{2}{n}, \\ u \in C^{1,\beta}(\overline{\mathbb{R}_+^n}) & \text{for all } 0 < \beta < \frac{n(2-s)}{n-2} \quad \text{if } s > 1 + \frac{2}{n}. \end{cases}$$

(ii) There is a constant C such that $|u(y)| \leq C(1 + |y|)^{1-n}$ and $|\nabla u(y)| \leq C(1 + |y|)^{-n}$.

(iii) $u(y', y_n)$ is axially symmetric with respect to the y_n -axis, i.e., $u(y', y_n) = u(|y'|, y_n)$.

Concerning the proof of the existence of a solution, assertions (i) and (ii) for the equation (2.2), see Egnell [2] and Ghoussoub-Robert [3, 4]. The existence of a solution for the equation (2.3) is obtained in Theorem 1.3. For the rest part of the proof for Lemma 2.2, we refer to Lin-Wadade [5].

Proof of Lemma 2.1. We first show (i). Ghoussoub-Robert [3, 4] proved that if $H(0) < 0$, then $\mu_s(\Omega)$ is attained by a positive function v_0 with

$$\int_{\Omega} |\nabla v_0|^2 dx = \mu_s(\Omega) \quad \text{and} \quad \int_{\Omega} \frac{v_0^{2^*}}{|x|^s} dx = 1.$$

Then since $\lambda > 0$, we have

$$\begin{aligned} \max_{t \geq 0} \Phi_{s,p}(tv_0) &= \max_{t \geq 0} \left(\frac{t^2}{2} \int_{\Omega} |\nabla v_0|^2 dx - \frac{\lambda t^{p+1}}{p+1} \int_{\Omega} v_0^{p+1} dx - \frac{t^{2^*}}{2^*} \int_{\Omega} \frac{v_0^{2^*}}{|x|^s} dx \right) \\ &< \max_{t \geq 0} \left(\frac{t^2}{2} \int_{\Omega} |\nabla v_0|^2 dx - \frac{t^{2^*}}{2^*} \int_{\Omega} \frac{v_0^{2^*}}{|x|^s} dx \right) = \max_{t \geq 0} \left(\frac{t^2}{2} \mu_s(\Omega) - \frac{t^{2^*}}{2^*} \right) = \left(\frac{1}{2} - \frac{1}{2^*} \right) \mu_s(\Omega)^{\frac{2^*}{2^*-2}}. \end{aligned}$$

Thus (i) is proved.

Next, we shall prove assertions (ii) and (iii). Since $\mu_s(\Omega) \leq \mu_s(\mathbb{R}_+^n)$, we consider two cases. First, if $\mu_s(\Omega) < \mu_s(\mathbb{R}_+^n)$, then $\mu_s(\Omega)$ can be achieved by some function in $H_0^1(\Omega)$, see [3, 4] and [5]. In this case, (1.2) can be proved by the same fashion used in the proof of the assertion (i). Hereafter, we only need to deal with the remaining case that $\mu_s(\Omega) = \mu_s(\mathbb{R}_+^n)$.

Without loss of generality, we may assume that in a neighborhood of 0, $\partial\Omega$ can be represented by $x_n = \varphi(x')$, $x' = (x_1, \dots, x_{n-1})$, where $\varphi(0) = 0$, $\nabla' \varphi(0) = 0$, $\nabla' = (\partial_1, \dots, \partial_{n-1})$, and the outer normal of $\partial\Omega$ is $-e_n$.

Let $u \in H_0^1(\mathbb{R}_+^n)$ be an entire positive solution of (2.2), and take U and $\tilde{U}$ to be neighborhoods of 0 such that $\Psi(U) = B_{r_0}(0)$ and $\Psi(\tilde{U}) = B_{\frac{r_0}{2}}(0)$, respectively. We define for $\varepsilon > 0$,

$$v_{\varepsilon}(x) := \varepsilon^{-\frac{n-2}{2}} u\left(\frac{\Psi(x)}{\varepsilon}\right) \quad \text{for } x \in \Omega \cap U, \quad \text{and} \quad \hat{v}_{\varepsilon} := \eta v_{\varepsilon} \quad \text{in } \Omega, \quad (2.4)$$

where $\eta \in C_c^\infty(U)$ is a positive cut-off function with $\eta \equiv 1$ in $\tilde{U}$, and

$$\Psi(x) := (x', x_n - \varphi(x')) \quad \text{for } x \in \overline{\Omega} \cap \overline{U}.$$

For $t \geq 0$, we have

$$\Phi_{s,p}(t\hat{v}_\varepsilon) \leq \frac{t^2}{2} \int_{\Omega} |\nabla \hat{v}_\varepsilon|^2 dx - \frac{\lambda t^{p+1}}{p+1} \int_{\Omega} \hat{v}_\varepsilon^{p+1} dx - \frac{t^{2^*}}{2^*} \int_{\Omega \cap \tilde{U}} \frac{v_\varepsilon^{2^*}}{|x|^s} dx. \quad (2.5)$$

In what follows, we investigate each integral in (2.5) precisely. By a change of the variable $\frac{\Psi(x)}{\varepsilon} = y$ and Lemma 2.2, we get

$$\begin{aligned} \int_{\Omega} |\nabla \hat{v}_\varepsilon|^2 dx &= \int_{\Omega \cap U} \eta^2 |\nabla v_\varepsilon|^2 dx - \int_{\Omega \cap U} \eta (\Delta \eta) v_\varepsilon^2 dx \\ &\leq \int_{\mathbb{R}_+^n} |\nabla u(y)|^2 dy - 2 \int_{B_{\frac{r_0}{\varepsilon}}^+} \eta (\Psi^{-1}(\varepsilon y))^2 \partial_n u(y) \nabla' u(y) \cdot (\nabla' \varphi)(\varepsilon y') dy \\ &\quad + \int_{B_{\frac{r_0}{\varepsilon}}^+} \eta (\Psi^{-1}(\varepsilon y))^2 (\partial_n u(y))^2 |(\nabla' \varphi)(\varepsilon y')|^2 dy - \varepsilon^2 \int_{B_{\frac{r_0}{\varepsilon}}^+} \eta (\Psi^{-1}(\varepsilon y)) (\Delta \eta) (\Psi^{-1}(\varepsilon y)) u(y)^2 dy \\ &= \int_{\mathbb{R}_+^n} |\nabla u(y)|^2 dy - 2 \int_{B_{\frac{r_0}{\varepsilon}}^+} \eta (\Psi^{-1}(\varepsilon y))^2 \partial_n u(y) \nabla' u(y) \cdot (\nabla' \varphi)(\varepsilon y') dy + O(\varepsilon^2). \end{aligned}$$

Using integration by parts and $\nabla' u \equiv 0$ on $\partial \mathbb{R}_+^n$, we obtain

$$\begin{aligned} I &:= -2 \int_{B_{\frac{r_0}{\varepsilon}}^+} \eta (\Psi^{-1}(\varepsilon y))^2 \partial_n u(y) \nabla' u(y) \cdot (\nabla' \varphi)(\varepsilon y') dy \\ &= -\frac{2}{\varepsilon} \int_{B_{\frac{r_0}{\varepsilon}}^+} \eta (\Psi^{-1}(\varepsilon y))^2 \partial_n u(y) \nabla' u(y) \cdot \nabla' [\varphi(\varepsilon y')] dy \\ &= \frac{4}{\varepsilon} \int_{B_{\frac{r_0}{\varepsilon}}^+} \eta (\Psi^{-1}(\varepsilon y)) \nabla' [\eta (\Psi^{-1}(\varepsilon y))] \cdot \partial_n u(y) \nabla' u(y) \varphi(\varepsilon y') dy \\ &\quad + \frac{2}{\varepsilon} \int_{B_{\frac{r_0}{\varepsilon}}^+} \eta (\Psi^{-1}(\varepsilon y))^2 \nabla' \partial_n u(y) \cdot \nabla' u(y) \varphi(\varepsilon y') dy \\ &\quad + \frac{2}{\varepsilon} \int_{B_{\frac{r_0}{\varepsilon}}^+} \eta (\Psi^{-1}(\varepsilon y))^2 \partial_n u(y) \sum_{i=1}^{n-1} \partial_{ii} u(y) \varphi(\varepsilon y') dy \\ &= \frac{2}{\varepsilon} \int_{B_{\frac{r_0}{\varepsilon}}^+} \eta (\Psi^{-1}(\varepsilon y))^2 \partial_n u(y) \sum_{i=1}^{n-1} \partial_{ii} u(y) \varphi(\varepsilon y') dy + O(\varepsilon^n). \end{aligned}$$

Applying the equation (2.2) and integration by parts, we have

$$\begin{aligned} I' &:= \frac{2}{\varepsilon} \int_{B_{\frac{r_0}{\varepsilon}}^+} \eta (\Psi^{-1}(\varepsilon y))^2 \partial_n u(y) \sum_{i=1}^{n-1} \partial_{ii} u(y) \varphi(\varepsilon y') dy \\ &= -\frac{2\mu_s(\mathbb{R}_+^n)}{2^* \varepsilon} \int_{B_{\frac{r_0}{\varepsilon}}^+} \eta (\Psi^{-1}(\varepsilon y))^2 \frac{\partial_n [u(y)^{2^*}]}{|y|^s} \varphi(\varepsilon y') dy \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{\varepsilon} \int_{B_{\frac{r_0}{\varepsilon}}^+} \eta (\Psi^{-1}(\varepsilon y))^2 \partial_n [(\partial_n u(y))^2] \varphi(\varepsilon y') dy \\
& = -\frac{2s\mu_s(\mathbb{R}_+^n)}{2^*\varepsilon} \int_{B_{\frac{r_0}{\varepsilon}}^+} \eta (\Psi^{-1}(\varepsilon y))^2 \frac{u(y)^{2^*} y_n}{|y|^{s+2}} \varphi(\varepsilon y') dy \\
& + \frac{1}{\varepsilon} \int_{B_{\frac{r_0}{\varepsilon}}^+ \cap \partial \mathbb{R}_+^n} \eta (\Psi^{-1}(\varepsilon y))^2 (\partial_n u(y))^2 \varphi(\varepsilon y') dS_y + O(\varepsilon^n) =: J_1 + J_2 + O(\varepsilon^n).
\end{aligned}$$

If $\partial \Omega$ is C^3 at 0, φ can be expanded by

$$\varphi(y') = \sum_{i=1}^{n-1} \alpha_i y_i^2 + O(|y'|^3). \quad (2.6)$$

Thus we see that

$$\begin{aligned}
J_1 & = -\frac{2s\mu_s(\mathbb{R}_+^n)}{2^*\varepsilon} \int_{B_{\frac{r_0}{\varepsilon}}^+} \eta (\Psi^{-1}(\varepsilon y))^2 \frac{u(y)^{2^*} y_n}{|y|^{s+2}} \varphi(\varepsilon y') dy \\
& = -\frac{2s\mu_s(\mathbb{R}_+^n)}{2^*\varepsilon} \int_{B_{\frac{r_0}{\varepsilon}}^+ \setminus B_{\frac{r_0/2}{\varepsilon}}^+} \eta (\Psi^{-1}(\varepsilon y))^2 \frac{u(y)^{2^*} y_n}{|y|^{s+2}} \varphi(\varepsilon y') dy \\
& - \frac{2s\mu_s(\mathbb{R}_+^n)}{2^*\varepsilon} \int_{B_{\frac{r_0/2}{\varepsilon}}^+} \frac{u(y)^{2^*} y_n}{|y|^{s+2}} \varphi(\varepsilon y') dy =: J_{1,1} + J_{1,2}, \quad \text{and}
\end{aligned}$$

$$|J_{1,1}| \leq C\varepsilon \int_{\{\frac{r_0}{2} \leq |\varepsilon y| < r_0\}} |y|^{2^*(1-n)+1-s} dy = O(\varepsilon^{\frac{n(n-s)}{n-2}}).$$

Moreover, note that

$$\begin{cases} \varepsilon \int_{\mathbb{R}_+^n \setminus B_{\frac{r_0/2}{\varepsilon}}^+} u(y)^{2^*} |y|^{1-s} dy = O(\varepsilon^{\frac{n(n-s)}{n-2}}), \\ \varepsilon^2 \int_{B_{\frac{r_0/2}{\varepsilon}}^+} u(y)^{2^*} |y|^{2-s} dy = O(\varepsilon^2), \end{cases} \quad (2.7)$$

which is integrable because $2^*(1-n) + 2 - s + n < 0$, i.e., $n^2 - (2+s)n + 4 > 0$. Thus by using (2.6) and (2.7), we get

$$\begin{aligned}
J_{1,2} & = -\frac{2s\varepsilon\mu_s(\mathbb{R}_+^n)}{2^*} \sum_{i=1}^{n-1} \alpha_i \int_{\mathbb{R}_+^n} \frac{u(y)^{2^*} y_i^2 y_n}{|y|^{2+s}} dy + O(\varepsilon^2) \\
& = -\frac{2s\varepsilon\mu_s(\mathbb{R}_+^n)}{2^*(n-1)} \int_{\mathbb{R}_+^n} \frac{u(y)^{2^*} |y'|^2 y_n}{|y|^{2+s}} dy \sum_{i=1}^{n-1} \alpha_i + O(\varepsilon^2) = -K_1 H(0) \varepsilon + O(\varepsilon^2),
\end{aligned}$$

where

$$H(0) := \frac{1}{n-1} \sum_{i=1}^{n-1} \alpha_i \quad \text{and} \quad K_1 := \frac{2s\mu_s(\mathbb{R}_+^n)}{2^*} \int_{\mathbb{R}_+^n} \frac{u(y)^{2^*} |y'|^2 y_n}{|y|^{2+s}} dy. \quad (2.8)$$

In the above estimate, we used the fact $u(y', y_n) = u(|y'|, y_n)$. Next, we see that

$$J_2 = \frac{1}{\varepsilon} \int_{B_{\frac{r_0}{\varepsilon}}^+ \cap \partial \mathbb{R}_+^n} \eta (\Psi^{-1}(\varepsilon y))^2 (\partial_n u(y))^2 \varphi(\varepsilon y') dS_y$$

$$\begin{aligned}
&= \frac{1}{\varepsilon} \int_{(B_{\frac{r_0}{\varepsilon}}^+ \setminus B_{\frac{r_0/2}{\varepsilon}}^+) \cap \partial \mathbb{R}_+^n} \eta (\Psi^{-1}(\varepsilon y))^2 (\partial_n u(y))^2 \varphi(\varepsilon y') dS_y \\
&+ \frac{1}{\varepsilon} \int_{B_{\frac{r_0/2}{\varepsilon}}^+ \cap \partial \mathbb{R}_+^n} (\partial_n u(y))^2 \varphi(\varepsilon y') dS_y =: J_{2,1} + J_{2,2}, \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
|J_{2,1}| &\leq \frac{C}{\varepsilon} \int_{\{\frac{r_0}{2} < |\varepsilon y'| \leq r_0\}} |(\partial_n u)(y', 0)|^2 |\varphi(\varepsilon y')| dy' \\
&\leq C\varepsilon \int_{\{\frac{r_0}{2} < |\varepsilon y'| \leq r_0\}} |y'|^{-2n+2} dy' = O(\varepsilon^n).
\end{aligned}$$

Moreover, note that $|(\partial_n u)(y', 0)|^2 |y'|^3 = O(|y'|^{-2n+3})$ for large $|y'|$ and $2n - 3 > n - 1$ for $n \geq 3$. Hence, it is integrable and

$$\begin{cases} \varepsilon \int_{\{|\varepsilon y'| > \frac{r_0}{2}\}} |(\partial_n u)(y', 0)|^2 |y'|^2 dy' = O(\varepsilon^n), \\ \varepsilon^2 \int_{\{|\varepsilon y'| \leq \frac{r_0}{2}\}} |(\partial_n u)(y', 0)|^2 |y'|^3 dy' = O(\varepsilon^2). \end{cases} \quad (2.9)$$

Thus by using (2.6) and (2.9), we get

$$\begin{aligned}
J_{2,2} &= \varepsilon \sum_{i=1}^{n-1} \alpha_i \int_{\mathbb{R}^{n-1}} ((\partial_n u)(y', 0))^2 y_i^2 dy' + O(\varepsilon^2) \\
&= \frac{\varepsilon}{n-1} \int_{\mathbb{R}^{n-1}} |(\nabla u)(y', 0)|^2 |y'|^2 dy' \sum_{i=1}^{n-1} \alpha_i + O(\varepsilon^2) = K_2 H(0) \varepsilon + O(\varepsilon^2),
\end{aligned}$$

where

$$K_2 := \int_{\mathbb{R}^{n-1}} |(\nabla u)(y', 0)|^2 |y'|^2 dy'. \quad (2.10)$$

After all, we get

$$\int_{\Omega} |\nabla \hat{v}_\varepsilon|^2 dx \leq \mu_s(\mathbb{R}_+^n) - K_1 H(0) \varepsilon + K_2 H(0) \varepsilon + O(\varepsilon^2). \quad (2.11)$$

Next, by changing the variable $\frac{\Psi(x)}{\varepsilon} = y$, we have

$$\int_{\Omega} \hat{v}_\varepsilon^{p+1} dx = \varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2}} \int_{\mathbb{R}_+^n} u^{p+1} dy + O(\varepsilon^{\frac{n(p+1)}{2}}). \quad (2.12)$$

Furthermore, the integral $\int_{\Omega \cap \tilde{U}} \frac{v_\varepsilon^{2^*}}{|x|^s} dx$ can be estimated as follows. By a change of the variable $\frac{\Psi(x)}{\varepsilon} = y$, we have

$$\int_{\Omega \cap \tilde{U}} \frac{v_\varepsilon^{2^*}}{|x|^s} dx = \int_{B_{\frac{r_0/2}{\varepsilon}}^+} \frac{u^{2^*}}{\left| \frac{\Psi^{-1}(\varepsilon y)}{\varepsilon} \right|^s} dy. \quad (2.13)$$

Since $\Psi^{-1}(y) = (y', y_n + \varphi(y'))$, it holds $|\Psi^{-1}(y)|^2 = |y|^2 + 2y_n \varphi(y') + (\varphi(y'))^2$, and then

$$\frac{1}{\left| \frac{\Psi^{-1}(\varepsilon y)}{\varepsilon} \right|^s} = \frac{1}{|y|^s} \cdot \frac{1}{\left(1 + \frac{2y_n \varphi(\varepsilon y')}{\varepsilon |y|^2} + \frac{(\varphi(\varepsilon y'))^2}{\varepsilon^2 |y|^2} \right)^{\frac{s}{2}}}$$

$$= \frac{1}{|y|^s} \left(1 - \frac{sy_n \varphi(\varepsilon y')}{\varepsilon |y|^2} - \frac{s(\varphi(\varepsilon y'))^2}{2\varepsilon^2 |y|^2} \right) + C \frac{1}{|y|^s} \left(\frac{2y_n \varphi(\varepsilon y')}{\varepsilon |y|^2} + \frac{(\varphi(\varepsilon y'))^2}{\varepsilon^2 |y|^2} \right)^2. \quad (2.14)$$

Thus from (2.13) and (2.14), we obtain

$$\begin{aligned} \int_{\Omega \cap \tilde{U}} \frac{v_\varepsilon^{2^*}}{|x|^s} dx &= \int_{\mathbb{R}_+^n} \frac{u^{2^*}}{|y|^s} dy - \frac{s}{\varepsilon} \int_{B_{\frac{r_0}{2}}^+} \frac{u(y)^{2^*} y_n \varphi(\varepsilon y')}{|y|^{2+s}} dy + O(\varepsilon^2) \\ &= 1 - s\varepsilon \sum_{i=1}^{n-1} \alpha_i \int_{\mathbb{R}_+^n} \frac{u(y)^{2^*} y_i^2 y_n}{|y|^{2+s}} dy + O(\varepsilon^2) \\ &= 1 - \frac{s\varepsilon}{n-1} \int_{\mathbb{R}_+^n} \frac{u(y)^{2^*} |y'|^2 y_n}{|y|^{2+s}} dy \sum_{i=1}^{n-1} \alpha_i + O(\varepsilon^2) = 1 - \frac{2^* K_1}{2\mu_s(\mathbb{R}_+^n)} H(0)\varepsilon + O(\varepsilon^2), \end{aligned}$$

where K_1 is the same positive constant as in (2.8).

After all, each integral can be estimated by

$$\begin{cases} \int_{\Omega} |\nabla \hat{v}_\varepsilon|^2 dx \leq \mu_s(\mathbb{R}_+^n) - K_1 H(0)\varepsilon + K_2 H(0)\varepsilon + O(\varepsilon^2), \\ \int_{\Omega} \hat{v}_\varepsilon^{p+1} dx = \varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2}} \int_{\mathbb{R}_+^n} u^{p+1} dy + O(\varepsilon^{\frac{n(p+1)}{2}}), \\ \int_{\Omega \cap \tilde{U}} \frac{v_\varepsilon^{2^*}}{|x|^s} dx = 1 - \frac{2^* K_1}{2\mu_s(\mathbb{R}_+^n)} H(0)\varepsilon + O(\varepsilon^2). \end{cases} \quad (2.15)$$

By (2.5) and (2.15), we have for $t \geq 0$,

$$\begin{aligned} \Phi_{s,p}(t\hat{v}_\varepsilon) &\leq \frac{t^2}{2} (\mu_s(\mathbb{R}_+^n) - K_1 H(0)\varepsilon + K_2 H(0)\varepsilon + O(\varepsilon^2)) \\ &\quad - \frac{\lambda t^{p+1}}{p+1} \left(\varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2}} \int_{\mathbb{R}_+^n} u^{p+1} dy + O(\varepsilon^{\frac{n(p+1)}{2}}) \right) - \frac{t^{2^*}}{2^*} \left(1 - \frac{2^* K_1}{2\mu_s(\mathbb{R}_+^n)} H(0)\varepsilon + O(\varepsilon^2) \right). \end{aligned} \quad (2.16)$$

We claim that the right-hand side has the maximum point t_m expressed by

$$\begin{aligned} t_m &= \mu_s(\mathbb{R}_+^n)^{\frac{1}{2^*-2}} + \frac{1}{2} \mu_s(\mathbb{R}_+^n)^{-\frac{2^*-3}{2^*-2}} K_1 H(0)\varepsilon + \frac{1}{2^*-2} \mu_s(\mathbb{R}_+^n)^{-\frac{2^*-3}{2^*-2}} K_2 H(0)\varepsilon \\ &\quad - \frac{\lambda}{2^*-2} \mu_s(\mathbb{R}_+^n)^{-\frac{2^*-2-p}{2^*-2}} \int_{\mathbb{R}_+^n} u^{p+1} dy \varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2}} + O(\varepsilon^2) + O(\varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2} + 1}) + O(\varepsilon^{n+2-(n-2)p}). \end{aligned} \quad (2.17)$$

Indeed, set

$$t_m = \mu_s(\mathbb{R}_+^n)^{\frac{1}{2^*-2}} + A(\varepsilon) \quad \text{with } A(\varepsilon) \rightarrow 0 \quad (2.18)$$

as $\varepsilon \rightarrow 0$. Since t_m is the maximum point, we have

$$\begin{aligned} &\mu_s(\mathbb{R}_+^n) - K_1 H(0)\varepsilon + K_2 H(0)\varepsilon + O(\varepsilon^2) - \lambda t_m^{p-1} \left(\varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2}} \int_{\mathbb{R}_+^n} u^{p+1} dy + O(\varepsilon^{\frac{n(p+1)}{2}}) \right) \\ &\quad - t_m^{2^*-2} \left(1 - \frac{2^* K_1}{2\mu_s(\mathbb{R}_+^n)} H(0)\varepsilon + O(\varepsilon^2) \right) = 0. \end{aligned} \quad (2.19)$$

By substituting (2.18) into (2.19), we see that

$$\mu_s(\mathbb{R}_+^n) - K_1 H(0)\varepsilon + K_2 H(0)\varepsilon + O(\varepsilon^2)$$

$$\begin{aligned}
& -\lambda \left(\mu_s(\mathbb{R}_+^n)^{\frac{1}{2^*-2}} + A(\varepsilon) \right)^{p-1} \left(\varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2}} \int_{\mathbb{R}_+^n} u^{p+1} dy + O(\varepsilon^{\frac{n(p+1)}{2}}) \right) \\
& - \left(\mu_s(\mathbb{R}_+^n)^{\frac{1}{2^*-2}} + A(\varepsilon) \right)^{2^*-2} \left(1 - \frac{2^* K_1}{2\mu_s(\mathbb{R}_+^n)} H(0)\varepsilon + O(\varepsilon^2) \right) \\
& = \mu_s(\mathbb{R}_+^n) - K_1 H(0)\varepsilon + K_2 H(0)\varepsilon + O(\varepsilon^2) \\
& - \lambda \left(\mu_s(\mathbb{R}_+^n)^{\frac{p-1}{2^*-2}} + (p-1)\mu_s(\mathbb{R}_+^n)^{\frac{p-2}{2^*-2}} A(\varepsilon) + O(A(\varepsilon)^2) \right) \left(\varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2}} \int_{\mathbb{R}_+^n} u^{p+1} dy + O(\varepsilon^{\frac{n(p+1)}{2}}) \right) \\
& - \left(\mu_s(\mathbb{R}_+^n) + (2^* - 2)\mu_s(\mathbb{R}_+^n)^{\frac{2^*-3}{2^*-2}} A(\varepsilon) + O(A(\varepsilon)^2) \right) \left(1 - \frac{2^* K_1}{2\mu_s(\mathbb{R}_+^n)} H(0)\varepsilon + O(\varepsilon^2) \right) \\
& = \frac{2^* - 2}{2} K_1 H(0)\varepsilon + K_2 H(0)\varepsilon - \lambda \mu_s(\mathbb{R}_+^n)^{\frac{p-1}{2^*-2}} \int_{\mathbb{R}_+^n} u^{p+1} dy \varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2}} + O(\varepsilon^2) + O(A(\varepsilon)^2) \\
& - A(\varepsilon) \left((2^* - 2)\mu_s(\mathbb{R}_+^n)^{\frac{2^*-3}{2^*-2}} - \frac{2^*(2^* - 2)}{2} \mu_s(\mathbb{R}_+^n)^{-\frac{1}{2^*-2}} K_1 H(0)\varepsilon + O(\varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2}}) \right) = 0,
\end{aligned}$$

where note that

$$\frac{n+2}{2} - \frac{(n-2)p}{2} \leq 2, \text{ i.e., } p \geq 1, \quad \text{and} \quad \frac{n+2}{2} - \frac{(n-2)p}{2} \leq \frac{n(p+1)}{2}, \text{ i.e., } p \geq \frac{1}{n-1}.$$

Then we have

$$\begin{aligned}
A(\varepsilon) & = \left((2^* - 2)\mu_s(\mathbb{R}_+^n)^{\frac{2^*-3}{2^*-2}} - \frac{2^*(2^* - 2)}{2} \mu_s(\mathbb{R}_+^n)^{-\frac{1}{2^*-2}} K_1 H(0)\varepsilon + O(\varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2}}) \right)^{-1} \\
& \times \left(\frac{2^* - 2}{2} K_1 H(0)\varepsilon + K_2 H(0)\varepsilon - \lambda \mu_s(\mathbb{R}_+^n)^{\frac{p-1}{2^*-2}} \int_{\mathbb{R}_+^n} u^{p+1} dy \varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2}} + O(\varepsilon^2) \right) + O(A(\varepsilon)^2) \\
& =: B(\varepsilon) + O(A(\varepsilon)^2).
\end{aligned}$$

Here, note that $\lim_{\varepsilon \rightarrow 0} \frac{B(\varepsilon)}{A(\varepsilon)} = 1$, which implies $A(\varepsilon) = B(\varepsilon) + O(B(\varepsilon)^2)$. Moreover, we have

$$\begin{aligned}
B(\varepsilon) & = \left(\frac{1}{2^* - 2} \mu_s(\mathbb{R}_+^n)^{-\frac{2^*-3}{2^*-2}} + \frac{2^*}{2(2^* - 2)} \mu_s(\mathbb{R}_+^n)^{-\frac{2 \cdot 2^* - 5}{2^*-2}} K_1 H(0)\varepsilon + O(\varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2}}) \right) \\
& \times \left(\frac{2^* - 2}{2} K_1 H(0)\varepsilon + K_2 H(0)\varepsilon - \lambda \mu_s(\mathbb{R}_+^n)^{\frac{p-1}{2^*-2}} \int_{\mathbb{R}_+^n} u^{p+1} dy \varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2}} + O(\varepsilon^2) \right) \\
& = \frac{1}{2} \mu_s(\mathbb{R}_+^n)^{-\frac{2^*-3}{2^*-2}} K_1 H(0)\varepsilon + \frac{1}{2^* - 2} \mu_s(\mathbb{R}_+^n)^{-\frac{2^*-3}{2^*-2}} K_2 H(0)\varepsilon + O(\varepsilon^2) \\
& - \frac{\lambda}{2^* - 2} \mu_s(\mathbb{R}_+^n)^{-\frac{2^*-2-p}{2^*-2}} \int_{\mathbb{R}_+^n} u^{p+1} dy \varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2}} + O(\varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2} + 1}) + O(\varepsilon^{n+2-(n-2)p}). \quad (2.20)
\end{aligned}$$

As a consequence, we obtain (2.17). By (2.16) and (2.17), we see that

$$\begin{aligned}
& \max_{t \geq 0} \Phi_{s,p}(t\hat{v}_\varepsilon) \\
& \leq \frac{1}{2} \left[\mu_s(\mathbb{R}_+^n)^{\frac{2}{2^*-2}} + 2\mu_s(\mathbb{R}_+^n)^{\frac{1}{2^*-2}} \left(\frac{1}{2} \mu_s(\mathbb{R}_+^n)^{-\frac{2^*-3}{2^*-2}} K_1 H(0)\varepsilon + \frac{1}{2^* - 2} \mu_s(\mathbb{R}_+^n)^{-\frac{2^*-3}{2^*-2}} K_2 H(0)\varepsilon \right. \right.
\end{aligned}$$

$$\begin{aligned}
& - \frac{\lambda}{2^* - 2} \mu_s(\mathbb{R}_+^n)^{-\frac{2^* - 2 - p}{2^* - 2}} \int_{\mathbb{R}_+^n} u^{p+1} dy \varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2}} \Big) + O(\varepsilon^2) + O(\varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2} + 1}) + O(\varepsilon^{n+2 - (n-2)p}) \Big] \\
& \times \left(\mu_s(\mathbb{R}_+^n) - K_1 H(0) \varepsilon + K_2 H(0) \varepsilon + O(\varepsilon^2) \right) - \frac{\lambda}{p+1} \left[\mu_s(\mathbb{R}_+^n)^{\frac{p+1}{2^* - 2}} + (p+1) \mu_s(\mathbb{R}_+^n)^{\frac{p}{2^* - 2}} \right. \\
& \times \left(\frac{1}{2} \mu_s(\mathbb{R}_+^n)^{-\frac{2^* - 3}{2^* - 2}} K_1 H(0) \varepsilon + \frac{1}{2^* - 2} \mu_s(\mathbb{R}_+^n)^{-\frac{2^* - 3}{2^* - 2}} K_2 H(0) \varepsilon \right. \\
& \left. \left. - \frac{\lambda}{2^* - 2} \mu_s(\mathbb{R}_+^n)^{-\frac{2^* - 2 - p}{2^* - 2}} \int_{\mathbb{R}_+^n} u^{p+1} dy \varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2}} \right) + O(\varepsilon^2) + O(\varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2} + 1}) + O(\varepsilon^{n+2 - (n-2)p}) \right] \\
& \times \left(\int_{\mathbb{R}_+^n} u^{p+1} dy \varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2}} + O(\varepsilon^{\frac{n(p+1)}{2}}) \right) \\
& - \frac{1}{2^*} \left[\mu_s(\mathbb{R}_+^n)^{\frac{2^*}{2^* - 2}} + 2^* \mu_s(\mathbb{R}_+^n)^{\frac{2^* - 1}{2^* - 2}} \left(\frac{1}{2} \mu_s(\mathbb{R}_+^n)^{-\frac{2^* - 3}{2^* - 2}} K_1 H(0) \varepsilon + \frac{1}{2^* - 2} \mu_s(\mathbb{R}_+^n)^{-\frac{2^* - 3}{2^* - 2}} K_2 H(0) \varepsilon \right. \right. \\
& \left. \left. - \frac{\lambda}{2^* - 2} \mu_s(\mathbb{R}_+^n)^{-\frac{2^* - 2 - p}{2^* - 2}} \int_{\mathbb{R}_+^n} u^{p+1} dy \varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2}} \right) + O(\varepsilon^2) + O(\varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2} + 1}) + O(\varepsilon^{n+2 - (n-2)p}) \right] \\
& \times \left(1 - \frac{2^* K_1}{2 \mu_s(\mathbb{R}_+^n)} H(0) \varepsilon + O(\varepsilon^2) \right) \\
& = \left(\frac{1}{2} - \frac{1}{2^*} \right) \mu_s(\mathbb{R}_+^n)^{\frac{2^*}{2^* - 2}} + \frac{1}{2} \mu_s(\mathbb{R}_+^n)^{\frac{2}{2^* - 2}} K_2 H(0) \varepsilon - \frac{\lambda}{p+1} \mu_s(\mathbb{R}_+^n)^{\frac{p+1}{2^* - 2}} \int_{\mathbb{R}_+^n} u^{p+1} dy \varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2}} \\
& + O(\varepsilon^2) + O(\varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2} + 1}) + O(\varepsilon^{n+2 - (n-2)p}). \tag{2.21}
\end{aligned}$$

By using (2.21), we can show (ii) and (iii) as follows. First, if $\lambda > 0$, $H(0) = 0$ and $1 < p < \frac{n+2}{n-2}$, i.e., $2 - \left(\frac{n+2}{2} - \frac{(n-2)p}{2} \right) > 0$, we have for small $\varepsilon > 0$,

$$\begin{aligned}
& \max_{t \geq 0} \Phi_{s,p}(t\hat{v}_\varepsilon) \\
& \leq \left(\frac{1}{2} - \frac{1}{2^*} \right) \mu_s(\mathbb{R}_+^n)^{\frac{2^*}{2^* - 2}} - \frac{\lambda}{p+1} \mu_s(\mathbb{R}_+^n)^{\frac{p+1}{2^* - 2}} \int_{\mathbb{R}_+^n} u^{p+1} dy \varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2}} + O(\varepsilon^2) + o(\varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2}}) \\
& < \left(\frac{1}{2} - \frac{1}{2^*} \right) \mu_s(\mathbb{R}_+^n)^{\frac{2^*}{2^* - 2}}.
\end{aligned}$$

Thus (ii) is proved.

Next, if $\lambda > 0$ and $\frac{n}{n-2} < p < \frac{n+2}{n-2}$, i.e., $1 - \left(\frac{n+2}{2} - \frac{(n-2)p}{2} \right) > 0$, we have for small $\varepsilon > 0$,

$$\begin{aligned}
& \max_{t \geq 0} \Phi_{s,p}(t\hat{v}_\varepsilon) \\
& \leq \left(\frac{1}{2} - \frac{1}{2^*} \right) \mu_s(\mathbb{R}_+^n)^{\frac{2^*}{2^* - 2}} - \frac{\lambda}{p+1} \mu_s(\mathbb{R}_+^n)^{\frac{p+1}{2^* - 2}} \int_{\mathbb{R}_+^n} u^{p+1} dy \varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2}} + O(\varepsilon) + o(\varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2}}) \\
& < \left(\frac{1}{2} - \frac{1}{2^*} \right) \mu_s(\mathbb{R}_+^n)^{\frac{2^*}{2^* - 2}}.
\end{aligned}$$

Thus (iii) is proved. $\square$

In the proof of the assertion (iv) of Theorem 1.1, we use the information derived from the positive solution of the equation

$$\begin{cases} \Delta v + \lambda v^{\frac{n+2}{n-2}} + \frac{v^{2^*-1}}{|x|^s} = 0 & \text{in } \mathbb{R}_+^n, \\ v = 0 & \text{on } \partial\mathbb{R}_+^n. \end{cases} \quad (2.22)$$

If the equation (2.22) has no positive solution, the proof of Theorem 1.1 is valid. In case (2.22) has a positive solution, we use the following lemma in our proof.

Lemma 2.3. *Suppose that Ω is a bounded domain in $\mathbb{R}^n$ with $0 \in \partial\Omega$, $\lambda > 0$, $\partial\Omega \in C^3$ at 0 and $H(0) < 0$. Then there exists a nonnegative function $v_0 \in H_0^1(\Omega) \setminus \{0\}$ such that $\Phi_{s,*}(v_0) < 0$ and*

$$\max_{t \geq 0} \Phi_{s,*}(tv_0) < c^*,$$

where

$$\Phi_{s,*}(u) := \Phi_{s, \frac{n+2}{n-2}}(u) := \int_{\Omega} \left(\frac{1}{2} |\nabla u|^2 - \frac{(n-2)\lambda}{2n} (u^+)^{\frac{2n}{n-2}} - \frac{1}{2^*} \frac{(u^+)^{2^*}}{|x|^s} \right) dx \quad \text{for } u \in H_0^1(\Omega),$$

$c^* := \Phi_{s,*}(v)$, and $v \in H_0^1(\mathbb{R}_+^n)$ is the positive solution of the equation (2.22).

Proof. Considering the proof of Lemma 2.1, we take $u = v \in H_0^1(\mathbb{R}_+^n)$ and define $\hat{v}_\varepsilon \in H_0^1(\Omega)$ by (2.4). From the estimates (2.15), we get for $t \geq 0$,

$$\begin{aligned} \Phi_{s,*}(t\hat{v}_\varepsilon) &\leq \frac{t^2}{2} \int_{\Omega} |\nabla \hat{v}_\varepsilon|^2 dx - \frac{\lambda t^{p+1}}{p+1} \int_{\Omega} \hat{v}_\varepsilon^{p+1} dx - \frac{t^{2^*}}{2^*} \int_{\Omega \cap \bar{U}} \frac{v_\varepsilon^{2^*}}{|x|^s} dx \\ &\leq \frac{t^2}{2} \left(\int_{\mathbb{R}_+^n} |\nabla v|^2 dy - K'_1 H(0) \varepsilon + K'_2 H(0) \varepsilon + O(\varepsilon^2) \right) \\ &\quad - \frac{\lambda t^{p+1}}{p+1} \left(\int_{\mathbb{R}_+^n} v^{p+1} dy + O(\varepsilon^{\frac{n(p+1)}{2}}) \right) - \frac{t^{2^*}}{2^*} \left(\int_{\mathbb{R}_+^n} \frac{v^{2^*}}{|y|^s} dy - \frac{2^* K'_1}{2} H(0) \varepsilon + O(\varepsilon^2) \right) \\ &= \frac{t^2}{2} \int_{\mathbb{R}_+^n} |\nabla v|^2 dy - \frac{\lambda t^{p+1}}{p+1} \int_{\mathbb{R}_+^n} v^{p+1} dy - \frac{t^{2^*}}{2^*} \int_{\mathbb{R}_+^n} \frac{v^{2^*}}{|y|^s} dy \\ &\quad + \frac{H(0)}{2} \left((K'_2 - K'_1) t^2 + K'_1 t^{2^*} \right) \varepsilon + O(\varepsilon^2) \\ &:= f_1(t) + \frac{H(0)\varepsilon}{2} f_2(t) + O(\varepsilon^2), \end{aligned} \quad (2.23)$$

where

$$K'_1 = \frac{2s}{2^*} \int_{\mathbb{R}_+^n} \frac{v^{2^*} |y'|^2 y_n}{|y|^{2+s}} dy \quad \text{and} \quad K'_2 = \int_{\mathbb{R}^{n-1}} |(\nabla v)(y', 0)|^2 |y'|^2 dy'.$$

Since $2^* > 2$, $\frac{2n}{n-2} > 2$ and

$$\int_{\mathbb{R}_+^n} |\nabla v|^2 dy = \lambda \int_{\mathbb{R}_+^n} v^{p+1} dy + \int_{\mathbb{R}_+^n} \frac{v^{2^*}}{|y|^s} dy,$$

we find

$$\sup_{t \geq 0} f_1(t) = f_1(1) = c^* \quad \text{and} \quad f_2(t) > 0 \quad \text{for } t > t_1,$$

where

$$t_1 := \begin{cases} 0 & \text{if } K'_2 > K'_1, \\ \left(\frac{K'_1 - K'_2}{K'_1} \right)^{\frac{1}{2^* - 2}} < 1 & \text{if } K'_2 < K'_1. \end{cases}$$

Hence, in case $H(0) < 0$ and ε small, we conclude

$$\Phi_{s,*}(t\hat{v}_\varepsilon) < f_1(1) = c^*.$$

Finally, we take $v_0 = t_0 \hat{v}_\varepsilon$ where t_0 is large enough so that $\Phi_{s,*}(v_0) < 0$. The lemma is proved. $\square$

Lemma 2.4. *Suppose that Ω is a bounded domain in $\mathbb{R}^n$ with $0 \in \partial\Omega$. If (i) $n = 3$ and $0 < s < 1$ or (ii) $n \geq 4$ and $0 < s < 2$, there exists a nonnegative function $v_0 \in H_0^1(\Omega) \setminus \{0\}$ such that $\Phi_{s,*}(v_0) < 0$ and*

$$\max_{t \geq 0} \Phi_{s,*}(tv_0) < \frac{1}{n} \lambda^{\frac{2-n}{2}} S_n^{\frac{n}{2}},$$

where

$$\Phi_{s,*}(u) := \int_{\Omega} \left(\frac{1}{2} |\nabla u|^2 - \frac{(n-2)\lambda}{2n} (u^+)^{\frac{2n}{n-2}} - \frac{1}{2^*} \frac{(u^+)^{2^*}}{|x|^s} \right) dx \quad \text{for } u \in H_0^1(\Omega)$$

and

$$S_n := \inf \left\{ \int_{\Omega} |\nabla u|^2 dx \mid u \in H_0^1(\Omega) \text{ and } \int_{\Omega} |u|^{\frac{2n}{n-2}} dx = 1 \right\}.$$

Remark. It is well-known that when $\Omega = \mathbb{R}^n$, S_n is achieved by the function

$$g(x) = C(1 + |x|^2)^{-\frac{n-2}{2}},$$

where C is a normalization constant.

Proof of Lemma 2.4. Let x_0 be an interior point of Ω such that $B(x_0, 3r) \subset \Omega$. Take $\phi(x) \in C_c^\infty(\Omega)$ be a cut off function with $\phi|_{B(x_0, r)} = 1$ and $\phi(x)|_{\Omega \setminus B(x_0, 2r)} = 0$. Consider

$$g_\varepsilon(x) := \varepsilon^{-\frac{n-2}{2}} g\left(\frac{x - x_0}{\varepsilon}\right) \phi(x) \in H_0^1(\Omega).$$

For $t \geq 0$, we have

$$\Phi_{s,*}(tg_\varepsilon) = \frac{t^2}{2} \int_{\Omega} |\nabla g_\varepsilon|^2 dx - \frac{(n-2)\lambda t^{\frac{2n}{n-2}}}{2n} \int_{\Omega} g_\varepsilon^{\frac{2n}{n-2}} dx - \frac{t^{2^*}}{2^*} \int_{\Omega} \frac{g_\varepsilon^{2^*}}{|x|^s} dx. \quad (2.24)$$

Using the integration by parts and a change of the variable $y = \frac{x - x_0}{\varepsilon}$, we get

$$\begin{aligned} & \int_{\Omega} |\nabla g_\varepsilon(x)|^2 dx \\ &= \varepsilon^{-n} \int_{B(x_0, 2r)} \left| \left(\nabla g \right) \left(\frac{x - x_0}{\varepsilon} \right) \right|^2 \phi(x) dx - \varepsilon^{2-n} \int_{B(x_0, 2r) \setminus B(x_0, r)} g \left(\frac{x - x_0}{\varepsilon} \right)^2 \phi(x) \Delta \phi(x) dx \\ &= \int_{B(0, \frac{2r}{\varepsilon})} |\nabla g(y)|^2 \phi(x_0 + \varepsilon y) dy - \varepsilon^2 \int_{B(0, \frac{2r}{\varepsilon}) \setminus B(0, \frac{r}{\varepsilon})} g(y)^2 \phi(x_0 + \varepsilon y) (\Delta \phi)(x_0 + \varepsilon y) dy. \end{aligned}$$

Direct calculation gives

$$\begin{cases} \int_{B(0, \frac{2r}{\varepsilon})} |\nabla g(y)|^2 dy = \int_{\mathbb{R}^n} |\nabla g(y)|^2 dy + O(\varepsilon^{n-2}), \\ \int_{\mathbb{R}^n \setminus B(0, \frac{r}{\varepsilon})} g(y)^2 dy = \begin{cases} O\left(\frac{1}{\varepsilon}\right) & \text{if } n = 3, \\ O(\log(\frac{1}{\varepsilon})) & \text{if } n = 4, \\ O(\varepsilon^{n-4}) & \text{if } n \geq 5. \end{cases} \end{cases}$$

Hence,

$$\int_{\Omega} |\nabla g_{\varepsilon}(x)|^2 dx = \int_{\mathbb{R}^n} |\nabla g(y)|^2 dy + \begin{cases} O(\varepsilon) & \text{if } n = 3, \\ O(\varepsilon^2 \log(\frac{1}{\varepsilon})) & \text{if } n = 4, \\ O(\varepsilon^{n-2}) & \text{if } n \geq 5. \end{cases}$$

Next, we have

$$\int_{\Omega} g_{\varepsilon}(x)^{\frac{2n}{n-2}} dx = \int_{\mathbb{R}^n} g(y)^{\frac{2n}{n-2}} dy + O(\varepsilon^n),$$

and

$$\int_{\Omega} \frac{g_{\varepsilon}(x)^{2^*}}{|x|^s} dx = \varepsilon^s \int_{B(0, \frac{2r}{\varepsilon})} \frac{(1 + |y|^2)^{-(n-s)}}{|x_0 + \varepsilon y|^s} \phi(x_0 + \varepsilon y)^{2^*} dy = O(\varepsilon^s).$$

Therefore,

$$\begin{aligned} \Phi_{s,*}(tg_{\varepsilon}) &= \frac{t^2}{2} \int_{\mathbb{R}^n} |\nabla g(y)|^2 dy - \frac{(n-2)\lambda t^{\frac{2n}{n-2}}}{2n} \int_{\mathbb{R}^n} g(y)^{\frac{2n}{n-2}} dy \\ &\quad - \varepsilon^s \int_{B(0, \frac{2r}{\varepsilon})} \frac{(1 + |y|^2)^{-(n-s)}}{|x_0 + \varepsilon y|^s} \phi(x_0 + \varepsilon y)^{2^*} dy + \begin{cases} O(\varepsilon) & \text{if } n = 3, \\ O(\varepsilon^2 \log(\frac{1}{\varepsilon})) & \text{if } n = 4, \\ O(\varepsilon^{n-2}) & \text{if } n \geq 5. \end{cases} \end{aligned}$$

Elementary calculus gives

$$\max_{t \geq 0} \left(\frac{t^2}{2} \int_{\mathbb{R}^n} |\nabla g(y)|^2 dy - \frac{(n-2)\lambda t^{\frac{2n}{n-2}}}{2n} \int_{\mathbb{R}^n} g(y)^{\frac{2n}{n-2}} dy \right) = \frac{1}{n} \lambda^{\frac{2-n}{2}} S_n^{\frac{n}{2}}.$$

To sum up,

$$\max_{t \geq 0} \Phi_{s,*}(tg_{\varepsilon}) < \frac{1}{n} \lambda^{\frac{2-n}{2}} S_n^{\frac{n}{2}},$$

if either (i) $n = 3$ and $0 < s < 1$ or (ii) $n \geq 4$ and $0 < s < 2$. This completes the proof. $\square$

We need the following lemma in the proof of Theorem 1.2.

Lemma 2.5. *Suppose that Ω is a bounded domain in $\mathbb{R}^n$ with $0 \in \partial\Omega$, $\lambda > 0$, $\partial\Omega \in C^3$ at 0, $H(0) < 0$ and $1 \leq p < \frac{n}{n-2}$. Then there exists a nonnegative function $v_0 \in H_0^1(\Omega) \setminus \{0\}$ such that $\Psi_{s,p}(v_0) < 0$ and*

$$\max_{0 \leq t \leq 1} \Psi_{s,p}(tv_0) < \left(\frac{1}{2} - \frac{1}{2^*} \right) \mu_s(\mathbb{R}_+^n)^{\frac{2^*}{2^*-2}},$$

where

$$\Psi_{s,p}(u) := \int_{\Omega} \left(\frac{1}{2} |\nabla u|^2 + \frac{\lambda}{p+1} (u^+)^{p+1} - \frac{1}{2^*} \frac{(u^+)^{2^*}}{|x|^s} \right) dx \quad \text{for } u \in H_0^1(\Omega).$$

Proof. As in the proof of Lemma 2.1, we use the entire solution $u \in H_0^1(\mathbb{R}_+^n)$ of the equation (2.2), and define $\hat{v}_{\varepsilon} \in H_0^1(\Omega)$ by (2.4). Then from the estimates (2.15), we obtain for $t \geq 0$,

$$\begin{aligned} \Psi_{s,p}(t\hat{v}_{\varepsilon}) &\leq \frac{t^2}{2} \int_{\Omega} |\nabla \hat{v}_{\varepsilon}|^2 dx + \frac{\lambda t^{p+1}}{p+1} \int_{\Omega} \hat{v}_{\varepsilon}^{p+1} dx - \frac{t^{2^*}}{2^*} \int_{\Omega} \frac{v_{\varepsilon}^{2^*}}{|x|^s} dx \\ &\leq \frac{t^2}{2} (\mu_s(\mathbb{R}_+^n) - K_1 H(0) \varepsilon + K_2 H(0) \varepsilon + O(\varepsilon^2)) \\ &\quad + \frac{\lambda t^{p+1}}{p+1} \left(\varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2}} \int_{\mathbb{R}_+^n} u^{p+1} dy + O(\varepsilon^{\frac{n(p+1)}{2}}) \right) - \frac{t^{2^*}}{2^*} \left(1 - \frac{2^* K_1}{2\mu_s(\mathbb{R}_+^n)} H(0) \varepsilon + O(\varepsilon^2) \right). \end{aligned} \quad (2.25)$$

It is easy to see that for each small $\varepsilon > 0$, there exists $t_0 > 0$ such that $\Psi_{s,p}(t_0 \hat{v}_\varepsilon) < 0$. Moreover, let t_m be the maximum point of the right-hand side of (2.25) in $(0, t_0)$. Then as in the same way to the proof of Lemma 2.1, we can find the expression of t_m by

$$\begin{aligned} t_m &= \mu_s(\mathbb{R}_+^n)^{\frac{1}{2^*-2}} + \frac{1}{2} \mu_s(\mathbb{R}_+^n)^{-\frac{2^*-3}{2^*-2}} K_1 H(0) \varepsilon + \frac{1}{2^*-2} \mu_s(\mathbb{R}_+^n)^{-\frac{2^*-3}{2^*-2}} K_2 H(0) \varepsilon + O(\varepsilon^2) \\ &+ \frac{\lambda}{2^*-2} \mu_s(\mathbb{R}_+^n)^{-\frac{2^*-2-p}{2^*-2}} \int_{\mathbb{R}_+^n} u^{p+1} dy \varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2}} + O(\varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2} + 1}) + O(\varepsilon^{n+2-(n-2)p}) \\ &= \mu_s(\mathbb{R}_+^n)^{\frac{1}{2^*-2}} + \frac{1}{2} \mu_s(\mathbb{R}_+^n)^{-\frac{2^*-3}{2^*-2}} K_1 H(0) \varepsilon + \frac{1}{2^*-2} \mu_s(\mathbb{R}_+^n)^{-\frac{2^*-3}{2^*-2}} K_2 H(0) \varepsilon + O(\varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2}}), \end{aligned} \quad (2.26)$$

where note that $1 \leq p < \frac{n}{n-2}$ implies $\frac{n+2}{2} - \frac{(n-2)p}{2} \leq 2 < \min\{\frac{n+2}{2} - \frac{(n-2)p}{2} + 1, n+2-(n-2)p\}$. By substituting (2.26) into (2.25), we eventually get

$$\begin{aligned} \max_{0 \leq t \leq t_0} \Psi_{s,p}(t \hat{v}_\varepsilon) &\leq \left(\frac{1}{2} - \frac{1}{2^*} \right) \mu_s(\mathbb{R}_+^n)^{\frac{2^*}{2^*-2}} + \frac{1}{2} \mu_s(\mathbb{R}_+^n)^{\frac{2^*}{2^*-2}} K_2 H(0) \varepsilon + O(\varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2}}) \\ &< \left(\frac{1}{2} - \frac{1}{2^*} \right) \mu_s(\mathbb{R}_+^n)^{\frac{2^*}{2^*-2}}, \end{aligned}$$

which is possible since $H(0) < 0$ and $\frac{n+2}{2} - \frac{(n-2)p}{2} - 1 > 0$, i.e., $p < \frac{n}{n-2}$. Thus Lemma 2.5 is proved. $\square$

3 Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1 by applying Lemma 2.1 and the mountain pass lemma of the following type:

Theorem C. *Let Φ be a C^1 -function on a Banach space E . Assume that there exist an open set $0 \in U \subset E$ and $\rho \in \mathbb{R}$ such that*

$$\begin{cases} \Phi(u) \geq \rho & \text{for all } u \in \partial U, \\ \Phi(0) < \rho, \quad \Phi(v) < \rho & \text{for some } v \neq 0. \end{cases} \quad (3.1)$$

Set

$$c := \inf_{P \in \mathcal{P}} \max_{w \in P} \Phi(w) \geq \rho,$$

where $\mathcal{P}$ denotes the class of continuous paths joining 0 to v . Then there exists a sequence $\{u_j\} \subset E$ such that

$$\begin{cases} \Phi(u_j) \rightarrow c, \\ \Phi'(u_j) \rightarrow 0 & \text{in } E^*. \end{cases}$$

Remark. Suppose $v_i \in H_0^1(\Omega)$ with $\Phi_{s,*}(v_i) < 0$ for $i = 1, 2$, where $\Phi_{s,*}$ is as defined in Lemma 2.3. Then $\Phi_{s,*}(tv_i) < 0$ for $t > 1$, and $\Phi_{s,*}(tv_i) \rightarrow -\infty$ as $t \rightarrow \infty$. Hence, there exists a continuous path $v(\eta) \in H_0^1(\Omega)$ with $v(0) = v_1$, $v(1) = v_2$ and $\Phi_{s,*}(v(\eta)) < 0$. Therefore, if we take

$$c := \inf_{P \in \mathcal{P}} \max_{w \in P} \Phi_{s,*}(w) > 0, \quad (3.2)$$

then c is independent of the choice of v as long as $\Phi_{s,*}(v) < 0$.

Proof of the Subcritical Case for Theorem 1.1. In what follows, take $E = H_0^1(\Omega)$ in Theorem C and check the condition (3.1). By the Sobolev inequality and the Sobolev-Hardy inequality, we have

$$\begin{cases} \int_{\Omega} (u^+)^{p+1} dx \leq \begin{cases} \frac{1}{\lambda_1(\Omega)} \|\nabla u\|_{L^2(\Omega)}^2 & \text{if } p = 1, \\ C \|\nabla u\|_{L^2(\Omega)}^{p+1} & \text{if } p > 1, \end{cases} \\ \int_{\Omega} \frac{(u^+)^{2^*}}{|x|^s} dx \leq C \|\nabla u\|_{L^2(\Omega)}^{2^*}. \end{cases}$$

Thus we see that

$$\begin{aligned} \Phi_{s,p}(u) &= \int_{\Omega} \left(\frac{1}{2} |\nabla u|^2 - \frac{\lambda}{p+1} (u^+)^{p+1} - \frac{1}{2^*} \frac{(u^+)^{2^*}}{|x|^s} \right) dx \\ &\geq \|\nabla u\|_{L^2(\Omega)}^2 \times \begin{cases} \left(\frac{1}{2} \left(1 - \frac{\lambda}{\lambda_1(\Omega)} \right) - C \|\nabla u\|_{L^2(\Omega)}^{2^*-2} \right) & \text{if } p = 1, \\ \left(\frac{1}{2} - C \|\nabla u\|_{L^2(\Omega)}^{p-1} - C \|\nabla u\|_{L^2(\Omega)}^{2^*-2} \right) & \text{if } p > 1. \end{cases} \end{aligned}$$

Noting that $1 - \frac{\lambda}{\lambda_1(\Omega)} > 0$, i.e., $\lambda < \lambda_1(\Omega)$ in the case $p = 1$, and taking $U := B_{r_0}(0)$ with small $r_0 > 0$, we have

$$\Phi_{s,p}(u) \geq \rho > 0 \quad \text{for all } u \in \partial U,$$

where

$$\rho := \begin{cases} r_0^2 \left(\frac{1}{2} \left(1 - \frac{\lambda}{\lambda_1(\Omega)} \right) - C r_0^{2^*-2} \right) & \text{if } p = 1, \\ r_0^2 \left(\frac{1}{2} - C r_0^{p-1} - C r_0^{2^*-2} \right) & \text{if } p > 1. \end{cases}$$

By Lemma 2.1, there exists a nonnegative function $v_0 \in H_0^1(\Omega) \setminus \{0\}$ such that

$$\sup_{t \geq 0} \Phi_{s,p}(t v_0) < \left(\frac{1}{2} - \frac{1}{2^*} \right) \mu_s(\Omega)^{\frac{2^*}{2^*-2}},$$

and set $v =: t_0 v_0$, where $t_0 > 0$ is chosen large enough so that $v \notin U$ and $\Phi_{s,p}(v) \leq 0$. Here, note that

$$\rho \leq c := \inf_{P \in \mathcal{P}} \max_{w \in P} \Phi_{s,p}(w) \leq \sup_{t \geq 0} \Phi_{s,p}(t v) < \left(\frac{1}{2} - \frac{1}{2^*} \right) \mu_s(\Omega)^{\frac{2^*}{2^*-2}}. \quad (3.3)$$

Since $\Phi_{s,p}(0) = 0$, by applying Theorem C, there exists a sequence $\{u_j\} \subset H_0^1(\Omega)$ such that

$$\begin{aligned} \Phi_{s,p}(u_j) &\rightarrow c \quad \text{and} \quad \Phi'_{s,p}(u_j) \rightarrow 0 \quad \text{in } H^{-1}(\Omega), \quad \text{i.e.,} \\ \int_{\Omega} \left(\frac{1}{2} |\nabla u_j|^2 - \frac{\lambda}{p+1} (u_j^+)^{p+1} - \frac{1}{2^*} \frac{(u_j^+)^{2^*}}{|x|^s} \right) dx &= c + o(1), \end{aligned} \quad (3.4)$$

and

$$-\Delta u_j - \lambda (u_j^+)^p - \frac{(u_j^+)^{2^*-1}}{|x|^s} =: \zeta_j \quad \text{with } \zeta_j \rightarrow 0 \quad \text{in } H^{-1}(\Omega). \quad (3.5)$$

We show the boundedness of $\{u_j\}$. Multiplying (3.5) by u_j , we obtain

$$\int_{\Omega} \left(|\nabla u_j|^2 - \lambda (u_j^+)^{p+1} - \frac{(u_j^+)^{2^*}}{|x|^s} \right) dx = \langle \zeta_j, u_j \rangle, \quad (3.6)$$

and (3.4) - $\frac{1}{2}$ (3.6) yields that

$$\int_{\Omega} \left(\lambda \left(\frac{1}{2} - \frac{1}{p+1} \right) (u_j^+)^{p+1} + \left(\frac{1}{2} - \frac{1}{2^*} \right) \frac{(u_j^+)^{2^*}}{|x|^s} \right) dx = c + o(1) - \frac{1}{2} \langle \zeta_j, u_j \rangle$$

$$\leq c + o(1) + \frac{1}{2} \|\zeta_j\|_{H^{-1}(\Omega)} \|u_j\|_{H_0^1(\Omega)} \leq C \left(1 + \|u_j\|_{H_0^1(\Omega)}\right). \quad (3.7)$$

By (3.4) and (3.7), we have

$$\|u_j\|_{H_0^1(\Omega)}^2 \leq C \left(1 + \|u_j\|_{H_0^1(\Omega)}\right),$$

which implies $\|u_j\|_{H_0^1(\Omega)} \leq C$. Then extracting a subsequence, still denoted by u_j , we see

$$\begin{cases} u_j \rightharpoonup u & \text{weakly in } H_0^1(\Omega), \\ u_j^+ \rightarrow u^+ & \text{strongly in } L^{p+1}(\Omega), \\ \frac{u_j^+}{|x|^{\frac{s}{2^*}}} \rightharpoonup \frac{u^+}{|x|^{\frac{s}{2^*}}} & \text{weakly in } L^{2^*}(\Omega). \end{cases}$$

Thus passing to the limit in (3.5) yields that

$$\Delta u + \lambda (u^+)^p + \frac{(u^+)^{2^*-1}}{|x|^s} = 0,$$

and then from the maximum principle, we obtain $u \geq 0$ in Ω .

Finally, we shall prove $u \neq 0$ in $H_0^1(\Omega)$, and suppose $u \equiv 0$ in Ω . Since $\int_{\Omega} |\nabla u_j|^2 dx$ and $\int_{\Omega} \frac{(u_j^+)^{2^*}}{|x|^s} dx$ are both bounded, we may assume

$$\lim_{j \rightarrow \infty} \int_{\Omega} |\nabla u_j|^2 dx =: C_1 \quad \text{and} \quad \lim_{j \rightarrow \infty} \int_{\Omega} \frac{(u_j^+)^{2^*}}{|x|^s} dx =: C_2.$$

Thus passing to the limit in (3.4) and (3.6), we get

$$\frac{C_1}{2} - \frac{C_2}{2^*} = c \quad \text{and} \quad C_1 - C_2 = 0,$$

and then we have

$$c = \left(\frac{1}{2} - \frac{1}{2^*}\right) C_1. \quad (3.8)$$

Here, we see that

$$\int_{\Omega} |\nabla u_j|^2 dx \geq \mu_s(\Omega) \left(\int_{\Omega} \frac{|u_j|^{2^*}}{|x|^s} dx \right)^{\frac{2}{2^*}} \geq \mu_s(\Omega) \left(\int_{\Omega} \frac{(u_j^+)^{2^*}}{|x|^s} dx \right)^{\frac{2}{2^*}},$$

and then

$$C_1 \geq \mu_s(\Omega) C_1^{\frac{2}{2^*}}, \quad \text{i.e.,} \quad C_1 \geq \mu_s(\Omega)^{\frac{2^*}{2^*-2}}. \quad (3.9)$$

By (3.8) and (3.9), we have

$$c \geq \left(\frac{1}{2} - \frac{1}{2^*}\right) \mu_s(\Omega)^{\frac{2^*}{2^*-2}}. \quad (3.10)$$

Thus combining (3.3) with (3.10) yields

$$\left(\frac{1}{2} - \frac{1}{2^*}\right) \mu_s(\Omega)^{\frac{2^*}{2^*-2}} \leq c < \left(\frac{1}{2} - \frac{1}{2^*}\right) \mu_s(\Omega)^{\frac{2^*}{2^*-2}},$$

which is a contradiction. Thus $v \neq 0$ in $H_0^1(\Omega)$. $\square$

Proof of the Critical Case for Theorem 1.1. We now consider the case that $p = \frac{n+2}{n-2}$. We divide the proof into two steps.

Step 1. By Lemma 2.4, there exists a nonnegative function $v_0 \in H_0^1(\Omega) \setminus \{0\}$ with $\Phi_{s,*}(v_0) < 0$ such that

$$\max_{t \geq 0} \Phi_{s,*}(tv_0) < \frac{1}{n} \lambda^{\frac{2-n}{2}} S_n^{\frac{n}{2}}.$$

Hence, it is easy to see there exists $\varepsilon_0 > 0$ such that $\Phi_{s,*}^\varepsilon(tv_0) < 0$ and

$$\max_{0 \leq t \leq 1} \Phi_{s,*}^\varepsilon(tv_0) < \frac{1}{n} \lambda^{\frac{2-n}{2}} S_n^{\frac{n}{2}} \quad \text{for } 0 \leq \varepsilon \leq \varepsilon_0,$$

where

$$\Phi_{s,*}^\varepsilon(u) := \int_{\Omega} \left(\frac{1}{2} |\nabla u|^2 - \frac{\lambda}{p_\varepsilon} (u^+)^{p_\varepsilon} - \frac{1}{2^*(s) - \varepsilon} \frac{(u^+)^{2^* - \varepsilon}}{|x|^s} \right) dx \quad \text{for } u \in H_0^1(\Omega),$$

and

$$p_\varepsilon := \frac{2n}{n-2} - \frac{2\varepsilon}{2-s}.$$

Note that $\Phi_{s,*}^0 = \Phi_{s,*}$.

Taking ε_0 small such that $p_{\varepsilon_0} - 2 > 0$ and $2^* - 2 - \varepsilon_0 > 0$, by the Sobolev-Hardy inequality,

$$\begin{aligned} \Phi_{s,*}^\varepsilon(u) &= \int_{\Omega} \left(\frac{1}{2} |\nabla u|^2 - \frac{\lambda}{p_\varepsilon} (u^+)^{p_\varepsilon} - \frac{1}{2^*(s) - \varepsilon} \frac{u^{2^* - \varepsilon}}{|x|^s} \right) dx \\ &\geq \frac{1}{2} \|\nabla u\|_{L^2(\Omega)}^2 - C \|\nabla u\|_{L^2(\Omega)}^{p_\varepsilon} - C \|\nabla u\|_{L^2(\Omega)}^{2^* - \varepsilon} \\ &= \|\nabla u\|_{L^2(\Omega)}^2 \left(\frac{1}{2} - C \|\nabla u\|_{L^2(\Omega)}^{p_\varepsilon - 2} - C \|\nabla u\|_{L^2(\Omega)}^{2^* - 2 - \varepsilon} \right), \end{aligned}$$

where the constant $C > 0$ is independent of ε . Therefore, there exist $r_0 > 0$ and $\rho_0 > 0$ such that

$$\Phi_{s,*}^\varepsilon(u) \geq r_0^2 \left(\frac{1}{2} - C r_0^{p_\varepsilon - 2} - C r_0^{2^* - 2 - \varepsilon} \right) =: \rho_\varepsilon \geq \rho_0 > 0$$

for any $u \in \partial B_{r_0}(0)$, where $B_{r_0}(0) = \{u \in H_0^1(\Omega) \mid \|\nabla u\|_{L^2(\Omega)} < r_0\}$. It is obvious that $v_0 \notin \overline{B_{r_0}(0)}$. Hence, we obtain

$$\rho_0 \leq c_\varepsilon := \inf_{P \in \mathcal{P}} \max_{u \in P} \Phi_{s,*}^\varepsilon(u) \leq \max_{0 \leq t \leq 1} \Phi_{s,*}^\varepsilon(tv_0) < \frac{1}{n} \lambda^{\frac{2-n}{2}} S_n^{\frac{n}{2}},$$

where $\mathcal{P}$ denotes the class of continuous paths in $H_0^1(\Omega)$ joining 0 to v_0 .

Since $\Phi_{s,*}^\varepsilon(0) = 0$, by applying Theorem C, there exists a double sequence $\{u_{\varepsilon,j}\} \subset H_0^1(\Omega)$ such that

$$\begin{aligned} \Phi_{s,*}^\varepsilon(u_{\varepsilon,j}) &\rightarrow c_\varepsilon \quad \text{and} \quad (\Phi_{s,*}^\varepsilon)'(u_{\varepsilon,j}) \rightarrow 0 \quad \text{in } H^{-1}(\Omega), \quad \text{i.e.,} \\ \int_{\Omega} \left(\frac{1}{2} |\nabla u_{\varepsilon,j}|^2 - \frac{\lambda}{p_\varepsilon} (u_{\varepsilon,j}^+)^{p_\varepsilon} - \frac{1}{2^*(s) - \varepsilon} \frac{(u_{\varepsilon,j}^+)^{2^* - \varepsilon}}{|x|^s} \right) dx &= c_\varepsilon + o(1), \end{aligned} \tag{3.11}$$

and

$$-\Delta u_{\varepsilon,j} - \lambda (u_{\varepsilon,j}^+)^{p_\varepsilon - 1} - \frac{(u_{\varepsilon,j}^+)^{2^* - 1 - \varepsilon}}{|x|^s} =: \zeta_{\varepsilon,j} \quad \text{with } \zeta_{\varepsilon,j} \rightarrow 0 \quad \text{in } H^{-1}(\Omega) \quad \text{as } j \rightarrow \infty. \tag{3.12}$$

Applying (3.12) on $u_{\varepsilon,j}$, we get

$$\int_{\Omega} \left(|\nabla u_{\varepsilon,j}|^2 - \lambda(u_{\varepsilon,j}^+)^{p_\varepsilon} - \frac{(u_{\varepsilon,j}^+)^{2^*-\varepsilon}}{|x|^2} \right) dx = \langle \zeta_{\varepsilon,j}, u_{\varepsilon,j} \rangle. \quad (3.13)$$

Similar to the subcritical case, for $0 \leq \varepsilon \leq \varepsilon_0$ with ε_0 small, we get the uniform boundedness of $\{u_{\varepsilon,j}\}$,

$$\|u_{\varepsilon,j}\|_{H_0^1(\Omega)} \leq C. \quad (3.14)$$

Hereafter, we assume $\varepsilon > 0$. Then, up to a subsequence, there exists a function $u_\varepsilon \in H_0^1(\Omega)$ such that as $j \rightarrow \infty$,

$$\begin{cases} u_{\varepsilon,j} \rightharpoonup u_\varepsilon & \text{weakly in } H_0^1(\Omega), \\ u_{\varepsilon,j}^+ \rightarrow u_\varepsilon^+ & \text{strongly in } L^{p_\varepsilon}(\Omega), \\ \frac{u_{\varepsilon,j}^+}{|x|^{\frac{s}{2^*-\varepsilon}}} \rightarrow \frac{u_\varepsilon^+}{|x|^{\frac{s}{2^*-\varepsilon}}} & \text{strongly in } L^{2^*-\varepsilon}(\Omega). \end{cases}$$

We claim that $u_\varepsilon^+ \neq 0$ in $H_0^1(\Omega)$. Suppose, on the contrary, $u_\varepsilon^+ = 0$ in $H_0^1(\Omega)$. Then passing to the limit $j \rightarrow \infty$ in (3.11) and (3.13) yields that

$$c_\varepsilon = \frac{1}{2} \int_{\Omega} |\nabla u_\varepsilon|^2 dx = 0,$$

which contradicts to $c_\varepsilon \geq \rho_0 > 0$. Thus, $u_\varepsilon^+ \neq 0$ in $H_0^1(\Omega)$. Hence, taking the limit $j \rightarrow \infty$ in (3.12), we see that

$$\Delta u_\varepsilon + \lambda(u_\varepsilon^+)^{p_\varepsilon-1} + \frac{(u_\varepsilon^+)^{2^*-1-\varepsilon}}{|x|^s} = 0 \quad \text{in } \Omega. \quad (3.15)$$

By the maximum principle, we find that $u_\varepsilon \geq 0$ in Ω . Inferring from (3.11) and (3.15), this positive solution u_ε satisfies

$$\begin{cases} \frac{1}{2} \int_{\Omega} |\nabla u_\varepsilon|^2 dx - \frac{\lambda}{p_\varepsilon} \int_{\Omega} u_\varepsilon^{p_\varepsilon} dx - \frac{1}{2^*-\varepsilon} \int_{\Omega} \frac{u_\varepsilon^{2^*-\varepsilon}}{|x|^s} dx = c_\varepsilon, \\ \int_{\Omega} |\nabla u_\varepsilon|^2 dx - \lambda \int_{\Omega} u_\varepsilon^{p_\varepsilon} dx - \int_{\Omega} \frac{u_\varepsilon^{2^*-\varepsilon}}{|x|^s} dx = 0. \end{cases} \quad (3.16)$$

Note that by (3.14), we have

$$\|u_\varepsilon\|_{H_0^1(\Omega)} \leq C.$$

Thus, by extracting a subsequence $\{u_j := u_{\varepsilon_j}\}_{j \in \mathbb{N}}$ with $\varepsilon_j \rightarrow 0$ as $j \rightarrow \infty$, there exists a function $u \in H_0^1(\Omega)$ such that

$$\begin{cases} u_j \rightharpoonup u & \text{weakly in } H_0^1(\Omega), \\ u_j^+ \rightharpoonup u^+ & \text{weakly in } L^{\frac{2n}{n-2}}(\Omega), \\ \frac{u_j^+}{|x|^{\frac{s}{2^*}}} \rightharpoonup \frac{u^+}{|x|^{\frac{s}{2^*}}} & \text{weakly in } L^{2^*}(\Omega). \end{cases}$$

Now, passing to the limit in (3.15) yields that

$$\Delta u + \lambda(u^+)^{\frac{n+2}{n-2}} + \frac{(u^+)^{2^*-1}}{|x|^s} = 0 \quad \text{in } \Omega.$$

By the maximum principle, we see that $u \geq 0$ in Ω . Therefore, u satisfies

$$\Delta u + \lambda u^{\frac{n+2}{n-2}} + \frac{u^{2^*-1}}{|x|^s} = 0 \quad \text{in } \Omega.$$

The rest part of the proof is to show $u \neq 0$ in $H_0^1(\Omega)$. Assume $u = 0$ in $H_0^1(\Omega)$. Then the blow-up occurs, i.e.,

$$m_j := u_j(x_j) = \max_{\Omega} u_j(x) \rightarrow \infty \quad \text{as } j \rightarrow \infty.$$

Otherwise, by (3.16) and the Lebesgue theorem, we have

$$0 = \lim_{j \rightarrow \infty} \frac{1}{2} \int_{\Omega} |\nabla u_j|^2 dx = c_0,$$

which contradicts to $c_0 > 0$.

We consider the scaling

$$v_j(y) := m_j^{-1} u_j(x_j + k_j y) \quad \text{for } y \in \Omega_j := \left\{ z \in \mathbb{R}^n \mid x_j + k_j z \in \Omega \right\},$$

where $k_j = m_j^{-\frac{p_j-2}{2}}$ and $p_j = \frac{2n}{n-2} - \frac{2\varepsilon_j}{2-s}$. By (3.15), v_j satisfies

$$\begin{cases} \Delta v_j + \lambda v_j^{p_j-1} + \frac{v_j^{2^*-1-\varepsilon_j}}{\left| \frac{x_j}{k_j} + y \right|^s} = 0 & \text{in } \Omega_j, \\ v_j = 0 & \text{on } \partial\Omega_j. \end{cases} \quad (3.17)$$

Let $\Omega_{\infty} = \lim_{j \rightarrow \infty} \Omega_j$. We distinguish into the following cases.

Case 1. If, up to a subsequence, $\frac{|x_j|}{k_j} \rightarrow \infty$, then $v_j(y)$ converges to some $v(y)$ uniformly in every compact subset of $\overline{\Omega_{\infty}}$, where $v(y) \in H_0^1(\Omega_{\infty})$ with $v(0) = 1$ is the solution of the equation

$$\begin{cases} \Delta v + \lambda v^{\frac{n+2}{n-2}} = 0 & \text{in } \Omega_{\infty}, \\ v = 0 & \text{on } \partial\Omega_{\infty}. \end{cases}$$

It is well-known that the above equation is only solvable for $\Omega_{\infty} = \mathbb{R}^n$. We easily see that

$$\begin{cases} C_1 := \lim_{j \rightarrow \infty} \int_{\Omega} |\nabla u_j|^2 dx = m_j^{\left(\frac{n-2}{2-s}\right)\varepsilon_j} \lim_{j \rightarrow \infty} \int_{\Omega_j} |\nabla v_j|^2 dy \geq \int_{\mathbb{R}^n} |\nabla v|^2 dy =: A_1, \\ C_2 := \lim_{j \rightarrow \infty} \int_{\Omega} u_j^{\frac{2n}{n-2}} dx = m_j^{\left(\frac{n}{2-s}\right)\varepsilon_j} \lim_{j \rightarrow \infty} \int_{\Omega_j} v_j^{\frac{2n}{n-2}} dy \geq \int_{\mathbb{R}^n} v^{\frac{2n}{n-2}} dy =: A_2, \\ C_3 := \lim_{j \rightarrow \infty} \int_{\Omega} \frac{u_j^{2^*}}{|x|^s} dx = m_j^{\left(\frac{n-s}{2-s}\right)\varepsilon_j} \lim_{j \rightarrow \infty} \int_{\Omega_j} \frac{v_j^{2^*}}{\left| \frac{x_j}{k_j} + y \right|^s} dy. \end{cases} \quad (3.18)$$

Furthermore, note that

$$\frac{C_1}{2} - \frac{(n-2)\lambda}{2n} C_2 - \frac{C_3}{2^*} = c_0, \quad C_1 - \lambda C_2 - C_3 = 0, \quad \text{and} \quad A_1 = \lambda A_2. \quad (3.19)$$

By (3.18) and (3.19), we have

$$c_0 = \left(\frac{1}{2} - \frac{1}{2^*} \right) C_1 + \lambda \left(\frac{1}{2^*} - \frac{n-2}{2n} \right) C_2 \geq \left(\frac{1}{2} - \frac{1}{2^*} \right) A_1 + \lambda \left(\frac{1}{2^*} - \frac{n-2}{2n} \right) A_2 = \frac{\lambda}{n} A_2.$$

On the other hand, by the Sobolev inequality, we see that

$$S_n A_2^{\frac{n-2}{n}} \leq A_1.$$

This leads to

$$A_2 \geq \lambda^{-\frac{n}{2}} S_n^{\frac{n}{2}}.$$

Hence,

$$c_0 \geq \frac{\lambda}{n} A_2 \geq \frac{1}{n} \lambda^{\frac{2-n}{2}} S_n^{\frac{n}{2}},$$

which contradicts to

$$c_0 \leq \max_{0 \leq t \leq 1} \Phi_{s,*}(tv_0) < \frac{1}{n} \lambda^{\frac{2-n}{2}} S_n^{\frac{n}{2}}.$$

Case 2. If, up to a subsequence, $\frac{x_j}{k_j} \rightarrow y_0 \in \mathbb{R}^n$, then Ω_∞ is a half space. Therefore, up to a linear transformation, v_j converges to some v uniformly in any compact set of $\overline{\mathbb{R}_+^n}$, where v is a positive solution of the equation

$$\begin{cases} \Delta v + \lambda v^{\frac{n+2}{n-2}} + \frac{v^{2^*-1}}{|y|^s} = 0 & \text{in } \mathbb{R}_+^n, \\ v = 0 & \text{on } \partial\mathbb{R}_+^n \end{cases} \quad (3.20)$$

with $v(y_0) = 1$ for some $y_0 \in \mathbb{R}_+^n$. If the equation (3.20) has no positive solution, this leads to a contradiction. Hence, the sequence $u_j(x)$ does not blow-up.

Step 2. However, if the equation (3.20) admits a positive solution, by Lemma 2.3 and Lemma 2.4, there exists a nonnegative function $v_0 \in H_0^1(\Omega) \setminus \{0\}$ with $\Phi_{s,*}(v_0) < 0$ such that

$$\max_{t \geq 0} \Phi_{s,*}(tv_0) < \min \left\{ c^*, \frac{1}{n} \lambda^{\frac{2-n}{2}} S_n^{\frac{n}{2}} \right\}.$$

Redoing Step 1, since the min-max value c_0 is independent of the choice of v_0 , we only need to deal with Case 2, see Remark after Theorem C. Let

$$B_1 = \int_{\mathbb{R}_+^n} |\nabla v(y)|^2 dy, \quad B_2 = \int_{\mathbb{R}_+^n} v^{\frac{2n}{n-2}} dy \quad \text{and} \quad B_3 = \int_{\mathbb{R}_+^n} \frac{v^{2^*}}{|y|^s} dy.$$

Noting that

$$\begin{cases} C_1 \geq B_1, & C_2 \geq B_2, & C_3 \geq B_3, \\ C_1 - \lambda C_2 - C_3 = 0, & B_1 - \lambda B_2 - B_3 = 0, \\ c_0 = \frac{C_1}{2} - \frac{(n-2)\lambda}{2n} C_2 - \frac{1}{2^*} C_3, \end{cases}$$

we see that

$$c_0 = \frac{\lambda}{n} C_2 + \left(\frac{1}{2} - \frac{1}{2^*} \right) C_3 \geq \frac{\lambda}{n} B_2 + \left(\frac{1}{2} - \frac{1}{2^*} \right) B_3 = \frac{B_1}{2} - \frac{(n-2)\lambda}{2n} B_2 - \frac{1}{2^*} B_3 = c^*.$$

This contradicts to $c_0 \leq \max_{0 \leq t \leq 1} \Phi_{s,*}(tv_0) < c^*$. Hence, we have proved $u \neq 0$ in $H_0^1(\Omega)$. The positivity of u is achieved by the strong maximum principle. The proof is complete. $\square$

4 Proof of Theorem 1.2

In this section, we shall prove Theorem 1.2. However, since $H(0) < 0$ implies $\mu_s(\Omega) < \mu_s(\mathbb{R}_+^n)$, we cannot apply Lemma 2.5 according to the argument of Theorem 1.1 to show the existence of a solution for the equation (1.3).

Proof of Theorem 1.2. For any small $\varepsilon \geq 0$, we let

$$\Psi_{s,p}^\varepsilon(u) := \int_{\Omega} \left(\frac{1}{2} |\nabla u|^2 + \frac{\lambda}{p+1} (u^+)^{p+1} - \frac{1}{2^* - \varepsilon} \frac{(u^+)^{2^* - \varepsilon}}{|x|^s} \right) dx \quad \text{for } u \in H_0^1(\Omega),$$

where $\Psi_{s,p}^0 = \Psi_{s,p}$. By $\lambda > 0$ and the Sobolev-Hardy inequality, we see that

$$\Psi_{s,p}^\varepsilon(u) \geq \frac{1}{2} \|\nabla u\|_{L^2(\Omega)}^2 - C \|\nabla u\|_{L^2(\Omega)}^{2^* - \varepsilon} = \|\nabla u\|_{L^2(\Omega)}^2 \left(\frac{1}{2} - C \|\nabla u\|_{L^2(\Omega)}^{2^* - 2 - \varepsilon} \right),$$

where $C > 0$ is independent of $\varepsilon \geq 0$. Then there exist positive constants r and ρ such that

$$\Psi_{s,p}^\varepsilon(u) \geq r^2 \left(\frac{1}{2} - C r^{2^* - 2 - \varepsilon} \right) =: \rho_\varepsilon \geq \rho > 0 \quad (4.1)$$

for all $u \in \partial B_r(0)$. On the other hand, by Lemma 2.5, there exists a nonnegative function $v_0 \in H_0^1(\Omega) \setminus \{0\}$ such that $\Psi_{s,p}^0(v_0) < 0$ and

$$\max_{0 \leq t \leq 1} \Psi_{s,p}^0(tv_0) < \left(\frac{1}{2} - \frac{1}{2^*} \right) \mu_s(\mathbb{R}_+^n)^{\frac{2^*}{2^* - 2}}. \quad (4.2)$$

By (4.1), (4.2) and the continuity of $\Psi_{s,p}^\varepsilon$ at $\varepsilon = 0$, we see that $\Psi_{s,p}^\varepsilon(v_0) < 0$ and

$$0 < \rho \leq c_\varepsilon := \inf_{P \in \mathcal{P}} \max_{w \in P} \Psi_{s,p}^\varepsilon(w) \leq \max_{0 \leq t \leq 1} \Psi_{s,p}^\varepsilon(tv) < \left(\frac{1}{2} - \frac{1}{2^*} \right) \mu_s(\mathbb{R}_+^n)^{\frac{2^*}{2^* - 2}} \quad (4.3)$$

for any small $\varepsilon \geq 0$. Since $\Psi_{s,p}^\varepsilon(0) = 0$, by applying Theorem C, there exists a sequence $\{u_{\varepsilon,j}\} \subset H_0^1(\Omega)$ such that

$$\begin{aligned} \Psi_{s,p}^\varepsilon(u_{\varepsilon,j}) &\rightarrow c_\varepsilon \quad \text{and} \quad (\Psi_{s,p}^\varepsilon)'(u_{\varepsilon,j}) \rightarrow 0 \quad \text{in } H^{-1}(\Omega) \quad \text{as } j \rightarrow \infty, \quad \text{i.e.,} \\ \int_{\Omega} \left(\frac{1}{2} |\nabla u_{\varepsilon,j}|^2 + \frac{\lambda}{p+1} (u_{\varepsilon,j}^+)^{p+1} - \frac{1}{2^* - \varepsilon} \frac{(u_{\varepsilon,j}^+)^{2^* - \varepsilon}}{|x|^s} \right) dx &= c_\varepsilon + o(1) \quad \text{as } j \rightarrow \infty, \end{aligned} \quad (4.4)$$

and

$$-\Delta u_{\varepsilon,j} + \lambda (u_{\varepsilon,j}^+)^p - \frac{(u_{\varepsilon,j}^+)^{2^* - 1 - \varepsilon}}{|x|^s} =: \zeta_{\varepsilon,j} \quad \text{with } \zeta_{\varepsilon,j} \rightarrow 0 \quad \text{in } H^{-1}(\Omega) \quad \text{as } j \rightarrow \infty. \quad (4.5)$$

Multiplying (4.5) by $u_{\varepsilon,j}$, we obtain

$$\int_{\Omega} \left(|\nabla u_{\varepsilon,j}|^2 + \lambda (u_{\varepsilon,j}^+)^{p+1} - \frac{(u_{\varepsilon,j}^+)^{2^* - \varepsilon}}{|x|^s} \right) dx = \langle \zeta_{\varepsilon,j}, u_{\varepsilon,j} \rangle. \quad (4.6)$$

We show the boundedness of $\{u_{\varepsilon,j}\}$ in $H_0^1(\Omega)$. (4.4) – $\frac{1}{2}$ (4.6) yields that

$$\begin{aligned} \int_{\Omega} \left(\lambda \left(\frac{1}{p+1} - \frac{1}{2} \right) (u_{\varepsilon,j}^+)^{p+1} + \left(\frac{1}{2} - \frac{1}{2^* - \varepsilon} \right) \frac{(u_{\varepsilon,j}^+)^{2^* - \varepsilon}}{|x|^s} \right) dx &= c_\varepsilon + o(1) - \frac{1}{2} \langle \zeta_{\varepsilon,j}, u_{\varepsilon,j} \rangle \\ &\leq C(1 + \|u_{\varepsilon,j}\|_{H_0^1(\Omega)}), \end{aligned}$$

and then we have

$$\int_{\Omega} \frac{(u_{\varepsilon,j}^+)^{2^* - \varepsilon}}{|x|^s} dx \leq \frac{\lambda(p-1)(2^* - \varepsilon)}{(p+1)(2^* - 2 - \varepsilon)} \int_{\Omega} (u_{\varepsilon,j}^+)^{p+1} dx + C(1 + \|u_{\varepsilon,j}\|_{H_0^1(\Omega)}), \quad (4.7)$$

where $C > 0$ can be taken independently of $j \in \mathbb{N}$ and $\varepsilon \geq 0$. Then by (4.4) and (4.7), we have

$$\begin{aligned} \frac{1}{2} \|\nabla u_{\varepsilon,j}\|_{L^2(\Omega)}^2 &\leq -\frac{\lambda}{p+1} \left(1 - \frac{p-1}{2^*-2-\varepsilon}\right) \int_{\Omega} (u_{\varepsilon,j}^+)^{p+1} dx + C(1 + \|u_{\varepsilon,j}\|_{H_0^1(\Omega)}) \\ &\leq C(1 + \|u_{\varepsilon,j}\|_{H_0^1(\Omega)}), \end{aligned} \quad (4.8)$$

where we used that $1 - \frac{p-1}{2^*-2-\varepsilon} > 0$ for any small $\varepsilon \geq 0$ by virtue of $1 - \frac{p-1}{2^*-2} > 0$, i.e., $p < 2^* - 1$. (4.8) implies

$$\|u_{\varepsilon,j}\|_{H_0^1(\Omega)} \leq C, \quad (4.9)$$

where $C > 0$ is independent of $j \in \mathbb{N}$ and also $\varepsilon \geq 0$.

Hereafter, we take $\varepsilon > 0$. Extracting a subsequence, still denoted by $u_{\varepsilon,j}$, we see that

$$\left\{ \begin{array}{ll} u_{\varepsilon,j} \rightharpoonup u_{\varepsilon} & \text{weakly in } H_0^1(\Omega), \\ u_{\varepsilon,j}^+ \rightarrow u_{\varepsilon}^+ & \text{strongly in } L^{p+1}(\Omega), \\ \frac{u_{\varepsilon,j}^+}{|x|^{\frac{2^*-\varepsilon}{2}}} \rightarrow \frac{u_{\varepsilon}^+}{|x|^{\frac{2^*-\varepsilon}{2}}} & \text{strongly in } L^{2^*-\varepsilon}(\Omega) \end{array} \right. \quad (4.10)$$

as $j \rightarrow \infty$. Then by passing to the limit $j \rightarrow \infty$ in (4.5), we get

$$-\Delta u_{\varepsilon} + \lambda (u_{\varepsilon}^+)^p - \frac{(u_{\varepsilon}^+)^{2^*-1-\varepsilon}}{|x|^s} = 0 \quad \text{in } \Omega,$$

and then by the maximum principle, we obtain $u_{\varepsilon} \geq 0$ in Ω .

It is easy to see that $u_{\varepsilon} \neq 0$ in $H_0^1(\Omega)$. As a consequence, for any small $\varepsilon > 0$, we get a positive solution $u_{\varepsilon} \in H_0^1(\Omega)$ satisfying

$$\Delta u_{\varepsilon} - \lambda u_{\varepsilon}^p + \frac{u_{\varepsilon}^{2^*-1-\varepsilon}}{|x|^s} = 0. \quad (4.11)$$

Next, for $\varepsilon > 0$, passing to the limit $j \rightarrow \infty$ in (4.4) and multiplying (4.11) by u_{ε} yield that

$$\left\{ \begin{array}{l} \frac{1}{2} \int_{\Omega} |\nabla u_{\varepsilon}|^2 dx + \frac{\lambda}{p+1} \int_{\Omega} u_{\varepsilon}^{p+1} dx - \frac{1}{2^*-\varepsilon} \int_{\Omega} \frac{u_{\varepsilon}^{2^*-\varepsilon}}{|x|^s} dx = c_{\varepsilon}, \\ \int_{\Omega} |\nabla u_{\varepsilon}|^2 dx + \lambda \int_{\Omega} u_{\varepsilon}^{p+1} dx - \int_{\Omega} \frac{u_{\varepsilon}^{2^*-\varepsilon}}{|x|^s} dx = 0. \end{array} \right. \quad (4.12)$$

Moreover, by taking a limit $j \rightarrow \infty$ in (4.9), we have

$$\|u_{\varepsilon}\|_{H_0^1(\Omega)} \leq C,$$

where $C > 0$ is independent of $\varepsilon > 0$. Thus by extracting a subsequence $\{u_j := u_{\varepsilon_j}\}$ with $\varepsilon_j \rightarrow 0$ as $j \rightarrow \infty$, we get

$$\left\{ \begin{array}{ll} u_j \rightharpoonup u & \text{weakly in } H_0^1(\Omega), \\ u_j \rightarrow u & \text{strongly in } L^{p+1}(\Omega), \\ \frac{u_j}{|x|^{\frac{s}{2^*}}} \rightharpoonup \frac{u}{|x|^{\frac{s}{2^*}}} & \text{weakly in } L^{2^*}(\Omega). \end{array} \right.$$

Thus passing to the limit $j \rightarrow \infty$ in (4.11) yields that

$$\Delta u - \lambda u^p + \frac{u^{2^*-1}}{|x|^s} = 0.$$

We shall prove $u \neq 0$ in $H_0^1(\Omega)$. Suppose $u = 0$ in $H_0^1(\Omega)$. Up to a subsequence, we let

$$C_1 := \lim_{j \rightarrow \infty} \int_{\Omega} |\nabla u_j|^2 dx \quad \text{and} \quad C_2 := \lim_{j \rightarrow \infty} \int_{\Omega} \frac{u_j^{2^* - \varepsilon_j}}{|x|^s} dx,$$

and then letting $j \rightarrow \infty$ in (4.12), we get

$$\frac{C_1}{2} - \frac{C_2}{2^*} = c_0 \quad \text{and} \quad C_1 - C_2 = 0, \quad \text{i.e.,} \quad \left(\frac{1}{2} - \frac{1}{2^*} \right) C_1 = c_0. \quad (4.13)$$

We easily see that $u_j(x_j) = \max_{\Omega} u_j \rightarrow \infty$ as $j \rightarrow \infty$. In what follows, we divide the proof into three steps. We let

$$\kappa_j := u_j(x_j)^{-\frac{2(2^* - 2 - \varepsilon_j)}{(n-2)(2^* - 2)}}. \quad (4.14)$$

Step 1. We claim $|x_j| = O(\kappa_j)$ as $j \rightarrow \infty$.

Suppose that up to a subsequence, $\lim_{j \rightarrow \infty} \frac{|x_j|}{\kappa_j} = \infty$. By scaling, set

$$v_j(y) := \frac{u_j(x_j + \kappa_j y)}{u_j(x_j)} \quad \text{for } y \in \Omega_j, \quad (4.15)$$

where

$$\Omega_j := \{y \in \mathbb{R}^n \mid x_j + \kappa_j y \in \Omega\}. \quad (4.16)$$

By (4.11) and (4.14), v_j satisfies

$$\begin{cases} \Delta v_j - \lambda \kappa_j^2 u_j(x_j)^{p-1} v_j^p + \left(\frac{\kappa_j}{|x_j|} \right)^s \frac{v_j^{2^* - 1 - \varepsilon_j}}{\left| \frac{x_j}{|x_j|} + \frac{\kappa_j}{|x_j|} y \right|^s} = 0 & \text{in } \Omega_j, \\ v_j = 0 & \text{on } \partial\Omega_j. \end{cases}$$

Furthermore, we have

$$\kappa_j^2 u_j(x_j)^{p-1} = \kappa_j^{2 - \frac{(n-2)(2^* - 2)(p-1)}{2(2^* - 2 - \varepsilon_j)}} \rightarrow 0 \quad \text{as } j \rightarrow \infty,$$

where note that $\kappa_j \rightarrow 0$ and $2 - \frac{(n-2)(p-1)}{2} > 0$, i.e., $p < \frac{n+2}{n-2}$. Thus v_j converges to some v smoothly in any compact set, and v satisfies $v(0) = 1$ and

$$\Delta v = 0 \quad \text{in } \mathbb{R}^n \quad (4.17)$$

provided that $\Omega_j \rightarrow \mathbb{R}^n$, or

$$\begin{cases} \Delta v = 0 & \text{in some half space } H, \\ v = 0 & \text{on } \partial H \end{cases} \quad (4.18)$$

provided that up to a linear transformation $\Omega_j \rightarrow H := \{y \in \mathbb{R}^n \mid y_n > -a\}$ for some $a > 0$. On the other hand, we have

$$\int_{\Omega_j} v_j^{\frac{2n}{n-2}} dy = \kappa_j^{\frac{n\varepsilon_j}{2^* - 2 - \varepsilon_j}} \int_{\Omega} u_j^{\frac{2n}{n-2}} dx \leq C,$$

and then $\int_{\mathbb{R}^n} v^{\frac{2n}{n-2}} dy$ is finite. This contradicts to $v(0) = 1$. The proof of Step 1 is complete.

Note that Step 1 implies that the origin is the only blow-up point.

Step 2. We claim that up to a subsequence, $\frac{x_j}{\kappa_j} \rightarrow y_0 \neq 0$ as $j \rightarrow \infty$.

Suppose that $\frac{x_j}{\kappa_j} \rightarrow 0$ as $j \rightarrow \infty$. As in the proof of Step 1, we define v_j and Ω_j in (4.15) and (4.16), respectively. Then by (4.11), v_j satisfies

$$\begin{cases} \Delta v_j - \lambda \kappa_j^2 u_j(x_j)^{p-1} v_j^p + \frac{v_j^{2^*-1-\varepsilon_j}}{\left|\frac{x_j}{\kappa_j} + y\right|^s} = 0 & \text{in } \Omega_j, \\ v_j = 0 & \text{on } \partial\Omega_j. \end{cases}$$

Since we already proved that $\kappa_j^2 u_j(x_j)^{p-1} \rightarrow 0$, v_j converges to some v smoothly in any compact set in $\overline{\mathbb{R}_+^n}$, and v satisfies

$$\begin{cases} \Delta v + \frac{v^{2^*-1}}{|y|^s} = 0 & \text{in } \mathbb{R}_+^n, \\ v = 0 & \text{on } \partial\mathbb{R}_+^n, \end{cases}$$

which is a contradiction to $v(0) = 1$. Thus Step 2 is proved.

Step 3. We complete the proof of Theorem 1.2 in this step. We note after a linear transformation, v_j converges to some v smoothly in any compact set in $\mathbb{R}_+^n$, and v satisfies

$$\begin{cases} \Delta v + \frac{v^{2^*-1}}{|y|^s} = 0 & \text{in } \mathbb{R}_+^n, \\ v = 0 & \text{on } \partial\mathbb{R}_+^n \text{ and } v(y_0) = \max_{\mathbb{R}_+^n} v = 1 \text{ for some } y_0 \in \mathbb{R}_+^n. \end{cases} \quad (4.19)$$

By (4.19), we have

$$\frac{\int_{\mathbb{R}_+^n} |\nabla v|^2 dy}{\left(\int_{\mathbb{R}_+^n} \frac{v^{2^*}}{|y|^s} dy\right)^{\frac{2}{2^*}}} = \left(\int_{\mathbb{R}_+^n} \frac{v^{2^*}}{|y|^s} dy\right)^{\frac{2^*-2}{2^*}} \geq \mu_s(\mathbb{R}_+^n),$$

and then

$$\int_{\mathbb{R}_+^n} |\nabla v|^2 dy = \int_{\mathbb{R}_+^n} \frac{v^{2^*}}{|y|^s} dy \geq \mu_s(\mathbb{R}_+^n)^{\frac{2^*}{2^*-2}}. \quad (4.20)$$

Furthermore, note that

$$C_1 = \lim_{j \rightarrow \infty} \int_{\Omega} |\nabla u_j|^2 dx = \lim_{j \rightarrow \infty} \kappa_j^{-\frac{(n-2)\varepsilon_j}{2^*-2-\varepsilon_j}} \int_{\Omega_j} |\nabla v_j|^2 dy \geq \lim_{j \rightarrow \infty} \int_{\Omega_j} |\nabla v_j|^2 dy \geq \int_{\mathbb{R}_+^n} |\nabla v|^2 dy. \quad (4.21)$$

Then by (4.13), (4.20) and (4.21), we have

$$c_0 \geq \left(\frac{1}{2} - \frac{1}{2^*}\right) C_1 \geq \left(\frac{1}{2} - \frac{1}{2^*}\right) \mu_s(\mathbb{R}_+^n)^{\frac{2^*}{2^*-2}},$$

which yields a contradiction to (4.3). Thus $u \neq 0$ in $H_0^1(\Omega)$, and Theorem 1.2 is proved. $\square$

5 Proof of Theorem 1.3

In order to prove Theorem 1.3, we first prove the following lemma.

Lemma 5.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with $n \geq 3$, $0 \in \partial\Omega$ and $0 < s < 2$. Then for $p = \frac{n+2}{n-2}$, the equation (1.2) has no positive solution provided that Ω is star-shaped with respect to the origin.*

Proof. Multiplying (1.2) by $x \cdot \nabla u$ and ∇u , respectively, and taking integrations, we obtain

$$\begin{cases} \frac{1}{2} \int_{\partial\Omega} (x \cdot \nu) \left(\frac{\partial u}{\partial \nu} \right)^2 dS_x + \frac{n-2}{2} \int_{\Omega} |\nabla u|^2 dx = \frac{n-2}{2} \left(\lambda \int_{\Omega} u^{\frac{2n}{n-2}} dx + \int_{\Omega} \frac{u^{2^*}}{|x|^s} dx \right), \\ \int_{\Omega} |\nabla u|^2 dx = \lambda \int_{\Omega} u^{\frac{2n}{n-2}} dx + \int_{\Omega} \frac{u^{2^*}}{|x|^s} dx, \end{cases}$$

where ν denotes the outward normal to $\partial\Omega$. Thus we derive the following Pohozaev identity

$$\int_{\partial\Omega} (x \cdot \nu) \left(\frac{\partial u}{\partial \nu} \right)^2 dS_x = 0.$$

Since Ω is star-shaped with respect to the origin, we deduce that $\frac{\partial u}{\partial \nu} \equiv 0$ on $\partial\Omega$. Hence,

$$\lambda \int_{\Omega} u^{\frac{n+2}{n-2}} dx + \int_{\Omega} \frac{u^{2^*-1}}{|x|^s} dx = - \int_{\Omega} \Delta u dx = 0,$$

which implies $u \equiv 0$ in Ω . □

Next, we take $\Omega = B_1(e_n)$, the unit ball centered at $e_n = (0, \dots, 1)$. It is obvious that Ω is star-shaped about the origin. Proceeding the same variational method as that in the proof for the assertion (iv) of Theorem 1.1, due to the nonexistence of a positive solution for the equation (1.2), Case 2 holds. Therefore, we get a positive solution of the equation (1.4). This proves Theorem 1.3.

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