

Construction of contact diffeomorphisms from Schwarzian derivatives

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I talk on my joint work with Tetsuya OZAWA([O-S]).

1 Contact Schwarzian derivative

On the affine 3-space $\mathbb{K}^3$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) with the usual coordinate (x, y, z) , we give the contact form $\alpha = dy - zdx$. Put

$$v_1 = \frac{\partial}{\partial x} + z \frac{\partial}{\partial y}, \quad v_2 = \frac{\partial}{\partial z}, \quad v_3 = \frac{\partial}{\partial y}, \quad v_4 = v_2 v_1 + v_1 v_2.$$

A local diffeomorphism ϕ is a *contact diffeomorphism*, if it satisfies $\phi^*(\alpha) = \rho\alpha$ for some nonvanishing function ρ . For a contact diffeomorphism $\phi : (x, y, z) \mapsto (X, Y, Z)$, we define the contact Schwarzian derivatives as follows: for $i, j, k = 1, 2$, set

$$s_{[ij,k]}(\phi) = v_i v_j(X) v_k(Z) - v_i v_j(Z) v_k(X),$$

and

$$S_{\{ijk\}}(\phi) = \frac{1}{3\Delta(\phi)} (s_{[ij,k]}(\phi) + s_{[jk,i]}(\phi) + s_{[ki,j]}(\phi)),$$

where $\Delta(\phi) = v_1(X)v_2(Z) - v_1(Z)v_2(X)$. We call the functions $S_{\{ijk\}}(\phi)$ the *contact Schwarzian derivatives* of the contact diffeomorphism ϕ . We denote the quadruple of functions by

$$S(\phi) = (S_{\{111\}}(\phi), S_{\{112\}}(\phi), S_{\{122\}}(\phi), S_{\{222\}}(\phi)).$$

Proposition 1.1. *The inverse ϕ^{-1} of a contact diffeomorphism $\phi : \mathbb{K}^3 \rightarrow \mathbb{K}^3$ maps the differential equation $Y''' = 0$ to*

$$y''' = S_{\{112\}}(\phi, x) + 3S_{\{111\}}(\phi, x)y'' + 3S_{\{222\}}(\phi, x)(y'')^2 + S_{\{122\}}(\phi, x)(y'')^3$$

By [S-Y], the condition that $y''' = f(x, y, y', y'')$ is mapped to $y''' = 0$ by a contact diffeomorphism is the vanishing of two curvatures A and b . We obtain that $b = 0$ is equivalent to $\partial^4 f / \partial x^4 = 0$. Let us consider

$$y''' = P + 3Qy' + 3R(y'')^2 + S(y'')^3,$$

where $P = P(x, y, y')$, $Q = Q(x, y, y')$, $R = R(x, y, y')$, $S = S(x, y, y')$. Then $b = 0$ and the condition $A = 0$ is equal to

$$\begin{aligned} v_3(P) &= 2(v_1 - 2Q)(M_{11}) + 4PM_4 \\ 3v_3(Q) &= 2(v_2 - 4R)(M_{11}) + 4(v_1 + Q)(M_4) + 4PM_{22} \\ 3v_3(R) &= 2(v_1 + 4Q)(M_{22}) + 4(v_2 - R)(M_4) - 4SM_{11} \\ v_3(S) &= 2(v_2 + 2R)(M_{22}) - 4SM_4. \end{aligned} \quad (\text{IC})$$

where we put

$$\begin{aligned} M_{11} &= -\frac{1}{4}(v_1(Q) - v_2(P) - 2Q^2 + 2PR) \\ M_4 &= -\frac{1}{4}(v_1(R) - v_2(Q) - QR + PS) \\ M_{22} &= -\frac{1}{4}(v_1(S) - v_2(R) - 2R^2 + 2QS). \end{aligned}$$

Theorem 1.1. *Four function P, Q, R, S on $\mathbb{K}^3$ is the Schwarzian derivatives of a contact diffeomorphism $\phi : \mathbb{K}^3 \rightarrow \mathbb{K}^3$;*

$$(P, Q, R, S) = S(\phi),$$

if and only if the system of the nonlinear differential equations (IC) is satisfied.

We seek a system of linear differential equations whose integrability equation is equal to (IC) and its solutions give the contact diffeomorphism. We call the linear system the linearization of (IC)

2 Fundamental system

Here is the linear differential system:

$$\begin{cases} v_1^2(\vartheta) = Qv_1(\vartheta) - Pv_2(\vartheta) + M_{11}\vartheta \\ v_4(\vartheta) = 2(Rv_1(\vartheta) - Qv_2(\vartheta) + M_4\vartheta) \\ v_2^2(\vartheta) = Sv_1(\vartheta) - Rv_2(\vartheta) + M_{22}\vartheta \end{cases} \quad (\text{Sp})$$

Theorem 2.1. *The necessary and sufficient condition for the linear PDE system (Sp) to have 4-dimensional solution space is equal to the nonlinear PDE system (IC).*

Proposition 2.1. *For any two solutions α and β of the PDE system (Sp), the function $I(\alpha, \beta)$ defined by*

$$I(\alpha, \beta) = \frac{1}{2}\alpha v_3(\beta) - \frac{1}{2}v_3(\alpha)\beta + v_1(\alpha)v_2(\beta) - v_2(\alpha)v_1(\beta) \quad (1)$$

is constant on (x, y, z) . Moreover this skew product $I(\alpha, \beta)$ is non-degenerate, and thus it defines a symplectic structure on the solution space $\mathcal{S}(P, Q, R, S)$ of (Sp), provided the dimension of $\mathcal{S}(P, Q, R, S)$ is equal to 4.

Theorem 2.2. *If a map $\phi : (x, y, z) \mapsto (X, Y, Z)$ is contact, then there exists a symplectic basis $\{\vartheta, \xi, \zeta, \eta\}$ of the solution space $\mathcal{S}(S(\phi))$ of the PDE system (Sp) such that ϕ is given by*

$$(x, y, z) \mapsto \left(\frac{\xi}{\vartheta}, \frac{1}{2} \left(\frac{\eta}{\vartheta} + \frac{\xi\zeta}{\vartheta^2} \right), \frac{\zeta}{\vartheta} \right). \quad (2)$$

Conversely, given a symplectic basis $\{\vartheta, \xi, \zeta, \eta\}$ of the solution space $\mathcal{S}(P, Q, R, S)$ of (Sp), the map ϕ defined by (2) is a contact diffeomorphism whose contact Schwarzian derivatives are equal to

$$S(\phi) = (P, Q, R, S).$$

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