

This article appeared as:

P. Li and S. Biwa, Flexural waves in a periodic non-uniform Euler-Bernoulli beam: Analysis for arbitrary contour profiles and applications to wave control, *International Journal of Mechanical Sciences*, 188 (2020), 105948. DOI: 10.1016/j.ijmecsci.2020.105948

Flexural waves in a periodic non-uniform Euler-Bernoulli beam: Analysis for arbitrary contour profiles and applications to wave control

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Abstract: The flexural wave in a periodic non-uniform Euler-Bernoulli beam with arbitrarily contoured profiles is studied by utilizing the power series expansion method. The convergence criterion that makes the power series expansion method applicable is also illustrated. The validation is carried out by comparing the theoretical results with that from the finite element analysis when the beam thickness varies in different forms. For a quadratic thickness variation, the first band gap evolution versus the structural parameter is investigated, based on which a flexural-wave-based low-pass filter for frequency shunting and a rectangular lens for energy focusing are designed. It is revealed in the frequency domain analysis that the flexural wave with a lower frequency can propagate further when it travels into the wave filter. The lens designed exhibits a good focusing phenomenon with the focusing size smaller than one wavelength, and has a good performance at a certain finite frequency range. The theoretical method and design scheme can provide effective guidance for the flexural wave control.

Keywords: non-uniform Euler-Bernoulli beam, power series expansion method, frequency shunting, rectangular lens for energy focusing.

1. Introduction

As the main carrier of vibration in thin-walled structures, flexural waves have always been an important topic in various branches of science and engineering, and many efforts have been made to clarify their propagation characteristics. More recently, the concept of phononic crystals (PCs) and metamaterials has been proposed. They belong to a kind of burgeoning artificial composites usually consisting of multiple elements, and can exhibit some novel properties beyond those of conventional materials. PCs and metamaterials have been successfully adopted to control flexural waves [1-13], e.g., prohibiting their propagation based on the Bragg scattering [1-3,8,12] or the local resonances [7,10], localizing vibrations by a point defect [6], manipulating the phase difference with the aid of metasurface [11,13], cloaking of hidden objects by a specific distribution of material parameters [5], focusing energy via a particular gradient-index (GRIN) PC [4,9], and so forth.

Not limited to multiple-phase materials, single-phase PCs and metamaterials with periodic or inhomogeneous geometry can also be used for the flexural wave control. A common method is to make holes [14-16] in the plate or implant stubs [17,18] on the plate surface, based on which several flexural-wave-based lenses can be designed and the wave propagation property can be controlled at will via modulating the hole/stub size [14-16]. In essence, flexural waves in beams and plates are dispersive, with their phase and group velocities depending on their thickness. As a consequence, thickness modulation can be viewed as an efficient means for the wave manipulation. For instance, a metasurface for steering the flexural wave front is realized through the sensitivity of velocity on the beam/plate thickness [19,20]. Acoustic black hole (ABH) phenomenon is another typical case, which can be realized via a particular thickness variation of a beam or plate [19,21-24], by which the wave can be slowed gradually, and the wave reflection is reduced evidently, even eliminated. Additionally, flexural-wave-based gradient lenses, including Luneburg lens, Maxwell Fish-Eye lens, Eaton lens, and 90° rotating lens, can be realized through different thickness variations in a single plate [25,26]. To some extent, the single-phase PCs and metamaterials are more suitable for the lens design, because the graded change of structural geometry is effective to reduce the unnecessary wave reflection. With the development of modern engineering technology, in particular with the occurrence and maturation of 3D printing techniques, fabricating a single material with complex geometric shapes has been realized [16,18]. Therefore, the single-phase PCs and metamaterials have great potential values for controlling flexural waves.

In order to achieve the structural design of beams and plates for wave control, exact analysis of the frequency spectrum is principal and crucial, based on which the wave propagation property can be determined. For a beam or plate with spatially varying thickness, the dynamic equation governing the deflection is a fourth-order partial differential equation with variable coefficients [27,28], which poses a great challenge to the analytical solution. Even for an Euler-Bernoulli beam with the rotary inertia and shear deformation neglected, theoretical solutions can be achieved only for specific geometric shapes. For example, the flexural wave can be analyzed in an exact manner when both the cross-section and the inertia moment vary in the same exponential function [29,30]. In this case, the exponential inhomogeneity does not substantially modify the character of the governing equation, and the solution is similar to that for a flat beam. When the beam thickness and/or width of the cross-section vary linearly along the length, the deflection can be expressed in terms of Bessel functions

[31], Frobenius series [32,33] or Chebyshev polynomials [34]. For a cantilever tapered Euler–Bernoulli beam with a rectangular or circular cross-section, the differential transform method can be utilized to solve the resonant frequencies [35,36]. Additionally, the tolerance average method that is suitable for low frequency regions can be applied for a non-uniform beam [37-40]. If the cross-section changes in a more complicated pattern, finding an exact solution of flexural waves in a non-uniform beam with variable profiles is very difficult, and the finite element method (FEM) seems to be the only choice [41-44]. Therefore, a general theoretical method applicable to the flexural waves in a beam with a random cross-section is still lacking, which justifies the motivation of the present study.

The objective of this paper is first to present a simple approach, i.e., the power series expansion method, to model and analyze the flexural waves in a tapered Euler-Bernoulli beam with an arbitrary cross-section. The solution method is introduced in detail in Section 2, in which the criterion of series convergence is also established. In Section 3, the band structures caused by periodic thickness variations of an Euler-Bernoulli beam is calculated based on this method, and its accuracy is demonstrated by comparison with the results obtained by the FEM. Based on these analytical results, the present study also aims to demonstrate a low-pass wave filter for frequency shunting and a rectangular lens for energy focusing which are designed via graded thickness variations. Their working performances are examined in the frequency domain by the FEM. Finally, the conclusions are summarized.

2. The power series expansion method for a non-uniform Euler-Bernoulli beam

An Euler-Bernoulli beam with symmetric contoured profiles is considered, as shown in Fig. 1. In order to theoretically analyze the flexural waves propagating in this beam, a rectangular coordinate system is established with the origin located at the mid-plane of the left edge of the beam. The length is represented by L , and the thickness variation is expressed as $h(x)$. For a sufficiently smooth thickness variation, $h(x)$ can be written into a power series according to Taylor series as

$$h(x) = \left[h_0 + h_1 \left(\frac{x}{L} \right) + h_2 \left(\frac{x}{L} \right)^2 + \dots \right] = \sum_{m=0}^{N_H} h_m \left(\frac{x}{L} \right)^m. \quad (1)$$

Here, N_H is the integer truncation parameter of the beam thickness. Similarly, the beam width $b(x)$ can also be expressed as

$$b(x) = \left[b_0 + b_1 \left(\frac{x}{L} \right) + b_2 \left(\frac{x}{L} \right)^2 + \dots \right] = \sum_{n=0}^{N_B} b_n \left(\frac{x}{L} \right)^n, \quad (2)$$

with the truncation parameter N_B of the beam width. Therefore, the beam area of the cross-section $A(x)$ and the inertia moment $I(x)$ can be obtained as

$$A(x) = 2h(x)b(x) = \sum_{s=0}^{N_A} A_s \left(\frac{x}{L} \right)^s, \quad (3)$$

$$I(x) = \frac{b(x)[2h(x)]^3}{12} = \sum_{m=0}^{N_I} I_m \left(\frac{x}{L} \right)^m. \quad (4)$$

Here the area truncation and the inertia moment truncation parameters are respectively $N_A = N_H + N_B$ and $N_I = 3N_H + N_B$. As a special case, a flat beam with a uniform width will have $N_H = N_B = N_A = N_I = 0$.

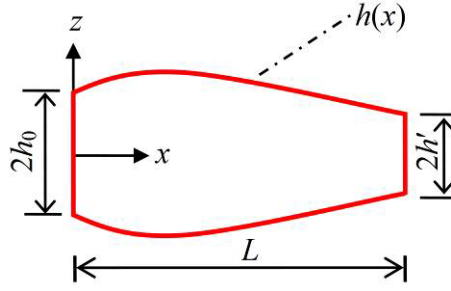


Figure 1 A non-uniform Euler-Bernoulli beam with symmetric contoured profiles.

For a non-uniform Euler-Bernoulli beam with symmetrically contoured thickness profiles, the deflection is decoupled from the extension, and its dynamic governing equation for the deflection $w(x, t)$ is [27-34]

$$\frac{\partial^2}{\partial x^2} \left(EI(x) \frac{\partial^2 w(x, t)}{\partial x^2} \right) + \rho A(x) \frac{\partial^2 w(x, t)}{\partial t^2} = 0, \quad (5)$$

in which E and ρ are Young's modulus and the mass density, and t stands for the time. Eq. (5) is a partial differential equation with variable coefficients, which is usually hard to solve analytically. The aim of this paper is to seek for a general theoretical solution which is suitable for an arbitrary $h(x)$ function.

Here, the power series expansion method is presented to semi-analytically solve this equation for the time-harmonic cases. The Taylor series expressions in Eqs. (3) and (4) suggest that the deflection $w(x, t)$ along the beam length direction can be expanded into the power series of x , i.e.,

$$w(x, t) = \left[W_0 + W_1 \left(\frac{x}{L} \right) + W_2 \left(\frac{x}{L} \right)^2 + \dots \right] \exp(i\omega t) = \sum_{n=0}^{N_W} W_n \left(\frac{x}{L} \right)^n \exp(i\omega t), \quad (6)$$

where N_W is the series truncation parameter, $\omega = 2\pi f$ is the angular frequency, and W_n ($n \geq 0$) are the expansion coefficients which will be determined below. Substitution of Eqs. (3) (4) and (6) into Eq. (5) yields the following relations about W_n :

$$\begin{aligned} & \sum_{m=0}^{N_I} I_m \left(\frac{x}{L} \right)^m \cdot \sum_{n=4}^{N_W} n(n-1)(n-2)(n-3) W_n \left(\frac{x}{L} \right)^{n-4} \\ & + 2 \sum_{m=1}^{N_I} m I_m \left(\frac{x}{L} \right)^{m-1} \cdot \sum_{n=3}^{N_W} n(n-1)(n-2) W_n \left(\frac{x}{L} \right)^{n-3} \\ & + \sum_{m=2}^{N_I} m(m-1) I_m \left(\frac{x}{L} \right)^{m-2} \cdot \sum_{n=2}^{N_W} n(n-1) W_n \left(\frac{x}{L} \right)^{n-2} \\ & - \frac{\omega^2}{E / \rho} \sum_{s=0}^{N_A} A_s \left(\frac{x}{L} \right)^s \cdot \sum_{n=0}^{N_W} W_n \left(\frac{x}{L} \right)^n = 0. \end{aligned} \quad (7)$$

Eq. (7) should be valid for arbitrary x , which requires that the coefficients of $(x/L)^n$ must be zero. This yields

$$\begin{aligned}
& \sum_{m=0}^{J_m} I_m (n-m+4)(n-m+3)(n-m+2)(n-m+1)W_{n-m+4} \\
& + \sum_{m=0}^{J_m} 2(m+1)I_{m+1}(n-m+3)(n-m+2)(n-m+1)W_{n-m+3} \\
& + \sum_{m=0}^{J_m} (m+2)(m+1)I_{m+2}(n-m+2)(n-m+1)W_{n-m+2} - \frac{\omega^2 L^4}{E/\rho} \sum_{s=0}^{J_s} A_s W_{n-s} = 0
\end{aligned} \tag{8}$$

with $J_m = \min(N_I, n)$ and $J_s = \min(N_A, n)$. From Eq. (8), the recursive linear relations about W_n can be achieved as

$$\begin{aligned}
W_{n+4} = & \frac{-1}{(n+4)(n+3)(n+2)(n+1)I_0} \left\{ \sum_{m=0}^{J_m} [I_{m+1}(n-m+3)(n-m+2)(n-m+1)(n+m+2)W_{n-m+3} \right. \\
& \left. + I_{m+2}(m+2)(m+1)(n-m+2)(n-m+1)W_{n-m+2}] - \frac{\omega^2 L^4}{E/\rho} \sum_{s=0}^{J_s} A_s W_{n-s} \right\},
\end{aligned} \tag{9}$$

or, explicitly,

$$\begin{aligned}
n=0: W_4 &= \frac{-1}{24I_0} \left(12I_1W_3 + 4I_2W_2 - \frac{\omega^2 L^4}{E/\rho} A_0W_0 \right), \\
n=1: W_5 &= \frac{-1}{120I_0} \left[72I_1W_4 + 36I_2W_3 + 12I_3W_2 - \frac{\omega^2 L^4}{E/\rho} (A_0W_1 + A_1W_0) \right], \\
n=2: W_6 &= \frac{-1}{360I_0} \left[240I_1W_5 + 144I_2W_4 + 72I_3W_3 + 24I_4W_2 - \frac{\omega^2 L^4}{E/\rho} (A_0W_2 + A_1W_1 + A_2W_0) \right], \\
& \vdots
\end{aligned} \tag{10}$$

Based on the above recursive linear relations, we can derive that there are totally four undetermined coefficients, i.e., W_0 , W_1 , W_2 , and W_3 . Once they are determined, other coefficients W_n for $n > 3$ can be calculated by using Eq. (9). Furthermore, it can be obtained from Eq. (9) that

$$\lim_{n \rightarrow \infty} \left| \frac{W_{n+4}}{W_{n+3}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+2)I_1}{(n+4)I_0} \right| = \left| \frac{I_1}{I_0} \right|. \tag{11}$$

Therefore, for any arbitrary profile with $h(x)$ and $b(x)$, this power series solution is convergent as long as $|I_1| < I_0$ can be ensured beforehand. This is the convergence criterion that makes the power series expansion method applicative for a non-uniform Euler-Bernoulli beam. This criterion for convergence can be reduced to $|h_1| < h_0/3$ for a beam with a constant width. In the following sections, this semi-analytical solution will be used to solve the flexural waves in a periodic beam with non-uniform profiles.

3 The thickness-induced band structures in a periodic Euler-Bernoulli beam

3.1 Bloch wave analysis

A periodic Euler-Bernoulli beam is considered in this section with its unit cell being the same as the non-uniform beam shown in Fig. 1. Therefore, L is the length of the unit cell. The deflection in the unit cell can be expressed in the form of Eq. (6), and the

Bloch theorem requires [45-50]

$$\begin{cases} w|_{x=L} = w|_{x=0} \exp(iKL), \\ \frac{\partial w}{\partial x}|_{x=L} = \frac{\partial w}{\partial x}|_{x=0} \exp(iKL), \\ EI(x) \frac{\partial^2 w}{\partial x^2}|_{x=L} = EI(x) \frac{\partial^2 w}{\partial x^2}|_{x=0} \exp(iKL), \\ \frac{\partial}{\partial x} \left[EI(x) \frac{\partial^2 w}{\partial x^2} \right]|_{x=L} = \frac{\partial}{\partial x} \left[EI(x) \frac{\partial^2 w}{\partial x^2} \right]|_{x=0} \exp(iKL). \end{cases} \quad (12)$$

where K is the wavenumber. Generally speaking, K is a complex value with its real part restricted in the first Brillouin zone $[-\pi/L, \pi/L]$. If K is real in certain frequency regions, the deflection of the flexural wave at $x = 0$ and $x = L$ differ only by a phase, which indicates that flexural waves can propagate without attenuation and these frequency regions belong to pass bands. On the contrary, when K is pure imaginary i.e., $K = iK'$ in other frequency regions, the deflection decays in the form of $e^{-K'L}$ when the wave travels across the unit cell. Therefore, waves can not travel effectively, and these frequency regions correspond to band gaps (BGs).

Substitution of Eq. (6) into Eq. (12) yields

$$\begin{cases} \sum_{n=0}^{T_W} W_n = W_0 \exp(iKL), \\ \sum_{n=1}^{N_W} n W_n = W_1 \exp(iKL), \\ \sum_{m=0}^{N_I} I_m \sum_{n=2}^{N_W} n(n-1) W_n = 2I_0 W_2 \exp(iKL), \\ \sum_{m=0}^{N_I} I_m \sum_{n=3}^{N_W} n(n-1)(n-2) W_n + \sum_{m=1}^{N_I} m I_m \sum_{n=2}^{N_W} n(n-1) W_n = (6I_0 W_3 + 2I_1 W_2) \exp(iKL). \end{cases} \quad (13)$$

The above equations implicitly contain four independent linear algebraic equations about W_0 , W_1 , W_2 , and W_3 , which can be written in a simplified form:

$$[Q_{4 \times 4}][W_0, W_1, W_2, W_3]^T = \mathbf{0}. \quad (14)$$

Here, $\mathbf{0}$ in Eq. (14) stands for a 4×1 zero vector.

In order to obtain non-trivial solutions of Eq. (14), the determinant of the coefficient matrix $Q_{4 \times 4}$ must be zero, i.e.,

$$\det|Q_{4 \times 4}| = 0, \quad (15)$$

which will be used to calculate the dispersion spectrum of flexural waves in the periodic Euler-Bernoulli beam. Eq. (15) is a transcendental equation that contains two unknown parameters, i.e., the frequency f and the wavenumber K . An exact relation between f and K is, however, hard to derive explicitly. Therefore, a numerical computation method is developed to solve Eq. (15). For simplification, only the roots with $\text{Real}(K)$ locating in the region of $[0, \pi/L]$ are determined, as the roots in the regions of $[-\pi/L, 0]$ can be given by symmetry. First, the K -space, whose real and imaginary components, i.e., $\text{Real}(K)$ and $\text{Imag}(K)$, are respectively limited in the regions of $[-\delta_1, \delta_1 + \pi/L]$ and $[-\delta_2, \mu]$, is divided into finite meshes with small intervals, so that each cross point corresponds to individual K value. Here, δ_1 , δ_2 and μ are real positive values that are

assumed artificially. Small δ_1 and δ_2 are introduced, so that the roots at Brillouin zone boundaries, i.e., $\text{Real}(K) = 0$ or π/L , can be included, and μ is the maximum attenuation factor expected within the frequency range of interest. Then, for a given f , the determinant of $Q_{4 \times 4}$ in every cross point is calculated. Finally, the root of Eq. (15) will be determined as a minimal point of $|\det Q_{4 \times 4}|$, where the absolute value of $|\det Q_{4 \times 4}|$ is smaller than those at its eight adjacent points.

In order to clearly demonstrate the generality and correctness of the power series expansion method, an aluminum plate with Young's modulus $E = 70$ GPa, Poisson's ratio $\nu = 0.33$ and mass density $\rho = 2700$ kg/m³ is chosen as a typical numerical case, and three kinds of thickness variations are adopted here. During the following simulations, the beam thickness at $x = 0$ is fixed as $2h_0 = 2$ mm, and the length is $L = 2$ cm. Case 1 is corresponding to a uniform beam with $h(x) = h_0$, whilst Cases 2 and 3 are inhomogeneous beams, with the thickness changing in a linear form $h(x) = h_0 + h_1(x/L)$ and in a quadratic form $h(x) = h_0 + h_2(x/L)^2$, respectively. Here and hereafter, beams of constant width, $b(x) = b_0$, are considered.

3.2 Convergence

The convergence of the power series expansion method is discussed first. Table 1 displays the non-dimensional value KL/π calculated from Eq. (15) with some selected truncation parameters for different frequencies. It can be seen from Table 1 that the power series method exhibits good convergence while the necessary number of series terms depends on the $h(x)$ variation and the frequency f . More series terms are needed for a dramatic quadratic change, compared with the uniform beam and the beam with a gentle linear variation, especially for a higher frequency. In this paper, 200 terms in the series are adopted, which can sufficiently ensure adequate accuracy in the following simulations.

Table 1 Non-dimensional values KL/π calculated from Eq. (15) for different truncations. The two values in each parenthesis are respectively the real and imaginary parts.

	Case 1: $h_0 = 1.0$ mm.		Case 2: $h_0 = 1.0$ mm, and $h_1 = 0.3$ mm.		Case 3: $h_0 = 1.0$ mm, and $h_2 = -0.8$ mm.	
f	N_W	KL/π	N_W	KL/π	N_W	KL/π
5 kHz	8	---	8	(1.0, 0.7720)	50	(1.0, 0.222)
	9	---	10	(0.631, 0.001)	60	(1.0, 0.181)
	10	(0.659, 0)	12	(0.630, 0)	80	(1.0, 0.162)
	11	(0.658, 0)	13	(0.630, 0)	100	(1.0, 0.160)
	12	(0.658, 0)	15	(0.630, 0)	120	(1.0, 0.160)
10 kHz	10	(1.0, 0.815)	8	(1.0, 0.315)	60	(0.807, 0.054)
	11	(0.933, 0.002)	10	(1.0, 0.421)	80	(0.799, 0.012)
	12	(0.930, 0.001)	13	(0.890, 0.001)	100	(0.798, 0.003)
	13	(0.931, 0)	14	(0.891, 0)	120	(0.797, 0)
	14	(0.931, 0)	15	(0.891, 0)	150	(0.797, 0)

20 kHz	10	(1.0, 0.712)	10	(1.0, 0.143)	80	(0.196, 0.054)
	12	(0.675, 0.031)	15	(0.746, 0.005)	100	(0.253, 0.012)
	14	(0.686, 0)	17	(0.743, 0.001)	120	(0.263, 0.002)
	15	(0.684, 0)	19	(0.742, 0)	150	(0.265, 0)
	16	(0.684, 0)	20	(0.742, 0)	180	(0.265, 0)

3.3 Band structure analysis

Based on the solution method developed above, the frequency spectra calculated from Eq. (15) for the three cases are depicted in Fig. 2, in which the blue and red curves are respectively the real and imaginary parts of K . Meanwhile, the results obtained by FEM using Comsol Multiphysics software after applying periodic boundary conditions are also given in Fig. 2, denoted by pink circles. For the FEM simulation, a two-dimensional plate model is established in the Solid Mechanics module with the profiles exactly the same as the three cases mentioned above, and the beam thickness is divided by 10 triangle elements, resulting in total of 2610, 2830 and 2106 elements for the whole unit cell in Figs. 2(a), 2(b) and 2(c), respectively. The upper and bottom surfaces are set as traction-free, and Floquet periodic boundary conditions are applied on the lateral interfaces. Letting the wavenumber K varying from 0 to π/L , the eigenfrequency problem is solved. The theoretical results coincide very well with those from FEM model in the low frequency region, which validates the correctness of the power series method. The discrepancy originates from the difference between the one-dimensional beam theory and the two-dimensional FEM model. With the increasing frequency, the discrepancy between the theoretical predictions and those by FEM appears evident, which is due to the approximate Euler-Bernoulli beam equation adopted here. Since the theory does not account for the rotary inertia and the shear effect, it is only suitable in the relatively low frequency region. As shown in Fig. 2(a), there is no BG for a uniform beam. With the thickness variation, some BGs caused by the geometry-induced impedance mismatch emerge, such as the one shown in Fig. 2(b). The region of BGs, represented by $\text{Imag}(K) \neq 0$, coincides with that of $\text{Real}(K) = 0$ or $\text{Real}(K) = \pi/L$, as anticipated. For a dramatic thickness change, the BGs become wider with larger attenuation factors, such as the BGs in 3.81-8.34 kHz and 23.46-34.44 kHz in Fig. 2(c). In these frequency regions, the shape of $\text{Imag}(K)$ indicates that these BGs originate from the Bragg scattering effect caused by the geometry-induced impedance mismatch [51]. It should also be stressed here that there is an additional evanescent branch in the imaginary region, represented by the olive line, which originates from the fourth-order partial differential governing equation of flexural waves. It is a near-field mode, not a traveling wave, which possesses a much larger spatial attenuation.

It should be stressed that the power series method has a high computational efficiency. Taking the quadratic form in Fig. 2(c) for example, if the triangle elements mentioned above are employed in the Comsol simulation, total 953s are needed to calculate the band structure below 60 kHz when using a Thinkpad T440 computer with an i5-4200U CPU. However, after setting the accuracy of $\text{Real}(KL/\pi)$ and $\text{Imag}(KL/\pi)$ in Eq. (15) as one hundredth, only 103s is needed with the proposed power series expansion method if the same computer is used. Obviously, the computational cost is saved with the power series method. Additionally, the branches corresponding to $\text{Imag}(KL/\pi)$ that are not calculated with Comsol are also obtained via this method.

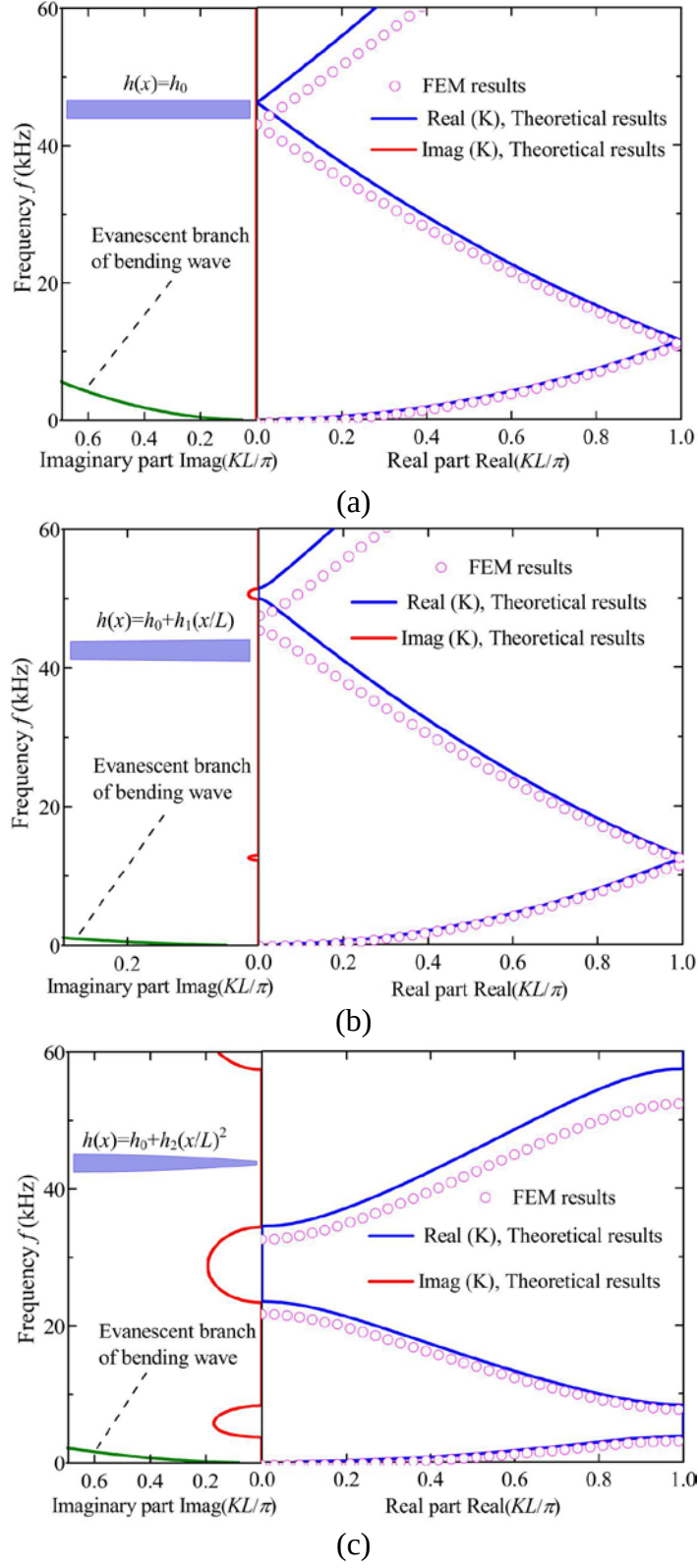


Figure 2 The frequency spectra comparison between the theoretical predictions from Eq. (15) and FEM simulations for different thickness variations: (a) Case 1 with $h_0 = 1.0$ mm; (b) Case 2 with $h_0 = 1.0$ mm and $h_1 = 0.3$ mm ; (c) Case 3 with $h_0 = 1.0$ mm and $h_2 = -0.8$ mm.

Taking Case 3 for instance, Fig. 3(a) shows the frequency spectra for some selected h_2 values, in which circles denote the FEM results. Beams with different quadratic height variations possess different phase velocities, and with the increase of the absolute value of h_2 , the phase velocity of the flexural wave decreases for a fixed frequency. In order to clearly show the evolution pattern of wave BG behaviors with the countered profile of a periodic Euler-Bernoulli beam, the variation of the normalized imaginary wavenumber $\text{Imag}(KL/\pi)$ with the quadratic thickness parameter is shown in Fig. 3(b). Here, the region with $\text{Imag}(K) \neq 0$ corresponds to the BG. Meanwhile, the BG evolution with h_2 calculated from FEM simulations are also shown and denoted by pink stars. Overall, the theoretical predictions are in agreement with those of FEM analysis, with very minor deviations. It can be observed that with the decreasing thickness at $x = L$, the BG becomes wider and the maximal attenuation factor in the BG becomes larger. In particular, the bottom edge frequency of this BG monotonically decreases from 11.46 kHz to 1.98 kHz when h_2 varies from 0 to -0.9 mm, which indicates the possibility to manipulate the flexural wave propagation via the quadratic thickness variations.

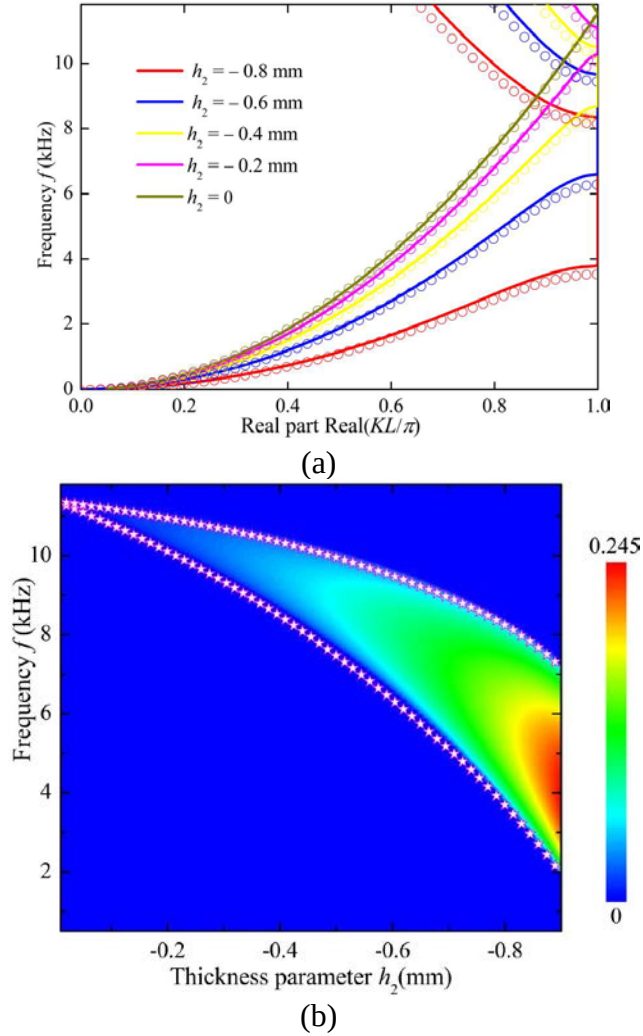


Figure 3 (a) The frequency spectra in the low frequency region for some selected h_2 values in Case 3, in which circles are results from FEM software; (b) The attenuation distribution of the first BG, i.e., $\text{Imag}(KL/\pi)$ for different h_2 values in Case 3. The BG edges from FEM simulations are denoted by pink stars.

4. Applications

4.1 The structural design for frequency shunting

In this section, a low-pass flexural wave filter is designed with the aid of a series of non-uniform Euler-Bernoulli beams with graded quadratic profiles, which is contained in the shadow region in Fig. 4. 100 sub-beams are distributed in series in the region of $0 \leq x \leq 2$ m, and the thickness in each sub-beam varies in the quadratic form with the parameter h_2 changing linearly from -0.009 mm to -0.9 mm at a constant step of -0.009 mm. From Fig. 3(b), the bottom edge of the first BG gradually decreases with decreasing h_2 . Therefore, it is anticipated that using this filter flexural waves with different frequencies will be stopped and separated at different positions.

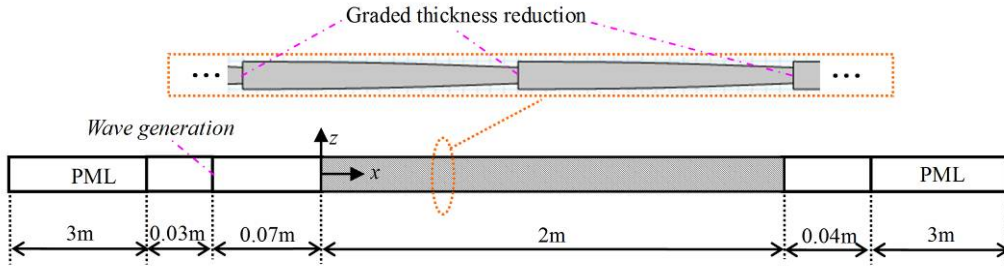


Figure 4 The schematic illustration of the low-pass flexural wave filter with graded contoured profiles.

In order to prove its working performance, the numerical model is established in the Comsol Multiphysics software, and the simulation in the frequency domain is carried out. A uniform displacement in the z direction with its amplitude $\bar{w} = 1\mu\text{m}$ is imposed at $x = -0.07$ m as the wave source. Additionally, two perfectly matched layers (PMLs) are arranged as shown in Fig. 4 to mimic the situation where the filter is connected to infinite beams.

Fig. 5 depicts the local deflection distribution for $1.5\text{ m} < x < 1.8\text{ m}$ at 4.0 kHz, 4.5 kHz, and 5.2 kHz, from which the frequency separation phenomenon can be observed clearly, i.e., the flexural wave with a lower frequency can propagate further. For quantitative examination, the deflection distributions are shown in Fig. 6 for different frequencies. The flexural wave with $f = 1.6$ kHz can go through this filter because 1.6 kHz is below the BG region in Fig. 3(b). For the working frequency $f = 4$ kHz, $|w|$ achieves its maximum at $x = 1.70$ m, beyond which the flexural wave will be reflected because of the BG and the amplitude decreases. At this position, the parameter of the sub-beam is $h_2 = -0.765$ mm, which coincides with the value $h_2 = -0.787$ mm predicted as the edge of BG from Fig. 3(b) with only small discrepancy. The same phenomenon can be obtained for the cases of 6 kHz and 8 kHz. Meanwhile, a flexural wave at higher frequency possesses a smaller attenuation factor in the BG, as shown in Fig. 3(b), and thus the wave decays more slowly after the maximal magnitude, compared with that at a low frequency. In particular for $f = 10$ kHz, a small but non-negligible amount of energy has transmitted through the filter.

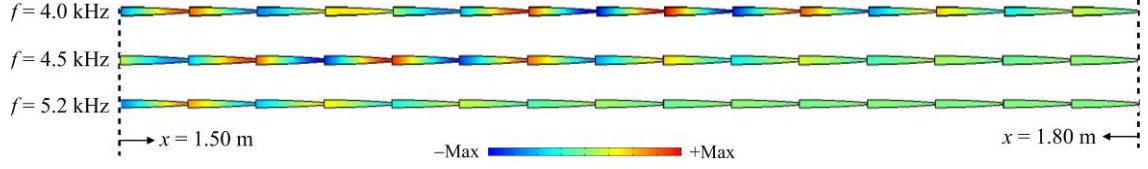


Figure 5 The local deflection distributions at 4.0 kHz, 4.5 kHz, and 5.2 kHz.

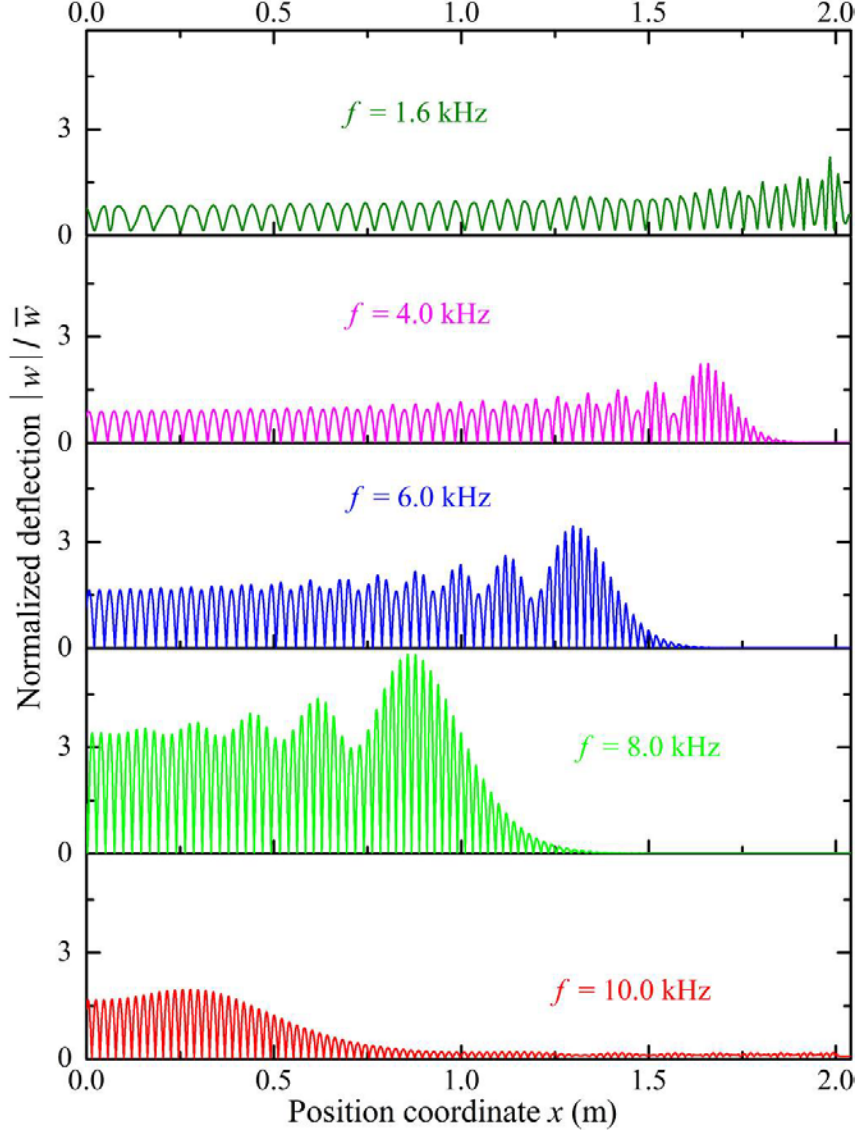


Figure 6 The normalized $|w|$ distribution along the x direction on the beam mid-plane.

The position corresponding to the maximum deflection in Fig. 6 is extracted, and its quantitative relation with the working frequency is shown in Fig. 7, in which the frequency separation phenomenon emerges again. Inspired by the notion of penetration depth in surface waves [52], the penetration depth of flexural wave d_p can be defined as the distance from the peak $|w|$ where the deflection amplitude decays to $|w|_{\max}/e$. The length d_p characterizes the distance that the flexural wave can travel in the BG after it

attains its maximum amplitude. If the detection position is beyond this distance, the incident flexural waves are viewed as being prohibited because nearly 87% wave energy has been reflected. The position corresponding to $|w|_{\max}/e$ versus the frequency f is also depicted in Fig. 7. The distance d_p between the positions of $|w|_{\max}$ and $|w|_{\max}/e$ increases with the increasing of f , due to the small attenuation in high frequency region shown in Fig. 3(b). For the potential application, the incident flexural wave with different frequencies can be received at different positions via this low-pass wave filter if the sensors or energy harvesters are arranged properly.

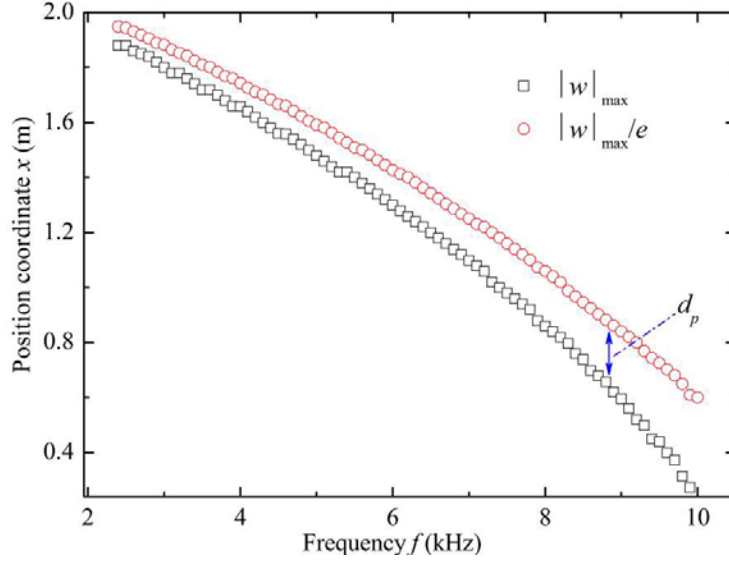


Figure 7 The quantitative relation between the frequency and the positions corresponding to $|w|_{\max}$ and $|w|_{\max}/e$.

4.2 A rectangular lens design for wave focusing

Reviewing Fig. 3(a), it can be deduced that the beams with different profiles possesses different phase velocities, based on which a lens can be designed for focusing a plane flexural wave. In optics, a convex aspherical lens can focus the light beams. Inspired by this, the design principle for flexural wave focusing is shown in Fig. 8. Multiple sub-beams with different h_2 values are distributed along the y direction, which compose the final lens. L' and L_f are the lens length and the focusing distance from the lens right edge, respectively. In order to make the incident plane wave focusing at a specific point, the waves through different channels should simultaneously arrive at this point, which requires [53,54]

$$\frac{L'}{c_0} + \frac{L_f}{c_f} = \frac{L'}{c_i} + \frac{\sqrt{L_f^2 + D_i^2}}{c_f}. \quad (16)$$

Here c_f is the phase velocity of the flexural wave in a flat plate at a fixed frequency f , and D_i is the distance of the i th sub-beam from the lens center line. Therefore, the phase velocity c_i in the i th sub-beam can be obtained as

$$c_i = \frac{c_f}{c_f/c_0 + (L_f - \sqrt{L_f^2 + D_i^2})/L'}. \quad (17)$$

After obtaining c_i , the h_2 value in individual sub-beam can be calculated with the aid of the analytical results in Fig. 3(a), and then the lens design can be finalized. The frequency of the incident flexural wave is assumed as $f = 5$ kHz, which then gives $c_f = 304$ m/s when the host plate thickness is $2h_0 = 2$ mm. Correspondingly, the wavelength is $\lambda = 6.08$ cm. L' is chosen as 0.3 m, approximately 5 times of the wavelength, which contains 15 unit sub-beams in the x direction, and L_f is set as 0.12 m, about the double wavelength.

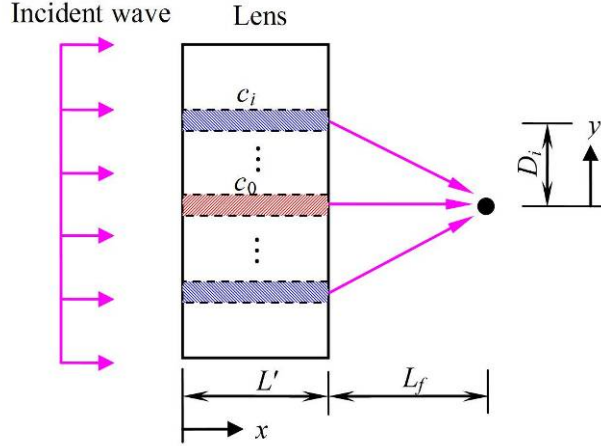


Figure 8 The design principle for focusing a plane flexural wave.

The beam width b in the y direction is adopted as 4 mm. On this condition, the lens designed contains 15×69 sub-units, with its top view shown in the blue region in Fig. 9(a). The lens centre is located at $(0.15 \text{ m}, 0, 0)$, and the forced displacement $\bar{w} = 1 \mu\text{m}$ is applied at $x = -0.03 \text{ m}$. The surrounding PMLs are arranged to prevent the wave reflections. The quadratic variation of sub-beam height is constant along the x direction and the h_2 variation in the y direction is depicted in Fig. 9(b), in which the phase velocity variation calculated from Eq. (17) is also included. The flexural wave is required to travel faster near the upper and bottom edges of the lens and slower near its center ($y = 0$). In order to reduce the mutual interference between different sub-beams, a series of tiny gaps with the width $0.02b$ have been inserted to separate them, so that the beam width has reduced as $0.98b$. For the simulations in the frequency domain, the beam and plate thicknesses are divided by four tetrahedral elements, and the maximum in-plane mesh size in the lens, the flat part of plate, and PMLs are set as 1.5 mm, 4 mm, and 6 mm, respectively, which ensures the calculation convergence. The total number of tetrahedral elements was 1857780.

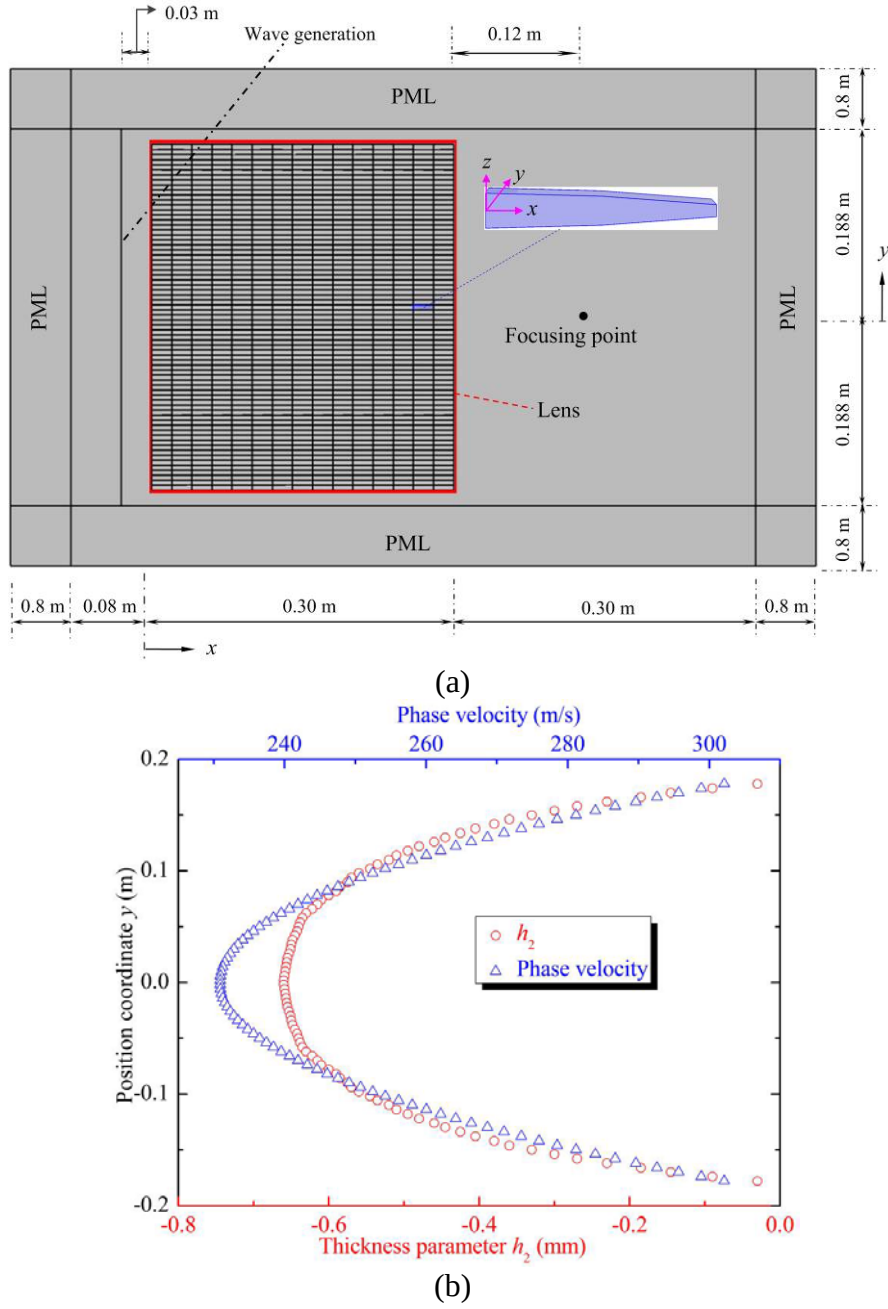


Figure 9 (a) The calculation model in the frequency domain (top view), with the lens for wave focusing contained in a rectangular region with its boundary shown in red; (b) The variations of phase velocity and h_2 value along the y direction.

Taking the value of $|w|^2$ as the performance index for focusing the flexural wave, Figure 10(a) illustrates its distribution after introducing the lens, and by contrast, the result in a flat plate without the lens is shown in Fig. 10(b). Additionally, Figs. 10(c) and 10(d) quantitatively illustrate its variations along the x and y directions, respectively. All of them clearly exhibit the wave focusing phenomenon. At the focusing point in Fig. 10(a), $|w|^2$ has been increased nearly 14 times of that in Fig. 10(b). The flexural wave is focused at $x = 0.4056$ m on the line along the plate thickness, which is nearly the same as the design value of $x = 0.42$ m. The small discrepancy may be due to the finite number of sub-beams distributed along the y direction which achieves the desired phase

velocity variation only approximately. The focusing size along the y direction, evaluated by -3 dB of $|w|$ or the quarter of peak $|w|^2$, is about 3.65 cm, i.e., 0.591λ , smaller than one wavelength, which is comparable to other focusing designs [55,56]. The focusing size along the x direction is 14.706 cm, equivalent to 2.419λ . It may be predicted that most of the energy is concentrated on a cubic region centered at the actual focal point (0.4056 m, 0, 0) with the length, width, height in the x, y, z directions respectively about 2.5λ , 0.6λ and 2 mm. As a whole, the working performance of this lens is satisfactory.

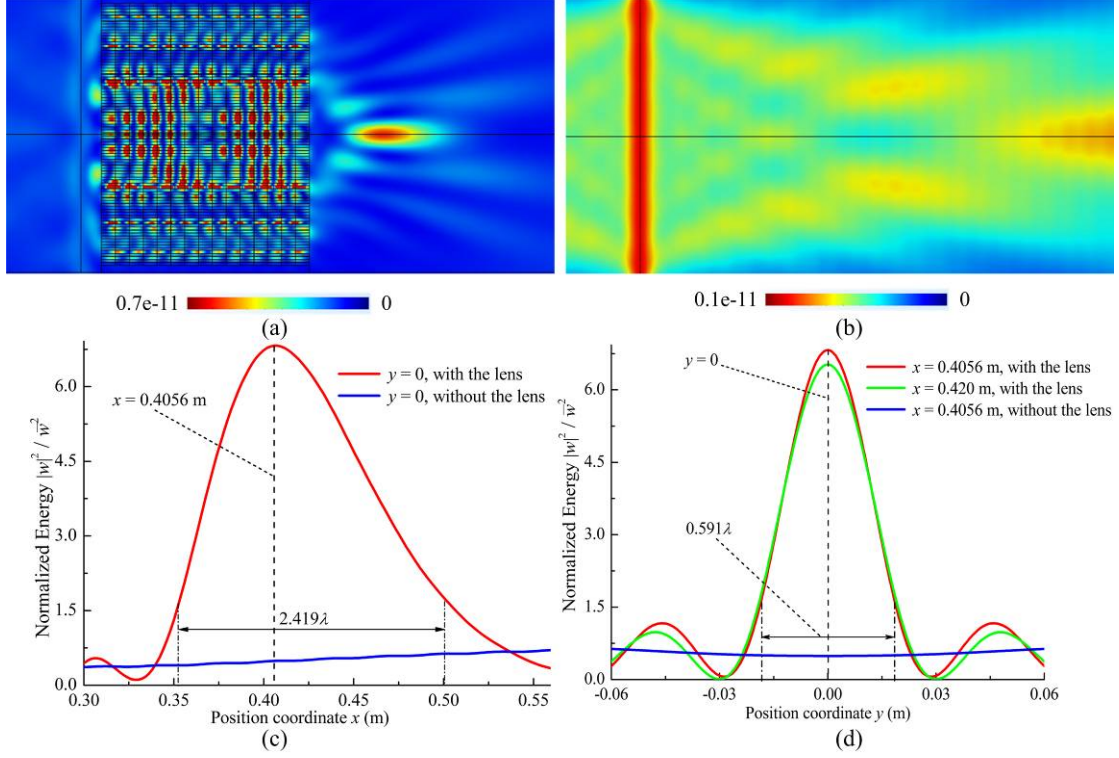


Figure 10 The simulation results in the frequency domain at $f = 5$ kHz: (a) the $|w|^2$ distribution in the plate with the lens (top view); (b) the $|w|^2$ distribution in the flat plate without the lens (top view); (c) the $|w|^2 / \bar{w}^2$ profile along the x direction at $z = 0$; (d) the $|w|^2 / \bar{w}^2$ profile along the y direction at $z = 0$.

Generally speaking, a smaller width b will make the phase velocity profile of Eq. (17) satisfied more accurately. However, it will increase the sub-beam number in the y direction, which may complicate the realization of this structure and increases the calculation cost. Oppositely, a larger b can decrease the sub-beam number, but with Eq. (17) achieved with less accuracy, and may affect the focusing phenomenon, as can be seen from the comparison between Fig. 10(a) when $b = 5$ mm and Fig. 11 when $b = 8$ mm. In Fig. 11, the peak value of $|w|^2$ at the focal position is reduced. Table 2 shows the focal position and the corresponding $|w|^2 / \bar{w}^2$ value, the focusing size along the x and y directions, and the sub-beam number for some selected b values. It can be seen that the sub-beam width in the range of [3 mm, 8.5 mm] give a satisfactory focusing phenomenon.

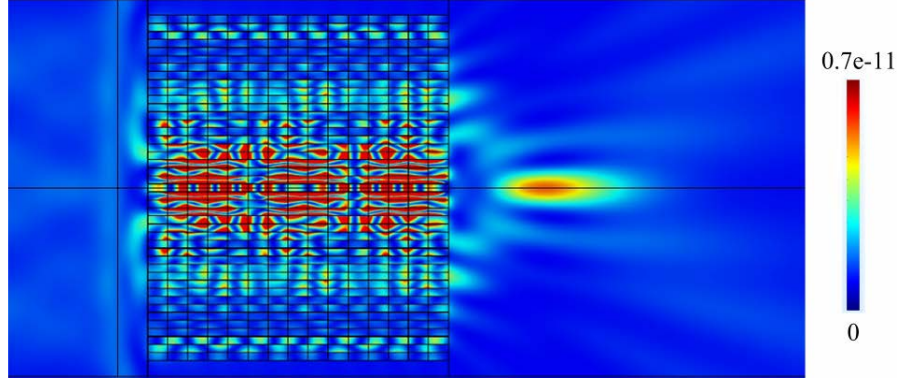


Figure 11 The $|w|^2$ distribution (top view) at $f = 5$ kHz when the beam width is chosen as $b = 8$ mm.

Table 2 The focusing performance comparison for different width values.

b (mm)	Focusing position	$ w ^2 / \bar{w}^2$ at the focusing position	Focusing size in the y direction (cm)	Focusing size in the x direction (cm)	Sub-beam number
3	$x = 0.4016$ m	7.245	3.636	14.731	117×15
3.5	$x = 0.3996$ m	6.868	3.642	14.721	99×15
4	$x = 0.4056$ m	6.823	3.65	14.724	89×15
4.5	$x = 0.3996$ m	6.416	3.608	14.282	77×15
5	$x = 0.3997$ m	6.014	3.592	13.541	69×15
5.5	$x = 0.4016$ m	6.359	3.676	14.431	62×15
6	$x = 0.4096$ m	5.902	3.76	14.756	57×15
6.5	$x = 0.3977$ m	5.885	3.67	15.08	53×15
7	$x = 0.3996$ m	6.333	3.636	14.719	49×15
7.5	$x = 0.3957$ m	5.996	3.624	14.36	45×15
8	$x = 0.3958$ m	6.255	3.64	14.919	43×15
8.5	$x = 0.3937$ m	6.448	3.678	14.943	39×15
9	$x = 0.4036$ m	4.79	3.892	17.9	37×15

Different curves in Fig. 3(a) are almost linear at $f = 5$ kHz, which means that the phase velocity of the flexural wave does not change significantly around the designed frequency. Therefore, it is anticipated that the lens can be suitable for other frequencies near 5 kHz. In order to examine the broadband property, the $|w|^2$ distributions at 5.4 kHz, 5.2 kHz, 4.2 kHz, and 3.6 kHz are shown in Fig. 12, as well as the actual focal positions. When the frequency is higher than 5 kHz, the slowing effect of the sub-beams at $y = 0$ is larger than those near the upper and bottom edges, which drags the flexural wave significantly, and makes the final focusing position moves towards the lens, as shown in Figs. 12(a) and 12(b). Oppositely, if the working frequency is lower than 5 kHz, the focusing position gradually moves far away from the lens, as shown Figs. 12(c) and 12(d).

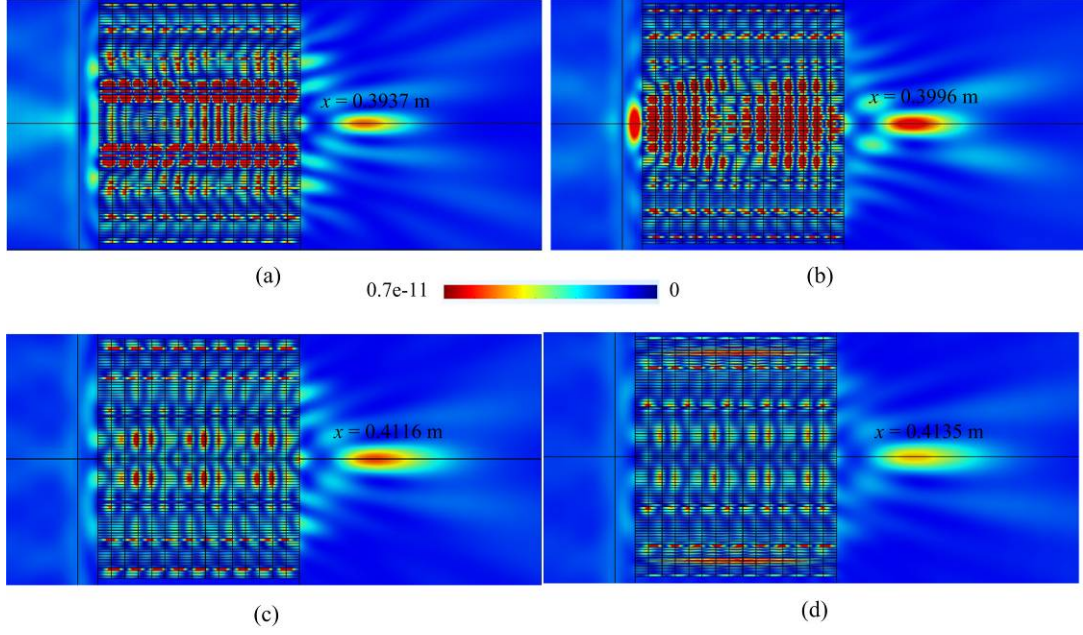


Figure 12 The $|w|^2$ distribution (top view) for other frequencies: (a) $f = 5.4$ kHz; (b) $f = 5.2$ kHz; (c) $f = 4.2$ kHz; (d) $f = 3.6$ kHz.

In order to quantitatively evaluate the wave focusing and determine the working frequency range of the lens, the following criterion is defined: the focal position should be located within a certain distance L_{cr} from that of 5 kHz. Different L_{cr} value will introduce different working frequency ranges. If L_{cr} is set as the quarter of the working wavelength, the working frequency range is obtained as [3.42 kHz, 5.48 kHz]. The part of working frequency range below 5 kHz is wider than that above 5 kHz, mainly due to the wave prohibiting property. Namely, as shown in the FEM results in Fig. 3(a), 5.67 kHz is the lower edge of the first BG, and waves with the frequency higher than this can not be transmitted through the lens.

5. Conclusions

In this study, the flexural wave propagation in a non-uniform Euler-Bernoulli beam with an arbitrary profile of the cross section has been solved by using the power series expansion method and utilized successfully in the theoretical analysis and numerical simulations of the corresponding periodic beams. As two applications of the flexural wave properties of such beams, a low-pass filter for frequency shunting and a rectangular lens for energy focusing have been designed via different graded quadratic thickness variations, and their working performance have been examined in the frequency domain. The conclusions of this study can be summarized as follows:

- (1) The power series expansion method has a good accuracy and convergence for analyzing flexural waves in a non-uniform Euler-Bernoulli beam. It has a high calculation efficiency, and is suitable for any thickness variation pattern as long as the convergence criterion for the inertia moment is satisfied, i.e., $|I_1| < I_0$.
- (2) The wave filter designed via gradient thickness variation can efficiently separate the incident flexural waves with different frequencies by stopping them at different positions.
- (3) It is demonstrated that the incident flexural wave can be focused successfully when

it travels into the rectangular lens designed, with the squared amplitude being increased nearly 14 times. The focusing capability is good, with the focusing size determined by -3 dB of deflection amplitude smaller than one wavelength.

As a final remark, the design scheme of the wave filter and the lens in this paper is based on a quadratic thickness variation. Not limited by this, other thickness variation profiles can also be suitable for the structural design. Furthermore, the flexural wave in a functionally graded (FG) Euler-Bernoulli beam has the same dynamic governing equation with Eq. (5), as long as $EI(x)$ and $\rho A(x)$ are respectively replaced by $E(x)I$ and $\rho(x)A$. Therefore, the power series method proposed in this paper is also suitable for an FG Euler-Bernoulli beam.

Acknowledgments

The authors gratefully acknowledge the supports of the Postdoctoral Fellowships for Research in Japan (FY2018 P18043) and the Grant-in-Aid for JSPS Fellows (18F18043) by the Japan Society for the Promotion of Science (JSPS), Natural Science Foundation of China (11972276), and the Opening Project from the State Key Laboratory for Strength and Vibration of Mechanical Structures (SV2018-KF-36).

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