

On a reverse of Cauchy-Schwarz inequalities in pre-inner product C^* -modules

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1. INTRODUCTION

This report is based on [3].

Let A be a positive operator on a Hilbert space H such that $mI \leq A \leq MI$ for some scalars $0 < m < M$. Then Kantorovich inequality [6, 4] says that

$$(1) \quad (Ax, x)(A^{-1}x, x) \leq \frac{(M+m)^2}{4Mm}$$

for every unit vector $x \in H$. This inequality (1) can be rephrased as follows:

$$\|Ax\| \|x\| \leq \frac{M+m}{2\sqrt{Mm}} (Ax, x)$$

for every vector $x \in H$. Therefore, Kantorovich inequality is just regarded as a reverse of Cauchy-Schwarz inequality

$$(Ax, x) \leq \|Ax\| \|x\|.$$

Dragomir [1] considered Kantorovich inequality (1) in the framework of an inner product space: Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space. Cauchy-Schwarz inequality says that

$$(2) \quad |\langle x, y \rangle| \leq \langle x, x \rangle^{\frac{1}{2}} \langle y, y \rangle^{\frac{1}{2}} \quad \text{for all } x, y \in H.$$

Dragomir showed the following Kantorovich type inequality for Cauchy-Schwarz inequality (2): If $x, y \in H$ and $\alpha, \beta \in \mathbb{C}$ satisfy the condition

$$\operatorname{Re} \langle \alpha y - x, x - \beta y \rangle \geq 0,$$

then

$$\langle x, x \rangle^{\frac{1}{2}} \langle y, y \rangle^{\frac{1}{2}} \leq \frac{|\alpha + \beta|}{2\sqrt{\operatorname{Re}(\alpha\bar{\beta})}} |\langle x, y \rangle|$$

and

$$(x, x)^{\frac{1}{2}} (y, y)^{\frac{1}{2}} - |(x, y)| \leq \frac{|\alpha - \beta|^2}{4|\alpha + \beta|} (y, y).$$

In this report, by virtue of the operator geometric mean and by using some ideas of [2], we shall consider Kantorovich type inequalities for Cauchy-Schwarz inequality in the framework of a pre-inner product C^* -module over a unital C^* -algebra, also see [9].

2. PRE-INNER PRODUCT C^* -MODULES

Let $\mathcal{A}$ be a unital C^* -algebra with the unit element e and the center $\mathcal{Z}(\mathcal{A})$. For $a \in \mathcal{A}$, we denote the real part of a by $\operatorname{Re} a = \frac{1}{2}(a + a^*)$. If $a \in \mathcal{A}$ is positive (that is selfadjoint with positive spectrum), then $a^{\frac{1}{2}}$ denotes a unique positive $b \in \mathcal{A}$ such that $b^2 = a$. For $a \in \mathcal{A}$, we denote the absolute value of a by $|a| = (a^*a)^{\frac{1}{2}}$. If $a \in \mathcal{Z}(\mathcal{A})$ is positive, then $a^{\frac{1}{2}} \in \mathcal{Z}(\mathcal{A})$. If $a, b \in \mathcal{A}$ are positive and $ab = ba$, then ab is positive and $(ab)^{\frac{1}{2}} = a^{\frac{1}{2}}b^{\frac{1}{2}}$.

Let $\mathcal{X}$ be an algebraic left $\mathcal{A}$ -module which is a complex linear space fulfilling $a(\lambda x) = (\lambda a)x = \lambda(ax)$ ($x \in \mathcal{X}, a \in \mathcal{A}, \lambda \in \mathbb{C}$). The space $\mathcal{X}$ is called a *(left) pre-inner product $\mathcal{A}$ -module* (or an *pre-inner product C^* -module over the unital C^* -algebra $\mathcal{A}$*) if there exists a mapping $\langle \cdot, \cdot \rangle: \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{A}$ satisfying

- (i) $\langle x, x \rangle \geq 0$,
- (ii) $\langle \lambda x + y, z \rangle = \lambda \langle x, z \rangle + \langle y, z \rangle$,
- (iii) $\langle ax, y \rangle = a \langle x, y \rangle$,
- (iv) $\langle y, x \rangle = \langle x, y \rangle^*$,

for all $x, y, z \in \mathcal{X}, a \in \mathcal{A}, \lambda \in \mathbb{C}$. Moreover, if

- (v) $x = 0$ whenever $\langle x, x \rangle = 0$,

then $\mathcal{X}$ is called an *inner product $\mathcal{A}$ -module*. In this case $\|x\| := \sqrt{\|\langle x, x \rangle\|}$, where the latter norm denotes the C^* -norm on $\mathcal{A}$. If this norm is complete, then $\mathcal{X}$ is called a *Hilbert $\mathcal{A}$ -module*. Any inner product space is an inner product $\mathbb{C}$ -module and any C^* -algebra $\mathcal{A}$ is a Hilbert C^* -module over itself via $\langle a, b \rangle = ab^*$ ($a, b \in \mathcal{A}$). For more details on Hilbert C^* -modules, see [8]. Notice that (iii) and (iv) imply $\langle x, ay \rangle = \langle x, y \rangle a^*$ for all $x, y \in \mathcal{X}, a \in \mathcal{A}$.

We discuss the Cauchy-Schwarz inequality and its reverse in a pre-inner product C^* -module over a unital C^* -algebra $\mathcal{A}$. Since the product of $\langle x, x \rangle$ and $\langle y, y \rangle$ are not selfadjoint in general, we would expect that the following Cauchy-Schwarz inequalities hold:

$$|\langle x, y \rangle|^2 \leq \operatorname{Re} \langle x, x \rangle \langle y, y \rangle \quad \text{for } x, y \in \mathcal{X}$$

and

$$\operatorname{Re} \langle x, y \rangle \leq \operatorname{Re} \langle x, x \rangle^{\frac{1}{2}} \langle y, y \rangle^{\frac{1}{2}} \quad \text{for } x, y \in \mathcal{X}.$$

But we have a counterexample. As a matter of fact, let $\mathcal{A} = M_2(\mathbb{C})$ be the C^* -algebra of 2×2 matrices with an inner product $\langle x, y \rangle = xy^*$ for $x, y \in \mathcal{A}$. Put $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Then we have $|\langle x, y \rangle|^2 \not\leq \operatorname{Re} \langle x, x \rangle \langle y, y \rangle$ and $\operatorname{Re} \langle x, y \rangle \not\leq \operatorname{Re} \langle x, x \rangle^{\frac{1}{2}} \langle y, y \rangle^{\frac{1}{2}}$.

In a pre-inner product C^* -module, the Cauchy-Schwarz inequality is firstly established by Lance [8]:

$$|\langle y, x \rangle|^2 = \langle x, y \rangle \langle y, x \rangle \leq \|\langle y, y \rangle\| \langle x, x \rangle$$

for $x, y \in \mathcal{X}$. Afterwards, Ilisević and Varosanec [5] showed another version:

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle$$

for $x, y \in \mathcal{X}$ and $\langle x, x \rangle \in \mathcal{Z}(\mathcal{A})$.

3. CAUCHY-SCHWARZ INEQUALITY AND ITS REVERSE

Let A and B be positive operators on a Hilbert space. Then the operator geometric mean $A \sharp B$ is defined by

$$A \sharp B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{\frac{1}{2}} A^{\frac{1}{2}}$$

if A is invertible, see [7]. The operator geometric mean has the symmetric property: $A \sharp B = B \sharp A$. If A commutes with B , then $A \sharp B = A^{\frac{1}{2}} B^{\frac{1}{2}}$. From viewpoint of (2), we would expect the following Cauchy-Schwarz inequality in a pre-inner product C^* -module:

$$(3) \quad |\langle x, y \rangle| \leq \langle x, x \rangle \sharp \langle y, y \rangle$$

holds for $x, y \in \mathcal{X}$. Unfortunately we also have a counterexample. If $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ mentioned above, then we have $|\langle x, y \rangle| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\langle x, x \rangle \sharp \langle y, y \rangle = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$. Therefore, we have $|\langle x, y \rangle| \not\leq \langle x, x \rangle \sharp \langle y, y \rangle$.

However, we have the following Cauchy-Schwarz type inequality:

Theorem 1. *Let $\mathcal{X}$ be a pre-inner product C^* -module over a unital C^* -algebra $\mathcal{A}$. Suppose that $x, y \in \mathcal{X}$ such that a polar decomposition $\langle x, y \rangle = u|\langle x, y \rangle|$ and $u \in \mathcal{A}$. Then*

$$|\langle x, y \rangle| \leq u^* \langle x, x \rangle u \sharp \langle y, y \rangle.$$

To prove a reverse of Cauchy-Schwarz type inequality in Theorem 1, we need the following lemma:

Lemma 2. *Let $\mathcal{X}$ be a pre-inner product C^* -module over a unital C^* -algebra $\mathcal{A}$. Suppose that $x, y \in \mathcal{X}$ such that there exist a partial isometry $u \in \mathcal{A}$ such that a polar decomposition $\langle x, y \rangle = u|\langle x, y \rangle|$ and*

$$(4) \quad \operatorname{Re} \langle Ay - u^*x, u^*x - ay \rangle \geq 0$$

for some $a, A \in \mathcal{Z}(\mathcal{A})$. Then

$$u^* \langle x, x \rangle u + \operatorname{Re}(Aa^*) \langle y, y \rangle \leq \operatorname{Re}(A + a) |\langle x, y \rangle|.$$

Remark 3. *The condition (4) in Lemma 2 is equivalent to*

$$\langle u^*x - \frac{A+a}{2}y, u^*x - \frac{A+a}{2}y \rangle \leq \frac{|A-a|^2}{4} \langle y, y \rangle.$$

Theorem 4. *Let $\mathcal{X}$ be a pre-inner product C^* -module over a unital C^* -algebra $\mathcal{A}$. Suppose that $x, y \in \mathcal{X}$ such that there exist a partial isometry $u \in \mathcal{A}$ such that a polar decomposition $\langle x, y \rangle = u|\langle x, y \rangle|$ and (4) holds for some elements $a, A \in \mathcal{Z}(\mathcal{A})$ and $\operatorname{Re}(Aa^*)$ is positive invertible and $\operatorname{Re}(A + a)$ is invertible. Then*

$$(i) \quad u^* \langle x, x \rangle u \sharp \langle y, y \rangle \leq \frac{\operatorname{Re}(A + a)}{2\sqrt{\operatorname{Re}(Aa^*)}} |\langle x, y \rangle|.$$

$$(ii) \quad u^* \langle x, x \rangle u \sharp \langle y, y \rangle - |\langle x, y \rangle| \leq \frac{(\operatorname{Re}(A + a))^2 - 4\operatorname{Re}(Aa^*)}{4\operatorname{Re}(A + a)} \langle y, y \rangle.$$

$$(iii) \quad u^* \langle x, x \rangle u \# \langle y, y \rangle - |\langle x, y \rangle| \leq \frac{(\operatorname{Re}(A + a))^2 - 4\operatorname{Re}(Aa^*)}{4\operatorname{Re}(Aa^*)\operatorname{Re}(A + a)} \langle x, x \rangle.$$

Finally, though the inequality (3) does not hold in general, we have reverse types of (3):

Theorem 5. *Let $\mathcal{X}$ be a pre-inner product C^* -module over a unital C^* -algebra $\mathcal{A}$. Suppose that $x, y \in \mathcal{X}$ such that*

$$\langle Ay - x, x - ay \rangle \geq 0 \quad \text{for some positive invertible } A, a \in \mathcal{Z}(\mathcal{A}).$$

Then

$$(i) \quad \langle x, x \rangle \# \langle y, y \rangle \leq \frac{A + a}{2\sqrt{Aa}} \operatorname{Re} \langle x, y \rangle.$$

$$(ii) \quad \langle x, x \rangle \# \langle y, y \rangle - \operatorname{Re} \langle x, y \rangle \leq \frac{(A - a)^2}{4(A + a)} \langle y, y \rangle.$$

$$(ii) \quad \langle x, x \rangle \# \langle y, y \rangle - \operatorname{Re} \langle x, y \rangle \leq \frac{(A - a)^2}{4Aa(A + a)} \langle x, x \rangle.$$

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