

A NOTE ON CERTAIN SUBORDINATIONS

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1. INTRODUCTION

Mocanu([1], Lemma 3) showed the following Lemma 1 in 1986:

LEMMA 1. If $p(z)$ is an analytic function in $|z| < 1$, with $p(0) = 1$, and if $\operatorname{Re}\{p(z) + zp'(z)\} > 0$ in $|z| < 1$, then

$$(1) \quad \left| \arg p(z) \right| < \frac{\pi}{3}$$

in $|z| < 1$.

Here we recall the definition of subordination. Let $p(z)$ and $q(z)$ be analytic functions in $|z| < 1$ and $p(0) = q(0)$. $p(z)$ is said to be subordinate to $q(z)$ (written $p(z) \prec q(z)$) if $p(z) = q(w(z))$, $|z| < 1$ for some analytic function $w(z)$ with $|w(z)| \leq |z|$ (See Duren [2], p-190).

Since a function $q(z) = \frac{1-z}{1+z}$ satisfies $q(0) = 1$ and $\left| \arg q(z) \right| < \frac{\pi}{2}$, the function $p(z)$ in Lemma 1 is represented by the subordination,

$$(2) \quad p(z) \prec \left(\frac{1-z}{1+z} \right)^{2/3}$$

In 1987 Miller and Mocanu ([3], Theorem 5) proved the following Lemma 2 by considering a differential subordinate system.

LEMMA 2. Let $\beta_0 = 1.218\cdots$ be the solution of

$$(3) \quad \beta \pi \frac{3\pi}{2} - \tan^{-1} \beta$$

and let

$$(4) \quad \alpha = \alpha(\beta) = \beta + \frac{2}{\pi} \tan^{-1} \beta$$

for $0 < \beta \leq \beta_0$. If $p(z)$ is an analytic function in $|z| < 1$, with $p(0) = 1$, then

$$(5) \quad p(z) + zp'(z) \prec \left(\frac{1+z}{1-z}\right)^\alpha \implies p(z) \prec \left(\frac{1+z}{1-z}\right)^\beta.$$

When $\alpha = 1$ in the equality (4), we have

$$(6) \quad 1 = \beta + \frac{2}{\pi} \tan^{-1} \beta \quad (0 < \beta \leq \beta_0),$$

$\beta^* = 0.638\cdots$, and β^* is the solution of (6). This shows that

$$(7) \quad \operatorname{Re}\{p(z) + zp'(z)\} > 0, \quad |z| < 1 \implies p(z) \prec \left(\frac{1+z}{1-z}\right)^\beta$$

and $\beta^* < \frac{2}{3}$. Hence Lemma 1 can be improved by such as the following

LEMMA 3. If $p(z)$ is an analytic function in $|z| < 1$, with $p(0) = 1$ and if $\operatorname{Re}\{p(z) + zp'(z)\} > 0$, $|z| < 1$, then we have

$$(7) \quad \left| \arg p(z) \right| < \frac{\pi}{2} \beta^*,$$

$|z| < 1$, where $\beta^* = 0.638\cdots$, and β^* is the solution of (6).

REMARK. It seems that an extremal function in Lemma 3 is

$$(8) \quad p(z) = \frac{2}{z} \log(1+z) - 1.$$

This function $p(z) = \frac{2}{z} \log(1+z) - 1$ satisfies

$$(i) \quad \operatorname{Re}\{p(z)\} > 2 \log 2 - 1.$$

$$(ii) \quad p(z) \prec \left(\frac{1-z}{1+z}\right)^{\beta_1},$$

where $\beta_1 = 0.503 \dots\dots$. But we can not prove these facts at present.

2. THEOREMS AND PROOFS

From Lemma 3 we can show the following

THEOREM 1. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $|z| < 1$ and

$\operatorname{Re}\{f'(z)\} > 0$ in $|z| < 1$, then

$$(9) \quad \left| \arg \frac{f(z)}{z} \right| = \left| \arg \int_0^r f'(\rho e^{i\theta}) d\rho \right| < \frac{\pi}{2} \beta^*,$$

where $z = re^{i\theta}$, $0 < r < 1$, and β^* is given in Lemma 3.

Proof. We put $p(z) = \frac{f(z)}{z}$ in Lemma 3, then we have $p(0) = 1$, and

$$(10) \quad \operatorname{Re}\{p(z) + z p'(z)\} = \operatorname{Re}\{f'(z)\} > 0 \text{ in } |z| < 1.$$

Therefore, we obtain the following relations

$$\begin{aligned} (11) \quad \left| \arg p(z) \right| &= \left| \arg \frac{f(z)}{z} \right| \\ &= \left| \arg \frac{1}{z} \int_0^z f'(t) dt \right| = \left| \arg \frac{1}{re^{i\theta}} \int_0^r f'(\rho e^{i\theta}) e^{i\theta} d\rho \right| \\ &= \left| \arg \frac{1}{r} \int_0^r f'(\rho e^{i\theta}) d\rho \right| \end{aligned}$$

$$= \left| \arg \int_0^r f'(\rho e^{i\theta}) d\rho \right| < \frac{\pi}{2} \beta^*,$$

where $z = r e^{i\theta}$, $0 < r < 1$, $t = \rho e^{i\theta}$ and $0 \leq \rho \leq r$.

First the function $f(z)$ in Theorem 1 are to be said close-to-convex functions in $|z| < 1$ and next Theorem 2 is a result for convex functions in $|z| < 1$.

THEOREM 2. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $|z| < 1$ and

$\operatorname{Re}\{1 + \frac{zf'(z)}{f'(z)}\} > 0$ in $|z| < 1$ (i.e., $f(z)$ are a univalent and convex functions) then we have

$$(12) \quad \left| \arg \int_0^r \left\{ 1 + \frac{\rho e^{i\theta} f'(\rho e^{i\theta})}{f'(\rho e^{i\theta})} \right\} d\rho \right| < \frac{\pi}{2} \beta^*,$$

where $z = r e^{i\theta}$, $0 < r < 1$, and β^* is given in Lemma 3.

Proof. We put $q(z) = \frac{zf'(z)}{f'(z)}$ and

$$(13) \quad p(z) = \frac{1}{z} \int_0^z \left\{ q(t) + \frac{t q'(t)}{q(t)} \right\} dt,$$

then we have

$$(14) \quad p(z) + z p'(z) = q(z) + \frac{z q'(z)}{q(z)} = 1 + \frac{zf'(z)}{f'(z)},$$

$p(0) = 1$, $q(0) = 1$ and $p(z)$, $q(z)$ are analytic in $|z| < 1$.

Here we again use a method similar to that in Theorem 1. Since

$$(15) \quad \operatorname{Re}\{p(z) + z p'(z)\} = \operatorname{Re}\left\{1 + \frac{zf'(z)}{f'(z)}\right\} > 0, \quad |z| < 1$$

by the hypotheses, we have the following relations

$$(16) \quad \left| \arg p(z) \right| = \left| \arg \frac{1}{z} \int_0^z \left\{ q(t) + \frac{t q'(t)}{q(t)} \right\} dt \right|$$

$$\begin{aligned}
&= \left| \arg \frac{1}{z} \int_0^z \left(1 + \frac{tf'(t)}{f'(t)} \right) dt \right| \\
&= \left| \arg \frac{1}{r} \int_0^r \left(1 + \frac{\rho e^{i\theta} f'(\rho e^{i\theta})}{f'(\rho e^{i\theta})} \right) d\rho \right| \\
&= \left| \arg \int_0^r \left(1 + \frac{\rho e^{i\theta} f'(\rho e^{i\theta})}{f'(\rho e^{i\theta})} \right) d\rho \right| < \frac{\pi}{2} \beta^*,
\end{aligned}$$

where $z = re^{i\theta}$, $0 < r < 1$, $t = \rho e^{i\theta}$, $0 \leq \rho \leq r$, and β^* is given in Lemma 3.

This completes the proof.

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