

On the Cyclicity of finite CM abelian varieties

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Abstract

Let A be an abelian variety over a number field F of dimension r , where $r \geq 1$ is an integer. Assume that $\text{End}_{\bar{F}} A \otimes \mathbb{Q} = K$, where K is a CM-field such that $[K : \mathbb{Q}] = 2r$. For $\wp$ a finite prime of F , we denote by $\mathbb{F}_{\wp}$ the residue field at $\wp$. If A has good reduction at $\wp$, let $\bar{A}$ be the reduction of A at $\wp$. Under GRH, we obtain ([V]) an asymptotic formula for the number of primes $\wp$ of F , with $N_{F/\mathbb{Q}}\wp \leq x$, for which $\bar{A}(\mathbb{F}_{\wp})$ has at most $2r - 1$ cyclic components.

1 The Main result

Consider A an abelian variety defined over a number field F , of conductor $\mathcal{N}$, and of dimension r , where $r \geq 1$ is an integer. Let Σ_F be the set of finite places of F , and for $\wp$ a prime of F , let $\mathbb{F}_{\wp}$ be the residue field at $\wp$. Let $\mathcal{P}_A$ be the set of primes $\wp \in \Sigma_F$ of good reduction for A , (i.e. $(N_{F/\mathbb{Q}}\wp, N_{F/\mathbb{Q}}\mathcal{N}) = 1$). For $\wp \in \mathcal{P}_A$, we denote by $\bar{A}$ the reduction of A at $\wp$.

We have that $\bar{A}(\mathbb{F}_{\wp}) \subseteq \bar{A}[m](\mathbb{F}_{\wp}) \subseteq (\mathbb{Z}/m\mathbb{Z})^{2r}$ for any positive integer m satisfying $|\bar{A}(\mathbb{F}_{\wp})| \mid m$. Hence

$$\bar{A}(\mathbb{F}_{\wp}) \simeq \mathbb{Z}/m_1\mathbb{Z} \times \mathbb{Z}/m_2\mathbb{Z} \times \cdots \times \mathbb{Z}/m_s\mathbb{Z}, \quad (1.1)$$

where $s \leq 2r$, $m_i \in \mathbb{Z}_{\geq 2}$, and $m_i \mid m_{i+1}$ for $1 \leq i \leq s-1$. Each $\mathbb{Z}/m_i\mathbb{Z}$ is called a cyclic component of $\bar{A}(\mathbb{F}_{\wp})$. If $s < 2r$, we say that $\bar{A}(\mathbb{F}_{\wp})$ has at most $(2r-1)$ cyclic components (thus if $r = 1$ this means that $\bar{A}(\mathbb{F}_{\wp})$ is cyclic). For $x \in \mathbb{R}$, define

$$f_{A,F}(x) = |\{\wp \in \mathcal{P}_A \mid N_{F/\mathbb{Q}}\wp \leq x, \bar{A}(\mathbb{F}_{\wp}) \text{ has at most } (2r-1) \text{ cyclic components}\}|.$$

Let $F(A[m])$ be the field obtained by adjoining to F the m -division points $A[m]$ of A .

We obtain (this is the main result of [V]; when $F = \mathbb{Q}$ and $r = 1$, i.e. when A is a CM elliptic curve over $\mathbb{Q}$, Theorem 1.1 is similar to Theorem 1.2 of [CM]):

Theorem 1.1. *Let A be an abelian variety over a number field F of dimension $r \geq 1$, of conductor $\mathcal{N}$, such that $\text{End}_{\bar{F}} A \otimes \mathbb{Q} = K$, where K is a CM-field satisfying $[K : \mathbb{Q}] = 2r$. Assume that the Generalized Riemann Hypothesis (GRH) holds for the Dedekind zeta functions of the division fields for A . Then we have*

$$f_{A,F}(x) = c_{A,F} \text{li } x + O_{A,F}(x^{\frac{5}{6}}(\log x)^{\frac{2}{3}}),$$

where $\text{li } x := \int_2^x \frac{1}{\log t} dt$, and

$$c_{A,F} = \sum_{m=1}^{\infty} \frac{\mu(m)}{[F(A[m]) : F]}.$$

Here $\mu(\cdot)$ is the Mobius function, and the implicit $O_{A,F}$ -constant depends on A and F .

2 Odds and ends

If F is a number field, we denote $G_F := \text{Gal}(\bar{F}/F)$. Let A be an abelian variety over F of dimension $r \geq 1$, and of conductor $\mathcal{N}$. We denote by $\mathcal{P}_A$ be the set of primes $\wp \in \Sigma_F$ of good reduction for A , (i.e. $(N_{F/\mathbb{Q}}\wp, N_{F/\mathbb{Q}}\mathcal{N}) = 1$). For $m \geq 1$ an integer, let $A[m]$ be the m -division points of A in $\bar{F}$. Then

$$A[m] \simeq (\mathbb{Z}/m\mathbb{Z})^{2r}.$$

If $F(A[m])$ is the field obtained by adjoining to F the elements of $A[m]$, then we have a natural injection

$$\Phi_m : \text{Gal}(F(A[m])/F) \hookrightarrow \text{Aut}(A[m]) \simeq \text{GL}_{2r}(\mathbb{Z}/m\mathbb{Z}).$$

For l a rational prime, define

$$T_l(A) = \varprojlim A[l^n].$$

The Galois group G_F acts on

$$T_l(A) \simeq \mathbb{Z}_l^{2r},$$

where $\mathbb{Z}_l$ is the l -adic completion of $\mathbb{Z}$ at l , and we obtain a representation

$$\rho_{A,l} : G_F \rightarrow \text{Aut}(T_l(A)) \simeq \text{GL}_{2r}(\mathbb{Z}_l),$$

which is unramified outside $lN_{F/\mathbb{Q}}\mathcal{N}$. If $\wp \in \mathcal{P}_A$, let $\sigma_{\wp}$ be the Artin symbol of $\wp$ in G_F , and let l be a rational prime satisfying $(l, N_{F/\mathbb{Q}}\wp) = 1$. We denote by $P_{A,\wp}(X) = X^{2r} + a_{2r-1,A}(\wp)X^{2r-1} + \dots + a_{1,A}(\wp)X + N_{F/\mathbb{Q}}\wp^r \in \mathbb{Z}[X]$ the characteristic polynomial of $\sigma_{\wp}$ on $T_l(A)$. Then $P_{A,\wp}(X)$ is independent of l . One can identify $T_l(A)$ with $T_l(\bar{A})$, where $\bar{A}$ is the reduction of A at $\wp$, and the action of $\sigma_{\wp}$ on $T_l(A)$ is the same as the action of the Frobenius $\pi_{\wp}$ of $\bar{A}$ on $T_l(\bar{A})$.

We say that an abelian variety A defined over a number field F of dimension r is CM (or has many complex multiplications) if $\text{End}_{\bar{F}}(A) \otimes \mathbb{Q} = K$, where K is a CM-field satisfying $[K : \mathbb{Q}] = 2r$. We denote by $\mathcal{F}$ the maximal totally real number field contained in K , and let $O_{\mathcal{F}}$ be the ring of integers of $\mathcal{F}$ and let O_K be the ring of integers of K . Let $\phi_1, \dots, \phi_r, \bar{\phi}_1, \dots, \bar{\phi}_r$, be the set of embeddings of K into $\mathbb{C}$, where $\bar{\phi}_i$ is the complex conjugate of ϕ_i . Then we have $[K : \mathcal{F}] = 2$, and $K = \mathcal{F}(\sqrt{-D})$ for some totally positive $D \in O_{\mathcal{F}}$.

Lemma 2.1. (Ribet [R]) Let A be a CM abelian variety defined over a number field F , of dimension r , of conductor N , and let m be a positive integer. Then

$$\phi(m)^2 \ll [F(A[m]) : F],$$

where $\phi(m)$ is the Euler function,

2. the extension $F(A[m])/F$ is ramified only at places dividing mN .

Lemma 2.2. (Shimura [SH]) Let A be a CM abelian variety defined over a number field F , of dimension r , and of conductor N . Then for all $\wp \in \mathcal{P}_A$, the characteristic polynomial $P_{A, \wp}(X)$ has roots $\pi_1(\wp), \dots, \pi_r(\wp), \bar{\pi}_1(\wp), \dots, \bar{\pi}_r(\wp)$, where $\bar{\pi}_i(\wp)$ is the complex conjugate of $\pi_i(\wp)$, and $\pi_i(\wp)\bar{\pi}_i(\wp) = N_{F/\mathbb{Q}}\wp$, for all $i = 1, \dots, r$. Moreover one can assume that $\pi_1(\wp) \in \text{End}_{\bar{F}}(A) \subseteq O_K$, and that for any $i = 1, \dots, r$, we have $\pi_i(\wp) = \phi_i(\pi_1(\wp))$.

One can prove the following results (see [V]):

Lemma 2.3. Let A be an abelian variety over a number field F , of conductor N . Let $\wp \in \mathcal{P}_A$, and let p be the rational prime below $\wp$. Let $q \neq p$ be a rational prime. Then $\bar{A}(\mathbb{F}_{\wp})$ contains a $(q, \dots, q)$ -type subgroup (q appears $2r$ -times), i.e. a subgroup isomorphic to $\mathbb{Z}/q\mathbb{Z} \times \dots \times \mathbb{Z}/q\mathbb{Z}$, iff $\wp$ splits completely in $F(A[q])$.

Lemma 2.4. Let A be a CM abelian variety defined over a number field F , of dimension r , and of conductor N . Let m be a positive integer. Then $\wp \in \mathcal{P}_A$, with $(N_{F/\mathbb{Q}}\wp, m) = 1$, splits completely in $F(A[m])$ iff $\frac{\pi_1(\wp)-1}{m} \in \text{End}_{\bar{F}}(A)$, where $\pi_1(\wp)$ appears in Lemma 2.2.

Lemma 2.5. Let A be an abelian variety over a number field F , of conductor N . Let $\wp \in \mathcal{P}_A$, and let p be the rational prime below $\wp$. Then $\bar{A}(\mathbb{F}_{\wp})$ contains at most $(2r-1)$ -cyclic components iff $\wp$ does not split completely in $F(A[q])$ for any rational prime $q \neq p$.

Lemma 2.6. With the same notations as above, for any $m \in \mathbb{N}^*$ and any $x \in \mathbb{R}$, we have that

$$S_m := |\{\wp \in \Sigma_F \mid N_{F/\mathbb{Q}}\wp \leq x, N_{F/\mathbb{Q}}\wp = (\alpha m + 1)^2 + D\beta^2 m^2,$$

for some $\alpha + \sqrt{-D}\beta \in O_K$, where $\alpha, \beta \in \mathcal{F}\}|$

$$\ll \frac{x^{\frac{3}{2}}}{m^3} + 1.$$

3 Chebotarev

Consider L/F a Galois extension of number fields, with Galois group G . We denote by n_L and d_L the degree and the discriminant of $L/\mathbb{Q}$, and by d_F the discriminant of $F/\mathbb{Q}$. Let $\mathcal{P}(L/F)$ be the set of rational primes p which lie below places of F which ramify in L/F .

Lemma 3.1. (Serre [SE]) *If L/F is Galois extension of number fields, then*

$$\log d_L \leq |G| \log d_F + n_L \left(1 - \frac{1}{|G|}\right) \sum_{p \in \mathcal{P}(L/F)} \log p + n_L \log |G|.$$

Let C be a conjugacy class in G . For a positive real number x , let

$$\pi_C(x, L/F) := |\{\varphi \in \Sigma_F | N_{F/\mathbb{Q}} \varphi \leq x, \varphi \text{ unramified in } L/F, \sigma_\varphi \in C\}|,$$

where σ_φ is a Frobenius element at φ . The Chebotarev density theorem says that

$$\pi_C(x, L/F) \sim \frac{|C|}{|G|} \text{li } x \sim \frac{|C|}{|G|} \frac{x}{\log x},$$

and moreover:

Lemma 3.2. (Serre [SE]) *Let L/F be a Galois extension of number fields. If the Dedekind zeta function of L satisfies the GRH, then*

$$|\pi_C(x, L/F) - \frac{|C|}{|G|} \text{li } x| \ll |C| x^{\frac{1}{2}} \left(\log x + \frac{\log |d_L|}{|G|} \right),$$

where the implied O -constant depends only on F .

4 Sketch of the proof of Theorem 1.1

Using §2 one obtains (see §4 of [V]), for $y = y(x)$ any real number with $y \leq 2x^{\frac{1}{2}}$, that

$$\begin{aligned} f_{A,F}(x) &= \sum_{m \leq 2x^{\frac{1}{2}}} \mu(m) \pi_1(x, F(A[m])/F) \\ &= \sum_{m \leq y} \mu(m) \pi_1(x, F(A[m])/F) + \sum_{y < m \leq 2x^{\frac{1}{2}}} \mu(m) \pi_1(x, F(A[m])/F) \\ &= \text{main} + \text{error}. \end{aligned} \tag{4.1}$$

Using §2 and Chebotarev, under GRH, one obtains (see §4 of [V])

$$\text{main} = \sum_{m \leq y} \frac{\mu(m)}{n(m)} \text{li } x + \sum_{m \leq y} O(x^{\frac{1}{2}} \log(m N_{F/\mathbb{Q}} \mathcal{N}x))$$

$$= \sum_{m \leq y} \frac{\mu(m)}{n(m)} \operatorname{li} x + O(yx^{\frac{1}{2}} \log(N_{F/\mathbb{Q}} \mathcal{N}x)), \quad (4.2)$$

where $n(m) := [F(A[m]) : F]$, and

$$\text{error} \ll \sum_{\substack{y < m \leq 2x^{\frac{1}{2}} \\ m \text{ square-free}}} \frac{x^{\frac{3}{2}}}{m^3} \ll \frac{x^{\frac{3}{2}}}{y^2}.$$

For

$$y := \frac{x^{\frac{1}{3}}}{(\log(N_{F/\mathbb{Q}} \mathcal{N}x))^{\frac{1}{3}}},$$

from §2 one gets (see §4 of [V])

$$\sum_{m > y} \frac{\mu(m)}{n(m)} \operatorname{li} x \ll \sum_{\substack{m > y \\ m \text{ square-free}}} \frac{(\log \log m)^2}{m^2} \operatorname{li} x \ll \frac{(\log \log y)^2}{y} \operatorname{li} x \ll x^{\frac{5}{6}}.$$

Hence

$$f_{A,F}(x) = \sum_{m=1}^{\infty} \frac{\mu(m)}{n(m)} \operatorname{li} x + O(x^{\frac{5}{6}} (\log(N_{F/\mathbb{Q}} \mathcal{N}x))^{\frac{2}{3}}).$$

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