

The transitivity of Conway's M_{13}

東北大学大学院・情報科学研究科

中嶋 康博 (Yasuhiro Nakashima)

Graduates School of Information Sciences, Tohoku University

1 Introduction

Mathieu groups M_{11} , M_{12} , M_{23} and M_{24} are the only nontrivial 4-transitive permutation groups, and M_{12} , M_{24} are the only nontrivial 5-transitive permutation groups. Conway introduced a set of permutations M_{13} on 13 letters, which contains Mathieu group M_{12} , and he claims that M_{13} is 6-transitive in some sense.

Martin and Sagan [2] generalized the concept of transitivity for a set of permutations. For a partition λ of a positive integer n , a subset D of the symmetric group S_n is said to be λ -transitive if there exists $r > 0$ such that for any partitions P, Q of shape λ , $\#\{\tau \in D \mid P^\tau = Q\} = r$. In particular, D is $(n - t, 1, \dots, 1)$ -transitive if and only if D is t -transitive.

Conway's M_{13} is not $(7, 1, 1, 1, 1, 1, 1)$ -transitive according to this definition, so Martin and Sagan raised a question to determine the full transitivity of M_{13} .

In this paper, we give a recursive definition of the elements of M_{13} , and answer the question of Martin and Sagan.

2 Construction of M_{13}

Let $\Omega := \{0, 1, \dots, 11, \infty\}$ be the set of points of a projective plane P of order 3, and $\Omega_{12} := \Omega - \{\infty\}$. For $a \in \Omega_{12}$ we define a permutation on Ω by $\sigma(a) := (\infty \ a)(b \ c)$ where $\{\infty, a, b, c\}$ is the line of P , determined by a, ∞ , and set $\sigma(\emptyset) = \sigma(\infty) = id_\Omega$. Recursively, for an integer k such that $k \geq 2$ and $a_1, a_2, \dots, a_k \in \Omega$ such that $a_1 \neq a_2 \neq \dots \neq a_k$, define

$$\sigma(a_1, a_2, \dots, a_k) := \tau(\infty \ a_k^\tau)(b^\tau \ c^\tau)$$

where $\tau = \sigma(a_1, \dots, a_{k-1})$ and $\{a_{k-1}, a_k, b, c\}$ is the line determined by a_{k-1}, a_k . Note that $\sigma(a)$ is the move $a|bc$, and a *triangular permutation* in the sense of [1] is of the form $\sigma(a, b, \infty)$. The sets M_{13}, M_{12} are defined as

$$\begin{aligned} M_{13} &:= \{\sigma(a_1, \dots, a_k) \mid k \in \mathbb{N}, a_i \in \Omega, a_i \neq a_{i+1} \ (1 \leq i \leq k-1)\}, \\ M_{12} &:= \{\tau \in M_{13} \mid \infty^\tau = \infty\}. \end{aligned}$$

The next proposition is useful to describe the elements of M_{13} .

Proposition 1. Let $a_1, \dots, a_k, b_1, \dots, b_l \in \Omega$ be such that $a_1 \neq \dots \neq a_k \neq \infty \neq b_1 \neq \dots \neq b_l$. Then

$$\sigma(a_1, \dots, a_k, \infty, b_1, \dots, b_l) = \sigma(b_1, \dots, b_l) \cdot \sigma(a_1, \dots, a_k, \infty).$$

Proof. We prove by induction on l . Let

$$\begin{aligned}\rho &= \sigma(b_1, \dots, b_{l-1}), \\ \pi &= \sigma(a_1, \dots, a_k, \infty),\end{aligned}$$

so $\sigma(a_1, \dots, a_k, \infty, b_1, \dots, b_{l-1}) = \rho\pi$ by the inductive hypothesis. Suppose that the line determined by b_{l-1}, b_l is $\{b_{l-1}, b_l, c, d\}$. Then

$$\begin{aligned}\sigma(a_1, \dots, a_k, \infty, b_1, \dots, b_l) &= \rho\pi(\infty b_l^{\rho\pi})(c^{\rho\pi} d^{\rho\pi}) \\ &= \sigma(b_1, \dots, b_l)(\infty b_l^{\rho})(c^{\rho} d^{\rho})\pi(\infty b_l^{\rho\pi})(c^{\rho\pi} d^{\rho\pi}) \\ &= \sigma(b_1, \dots, b_l)\pi(\infty^{\pi} b_l^{\rho\pi})(c^{\rho\pi} d^{\rho\pi})(\infty b_l^{\rho\pi})(c^{\tau} d^{\rho\pi}) \\ &= \sigma(b_1, \dots, b_l) \cdot \sigma(a_1, \dots, a_k, \infty).\end{aligned}$$

□

The following propositions are obvious.

Proposition 2. If i is an integer such that $1 \leq i \leq k$ and $x \in \Omega - \{a_i\}$, then

$$\sigma(a_1, \dots, a_k) = \sigma(a_1, \dots, a_i, x, a_i, a_{i+1}, \dots, a_k).$$

Proposition 3. For $a, b \in \Omega_{12}$ such that $\{a, b, \infty\}$ is contained in a line,

$$\sigma(a, \infty) = \sigma(a, b, \infty) = id_{\Omega}.$$

With these propositions, we prove the following theorem.

Theorem 4. M_{12} is the group generated by triangular permutations, and

$$M_{13} = \prod_{a \in \Omega} \sigma(a)M_{12}.$$

Proof. Let $\alpha = \sigma(a_1, \dots, a_k)$. Then $\alpha \in M_{12}$ if and only if $a_k = \infty$. For i in $\{1, \dots, k-1\}$, if $a_i \neq \infty$ then we insert ∞, a_i between a_i and a_{i+1} by Proposition 2. So by Propositions 1 and 3, α is written as a product of triangular permutations.

If $a_k = \infty$, then $\alpha \in M_{12}$. Otherwise, Proposition 2 implies

$$\alpha = \sigma(a_1, \dots, a_k, \infty, a_k)$$

so $\alpha \in \sigma(a_k)M_{12}$ by Proposition 1. □

3 Transitivity of M_{13}

An integer tuple $\lambda = (\lambda_1, \dots, \lambda_k)$ is called a partition of a positive integer n if $\lambda_i \geq \lambda_{i+1} \geq 0$ and $\sum_{i=1}^k \lambda_i = n$. A partition $P = (P_1, \dots, P_k)$ of the set $\Omega_n := \{1, \dots, n\}$ is said to have shape $\lambda = (\lambda_1, \dots, \lambda_k)$ of n , if $|P_i| = \lambda_i$.

Definition 5. Let n be an integer and D be a set of permutations on Ω_n . For a partition λ of n , we say that D is λ -transitive if there exists $r > 0$ such that for any partitions P, Q of shape λ , $\#\{\tau \in D \mid P^\tau = Q\} = r$.

For example, a permutation group G is t -transitive on Ω_n if and only if G is $(n-t, \underbrace{1, \dots, 1}_t)$ -transitive, where 1^t means $\underbrace{1, \dots, 1}_t$. We first prove the following general result.

Lemma 6. For each $i \in \Omega_{n+1} = \{1, \dots, n+1\}$, let a_i be a permutation on Ω_{n+1} such that $i^{a_i} = n+1$, and G be a permutation group on $\Omega_n = \{1, \dots, n\}$. If G is $(n-t, 1^t)$ -transitive on Ω_n and

$$D = \bigcup_{i \in \Omega_{n+1}} a_i G,$$

then D is a $(n-t+1, 1^t)$ -transitive set on Ω_{n+1} .

Proof. For t -tuples $X = (x_1, \dots, x_t), Y = (y_1, \dots, y_t)$ of distinct elements of Ω_{n+1} , we define

$$D_Y^X := \{\tau \in D \mid X^\tau = Y\}.$$

First, we suppose that Y contains $n+1$, for example, $y_1 = n+1$. Then $D_Y^X \subset a_{x_1} G$, and $\{x_k^{a_{x_1}} \mid 2 \leq k \leq t\}, \{y_k \mid 2 \leq k \leq t\} \subset \Omega_n$. By the $(n-(t-1), 1^{t-1})$ -transitivity of G on Ω_n ,

$$\begin{aligned} |D_Y^X| &= \#\{g \in G \mid (x_k^{a_{x_1}})^g = y_k \ (2 \leq k \leq t)\} \\ &= \frac{|G|}{n \cdot (n-1) \cdots (n-(t-2))}. \end{aligned}$$

Next, we assume that $n+1$ does not appear in Y . For an integer i such that $1 \leq i \leq t$, if $a_i g \in D_Y^X \cup a_i G$ then $i \notin X$ and $\{x_1^{a_i}, \dots, x_t^{a_i}\}, Y \subset \Omega_n$, so by the $(n-t, 1^t)$ -transitivity of G on Ω_n ,

$$\begin{aligned} |D_Y^X| &= \sum_{i \in \Omega_{n+1} - X} \#\{g \in G \mid (X^{a_i})^g = Y\} \\ &= |\Omega_{n+1} - X| \cdot \frac{|G|}{n \cdot (n-1) \cdots (n-(t-1))} \\ &= \frac{|G|}{n \cdot (n-1) \cdots (n-(t-2))}. \end{aligned}$$

□

By this lemma, we see that M_{13} is $(8, 1^5)$ -transitive. We will show that M_{13} is not $(7, 6)$ -transitive.

If M_{13} is $(7, 6)$ -transitive, then for any 6-element sets P, Q of Ω_{13} ,

$$\#\{\tau \in M_{13} \mid P^\tau = Q\} = \frac{|G|}{\binom{13}{6}} = 720.$$

It is known that M_{12} leaves the set of hexads invariant (see [1] for details). We define $H := \{h^{\sigma(a)} \mid a \in \Omega, h : \text{hexad}\}$. If $h = \{1, 2, 3, 4, 5, 6\}$ then $h^{\sigma(7)} = h^{\sigma(8)}$ and

$$|H| < |\Omega| \cdot 132 = \binom{13}{6},$$

so there is a 6-element set P that is not contained in H . Taking Q as a hexad, we obtain

$$\#\{\tau \in M_{13} \mid P^\tau = Q\} = 0.$$

Therefore M_{13} is not $(7, 6)$ -transitive.

We need to introduce the dominance order on partitions of n , in order to state a result of Martin and Sagan [2]. For two integer partitions $\lambda = (\lambda_1, \dots, \lambda_k), \mu = (\mu_1, \dots, \mu_l)$, we define

$$\lambda \trianglelefteq \mu \iff \sum_{i=1}^j \lambda_i \leq \sum_{i=1}^j \mu_i, \text{ for any positive integer } j$$

where $\lambda_i = 0$ for $i \geq k$ and $\mu_i = 0$ for $i \geq l$.

Theorem 7 (Martin and Sagan [2]). *If a set D is λ -transitive and $\lambda \trianglelefteq \mu$, then D is μ -transitive.*

Using this theorem, we can determine the transitivity of M_{13} .

Theorem 8. *Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition of 13. Then M_{13} is λ -transitive if and only if $\lambda_1 \geq 8$.*

Proof. If $\lambda_1 \geq 8$ then $\lambda \trianglelefteq (8, 1^5)$. Since M_{13} is $(8, 1^5)$ -transitive, M_{13} is also λ -transitive by Theorem 7. Suppose M_{13} is λ -transitive for some λ with $\lambda_1 \leq 7$. Then M_{13} is $(7, 6)$ -transitive by Theorem 7 again since $\lambda \trianglelefteq (7, 6)$. This is a contradiction. $\square$

References

- [1] J. H. Conway, M_{13} , Surveys in combinatorics, 1997 (London), pp. 1–11, London Math. Soc. Lecture Note Ser., 241, Cambridge Univ. Press, Cambridge, 1997.
- [2] W. J. Martin and B. E. Sagan, A new notion of transitivity for sets of permutations, to appear in J. London Math. Soc.