

Riesz decomposition and limits at infinity for p -precise functions on a half space

広島大総合科学部 水田義弘 (Yoshihiro Mizuta)
 広島大教育学部 下村 哲 (Tetsu Shimomura)

1 Introduction

Let u be a nonnegative superharmonic function on $D = \{x = (x_1, \dots, x_{n-1}, x_n) \in \mathbf{R}^n; x_n > 0\}$, where $n \geq 2$. Then it is known (cf. Lelong-Ferrand [6]) that u is uniquely decomposed as

$$u(x) = ax_n + \int_D G(x, y) d\mu(y) + \int_{\partial D} P(x, y) d\nu(y),$$

where a is a nonnegative number, μ (resp. ν) is a nonnegative measure on D (resp. ∂D), G is the Green function for D and P is the Poisson kernel for D . The first author showed in [9] that if $0 \leq \beta \leq 1$, $1 - n \leq \gamma < 1$ and $\int_D y_n^\gamma d\mu(y) + \int_{\partial D} |y|^{\gamma-1} d\nu(y) < \infty$, then

$$\lim_{|x| \rightarrow \infty, x \in D - E'} x_n^{-\beta} |x|^{n+\gamma-2+\beta} [u(x) - ax_n] = 0$$

with a suitable exceptional set $E' \subset D$. For related results, we also refer the reader to Essén-Jackson [3, Theorem 4.6], Aikawa [1] and Miyamoto-Yoshida [8].

Our main aim in this paper is to establish the analogue of these results for locally p -precise functions u in D satisfying

$$\int_D |\nabla u(x)|^p x_n^\gamma dx < \infty, \tag{1}$$

where ∇ denotes the gradient, $1 < p < \infty$ and $-1 < \gamma < p - 1$ (see Ohtsuka [15] and Ziemer [17] for locally p -precise functions).

2 Fine limits at infinity

Denote by $\mathbf{D}^{p,\gamma}$ the space of all locally p -precise functions on D satisfying (1). Consider the kernel function

$$K_\gamma(x, y) = |x - y|^{1-n} y_n^{-\gamma/p}.$$

To evaluate the size of exceptional sets, we use the capacity

$$C_{K_\gamma, p}(E; G) = \inf \int_D g(y)^p dy,$$

where E is a subset of an open set G in D and the infimum is taken over all nonnegative measurable functions g such that $g = 0$ outside G and

$$\int_D K_\gamma(x, y)g(y)dy \geq 1 \quad \text{for all } x \in E.$$

We say that $E \subset D$ is (K_γ, p) -thin at infinity if

$$\sum_{i=1}^{\infty} 2^{-i(n+\gamma-p)} C_{K_\gamma, p}(E_i; D_i) < \infty, \quad (2)$$

where $E_i = \{x \in E : 2^i \leq |x| < 2^{i+1}\}$ and $D_i = \{x \in D : 2^{i-1} < |x| < 2^{i+2}\}$.

Our first aim in this paper is to establish the following theorem.

THEOREM 1 (cf. [4]). *Let $p > 1$, $-1 < \gamma < p - 1$ and $n + \gamma - p \geq 0$. If $u \in \mathbf{D}^{p, \gamma}$, then there exist a set $E \subset D$ and a number A such that E is (K_γ, p) -thin at infinity;*

$$\lim_{|x| \rightarrow \infty, x \in D-E} |x|^{(n+\gamma-p)/p} [u(x) - A] = 0$$

in case $n + \gamma - p > 0$ and

$$\lim_{|x| \rightarrow \infty, x \in D-E} (\log |x|)^{-1/p'} [u(x) - A] = 0$$

in case $n + \gamma - p = 0$, where $p' = p/(p - 1)$.

In fact, if $1 \leq q < p$ and $q < p/(1 + \gamma)$, then Hölder's inequality gives

$$\int_G |\nabla u(x)|^q dx \leq \left(\int_G x_n^{-\gamma q/(p-q)} dx \right)^{1-q/p} \left(\int_G |\nabla u(x)|^p x_n^\gamma dx \right)^{q/p} < \infty$$

for every bounded open set $G \subset D$. Hence we can find a locally q -precise extension $\bar{u}$ to $\mathbf{R}^n$ such that $\bar{u}(x', x_n) = u(x', x_n)$ for $x_n > 0$ and $\bar{u}(x', x_n) = u(x', -x_n)$ for $x_n < 0$. We denote by $B(x, r)$ the open ball centered at x with radius $r > 0$. In view of [13], we can find a number a such that

$$\bar{u}(x) = c_n \sum_{i=1}^n \int_{B(0,1)} \frac{x_i - y_i}{|x - y|^n} \frac{\partial \bar{u}}{\partial y_i}(y) dy + c_n \sum_{i=1}^n \int_{\mathbf{R}^n - B(0,1)} \left(\frac{x_i - y_i}{|x - y|^n} - \frac{-y_i}{|y|^n} \right) \frac{\partial \bar{u}}{\partial y_i}(y) dy + a$$

for almost every $x \in \mathbf{R}^n$. Here we see that the equality holds for every $x \in D$ except that in a set of $C_{K_\gamma, p}$ -capacity zero. Now Theorem 1 is a consequence of [4].

3 Riesz decomposition

We denote by $\mathbf{D}_0^{p, \gamma}$ the space of all functions $u \in \mathbf{D}^{p, \gamma}$ having vertical limit zero at almost every boundary point of D , and by $\mathbf{HD}^{p, \gamma}$ the space of all harmonic functions on D in $\mathbf{D}^{p, \gamma}$. As in Deny-Lions [2], we have the following Riesz decomposition of $u \in \mathbf{D}^{p, \gamma}$.

THEOREM 2. A function $u \in \mathbf{D}^{p,\gamma}$ is uniquely represented as

$$u = u_0 + h, \quad (3)$$

where $u_0 \in \mathbf{D}_0^{p,\gamma}$ and $h \in \mathbf{HD}^{p,\gamma}$. More precisely, for fixed $\xi \in D$,

$$\begin{aligned} u_0(x) &= c_n \sum_{i=1}^n \int_D \left(\frac{x_i - y_i}{|x - y|^n} - \frac{\bar{x}_i - y_i}{|\bar{x} - y|^n} \right) \frac{\partial u}{\partial y_i}(y) dy, \\ h(x) &= 2c_n \sum_{i=1}^n \int_D \left(\frac{\bar{x}_i - y_i}{|\bar{x} - y|^n} - \frac{\bar{\xi}_i - y_i}{|\bar{\xi} - y|^n} \right) \frac{\partial u}{\partial y_i}(y) dy + A, \end{aligned}$$

where $\bar{x} = (x_1, \dots, x_{n-1}, -x_n)$ for $x = (x_1, \dots, x_{n-1}, x_n)$, $c_n = \Gamma(n/2)/(2\pi^{n/2})$ and A is a constant depending on u and ξ .

As applications we are concerned with the limits at infinity of functions in $\mathbf{D}_0^{p,\gamma}$ and $\mathbf{HD}^{p,\gamma}$.

Consider the kernel function

$$k_{\beta,\gamma}(x, y) = x_n^{1-\beta} y_n^{-\gamma/p} |x - y|^{1-n} |\bar{x} - y|^{-1}$$

for x and y in D . To evaluate the size of exceptional sets, we use the capacity

$$C_{k_{\beta,\gamma,p}}(E; G) = \inf \int_D g(y)^p dy,$$

where E is a subset of an open set G in D and the infimum is taken over all nonnegative measurable functions g such that $g = 0$ outside G and

$$\int_D k_{\beta,\gamma}(x, y) g(y) dy \geq 1 \quad \text{for all } x \in E.$$

We say that $E \subset D$ is $(k_{\beta,\gamma,p})$ -thin at infinity if

$$\sum_{i=1}^{\infty} 2^{-i(n+\gamma-(1-\beta)p)} C_{k_{\beta,\gamma,p}}(E_i; D_i) < \infty. \quad (4)$$

THEOREM 3. Let $p > 1$, $-1 < \gamma < p - 1$ and $0 \leq \beta \leq 1$. If $u \in \mathbf{D}_0^{p,\gamma}$, then there exists a set $E \subset D$ such that E is $(k_{\beta,\gamma,p})$ -thin at infinity and

$$\lim_{|x| \rightarrow \infty, x \in D-E} x_n^{-\beta} |x|^{(n+\gamma-(1-\beta)p)/p} u(x) = 0.$$

THEOREM 4. Let $p > 1$, $-1 < \gamma < p - 1$ and $n + \gamma - p \geq 0$. If $h \in \mathbf{HD}^{p,\gamma}$, then there exist a number A such that

$$\lim_{|x| \rightarrow \infty, x \in D} x_n^{(n+\gamma-p)/p} [h(x) - A] = 0$$

in case $n + \gamma - p > 0$ and

$$\lim_{|x| \rightarrow \infty, x \in D} (\max\{\log(1/x_n), \log|x|\})^{-1/p'} [h(x) - A] = 0$$

in case $n + \gamma - p = 0$.

REMARK 1. Let $p > 1$, $-1 < \gamma < p - 1$ and $n + \gamma - p > 0$. Then we can find a function $h \in \mathbf{HD}^{p,\gamma}$ such that

$$\limsup_{|x| \rightarrow \infty, x \in D} |x|^{(n+\gamma-p)/p} h(x) = \infty$$

and

$$\lim_{|x| \rightarrow \infty, x \in D} x_n^{(n+\gamma-p)/p} h(x) = 0.$$

For proofs of these theorems, we refer to [14].

4 Examples of thin sets at infinity

We are concerned with the measure condition on sets which are thin at infinity.

For a measurable set $E \subset \mathbf{R}^n$, denote by $|E|$ the Lebesgue measure of E . Then we can prove that

$$|E|^{(1-(1-\beta)/n)p} \leq MC_{k_{\beta,\gamma,p}}(E; D_0) \quad (5)$$

and

$$C_{k_{\beta,\gamma,p}}(rE; rD_0) = r^{n+\gamma-(1-\beta)p} C_{k_{\beta,\gamma,p}}(E; D_0) \quad (6)$$

whenever $E \subset D \cap B(0, 2) - B(0, 1)$ and $r > 0$. Hence we have the following result.

PROPOSITION 1. Let $0 \leq \beta \leq 1$ and $-1 < \gamma < p - 1$. If (4) holds, then

$$\sum_{i=1}^{\infty} \left(\frac{|E_i|}{|B_i|} \right)^{(1-(1-\beta)/n)p} < \infty,$$

where $E_i = E \cap B_{i+1} - B_i$ with $B_i = B(0, 2^i) \cap D$.

If E is well situated, then we have stronger results as in the following.

PROPOSITION 2. Let $0 \leq \beta \leq 1$ and $-1 < \gamma < p - 1$. Set $F = \cup_{j=1}^{\infty} B_j$, where $B_j = B(x_j, s_j)$ with $2^j \leq |x_j| < 2^{j+1}$ and $r_j = (x_j)_n > 2s_j$. If $p < n$ and F is $(k_{\beta,\gamma,p})$ -thin at infinity, then

$$\sum_{j=1}^{\infty} \left(\frac{s_j}{2^j} \right)^{n-p} \left(\frac{r_j}{2^j} \right)^{\beta p + \gamma} < \infty; \quad (7)$$

conversely, if (7) holds, then F is $(k_{\beta,\gamma}, p)$ -thin at infinity.

PROOF. First we show that if $p < n$, then

$$s^{n-p} r^{\beta p + \gamma} \leq MC_{k_{\beta,\gamma},p}(B; D_0) \quad (8)$$

for $B = B(x_0, s)$ with $1 \leq |x_0| < 2$ and $r = (x_0)_n > 2s$. Let g be a nonnegative measurable function such that $g = 0$ outside D_0 and

$$\int_D k_{\beta,\gamma}(x, y) g(y) dy \geq 1$$

for every $x \in B$. Then we have by Fubini's theorem

$$\begin{aligned} |B| &\leq \int_B \left(\int_{D_0} k_{\beta,\gamma}(x, y) g(y) dy \right) dx \\ &= \int_{D_0} g(y) y_n^{-\gamma/p} \left(\int_B x_n^{1-\beta} |x - y|^{1-n} |\bar{x} - y|^{-1} dx \right) dy \\ &\leq M r^{1-\beta} \int_{D_0} g(y) y_n^{-\gamma/p} \left(\int_B |x - y|^{1-n} |\bar{x} - y|^{-1} dx \right) dy. \end{aligned}$$

We set

$$I(y) = \int_B |x - y|^{1-n} |\bar{x} - y|^{-1} dx$$

and

$$J = \int_{D_0} g(y) y_n^{-\gamma/p} \left(\int_B |x - y|^{1-n} |\bar{x} - y|^{-1} dx \right) dy.$$

If $|y - x_0| < 3s/2$, then

$$I(y) \leq r^{-1} \int_B |x - y|^{1-n} dx \leq M r^{-1} s,$$

so that we have by Hölder's inequality

$$\begin{aligned} J_1 &= \int_{\{y \in D_0: |y - x_0| < 3s/2\}} g(y) y_n^{-\gamma/p} I(y) dy \\ &\leq M r^{-1} s \int_{\{y \in D_0: |y - x_0| < 3s/2\}} g(y) y_n^{-\gamma/p} dy \\ &\leq M r^{-1} s \left(\int_{\{y \in D_0: |y - x_0| < 3s/2\}} y_n^{-\gamma p'/p} dy \right)^{1/p'} \left(\int_{D_0} g(y)^p dy \right)^{1/p} \\ &\leq M s^n r^{-1-\gamma/p} s^{1-n/p} \left(\int_{D_0} g(y)^p dy \right)^{1/p}. \end{aligned}$$

If $|y - x_0| \geq 3s/2$ and $y_n \leq x_n/2$, then $|x - y| \geq M(|x' - y| + x_n) \geq M(|x'_0 - y| + r)$, so that

$$\begin{aligned} I_2(y) &= \int_{\{x \in B: y_n \leq x_n/2\}} |x - y|^{1-n} |\bar{x} - y|^{-1} dx \\ &\leq M(|x'_0 - y| + r)^{-n} s^n. \end{aligned}$$

Hence we have by Hölder's inequality

$$\begin{aligned} J_2 &= \int_{\{y \in D_0: |x_0 - y| \geq 3s/2\}} g(y) y_n^{-\gamma/p} I_2(y) dy \\ &\leq M s^n \int_{D_0} g(y) y_n^{-\gamma/p} (|x'_0 - y| + r)^{-n} dy \\ &\leq M s^n \left(\int_{D_0} y_n^{-\gamma p'/p} (|x'_0 - y| + r)^{-p'n} dy \right)^{1/p'} \left(\int_{D_0} g(y)^p dy \right)^{1/p} \\ &\leq M s^n r^{-\gamma/p - n/p} \left(\int_{D_0} g(y)^p dy \right)^{1/p} \\ &\leq M s^n s^{1-n/p} r^{-1-\gamma/p} \left(\int_{D_0} g(y)^p dy \right)^{1/p}, \end{aligned}$$

since $p < n$. If $|y - x_0| \geq 3s/2$ and $y_n > x_n/2$, then $|x - y| \geq M(|x_0 - y| + s)$ and $|\bar{x} - y| \geq M(|x_0 - y| + r)$, so that

$$\begin{aligned} I_3(y) &= \int_{\{x \in B: y_n > x_n/2\}} |x - y|^{1-n} |\bar{x} - y|^{-1} dx \\ &\leq M(|x_0 - y| + s)^{1-n} (|x_0 - y| + r)^{-1} s^n. \end{aligned}$$

Consequently, it follows that

$$\begin{aligned} J_3 &= \int_{\{y \in D_0: |x_0 - y| \geq 3s/2, y_n > r/4\}} g(y) y_n^{-\gamma/p} I_3(y) dy \\ &\leq M s^n \int_{\{y \in D_0: |y - x_0| \geq 3s/2, y_n > r/4\}} g(y) y_n^{-\gamma/p} (|x_0 - y| + s)^{1-n} (|x_0 - y| + r)^{-1} dy. \end{aligned}$$

Setting $t = |x_0 - y|$ and $|(x_0)_n - y_n| = t \cos \theta$, we note that

$$(t + r) \cos \theta \leq |(x_0)_n - y_n| + (x_0)_n \leq 3y_n < 3(r + t)$$

when $y_n > r/4$. Using Hölder's inequality and applying the polar coordinates about x_0 , we have

$$\begin{aligned} J_3 &\leq M s^n \left(\int_{3s/2}^{\infty} (t + s)^{p'(1-n)} (t + r)^{p'(-\gamma/p-1)} t^{n-1} dt \right)^{1/p'} \left(\int_{D_0} g(y)^p dy \right)^{1/p} \\ &\leq M s^n s^{1-n/p} r^{-1-\gamma/p} \left(\int_{D_0} g(y)^p dy \right)^{1/p} \end{aligned}$$

since $p < n$. Therefore we obtain

$$|B| \leq Mr^{-\beta-\gamma/p} s^{(p-n)/p} s^n \left(\int_{D_0} g(y)^p dy \right)^{1/p}.$$

Hence it follows from the definition of $C_{k_{\beta,\gamma,p}}$ that

$$r^{\beta p + \gamma} s^{n-p} \leq MC_{k_{\beta,\gamma,p}}(B; D_0),$$

as required.

To obtain the converse inequality, note that for $x \in B$

$$\begin{aligned} \int_B x_n^{1-\beta} |x-y|^{1-n} |\bar{x}-y|^{-1} y_n^{-\gamma/p} dy &\geq Mr^{-\beta-\gamma/p} \int_B |x-y|^{1-n} dy \\ &\geq Mr^{-\beta-\gamma/p} s, \end{aligned}$$

so that

$$C_{k_{\beta,\gamma,p}}(B; D_0) \leq Mr^{(\beta+\gamma/p)p} s^{-p} \int_B dy = Mr^{\beta p + \gamma} s^{n-p}.$$

Thus the proof is completed.

PROPOSITION 3. Let $0 \leq \beta \leq 1$ and $-1 < \gamma < p-1$. Set $V = \cup_{j=1}^{\infty} B(x_j, r_j) \cap D$ with $x_j \in \partial D$, $2^j \leq |x_j| < 2^{j+1}$ and $0 < r_j \leq 2^{j+1}$. If V is $(k_{\beta,\gamma,p})$ -thin at infinity, then

$$\sum_{j=1}^{\infty} \left(\frac{r_j}{2^j} \right)^{n+\gamma-(1-\beta)p} < \infty; \quad (9)$$

conversely, if $\gamma > (1-\beta)p$ and (9) holds, then V is $(k_{\beta,\gamma,p})$ -thin at infinity.

PROOF. First we show that if $B_+ = B(x_0, r) \cap D$ with $x_0 \in \partial D$, $1 \leq |x_0| < 2$ and $0 < r \leq 2$, then

$$r^{n+\gamma-(1-\beta)p} \leq MC_{k_{\beta,\gamma,p}}(B_+; D_0). \quad (10)$$

Let g be a nonnegative measurable function such that $g = 0$ outside D_0 and

$$\int_D k_{\beta,\gamma}(x, y) g(y) dy \geq 1$$

for every $x \in B_+$. Then we have by Fubini's theorem

$$\begin{aligned} |B_+| &\leq \int_{B_+} \left(\int_{D_0} k_{\beta,\gamma}(x, y) g(y) dy \right) dx \\ &= \int_{D_0} g(y) y_n^{-\gamma/p} \left(\int_{B_+} x_n^{1-\beta} |x-y|^{1-n} |\bar{x}-y|^{-1} dx \right) dy. \end{aligned}$$

Here we see that if $|x_0 - y| > 2r$, then

$$\int_{B_+} x_n^{1-\beta} |x - y|^{1-n} |\bar{x} - y|^{-1} dx \leq M |x_0 - y|^{-n} r^{1-\beta+n}$$

and that if $|x_0 - y| \leq 2r$, then

$$\begin{aligned} \int_{B_+} x_n^{1-\beta} |x - y|^{1-n} |\bar{x} - y|^{-1} dx &\leq M r^{1-\beta} \int_{B_+} |x - y|^{1-n} (|x - y| + y_n)^{-1} dx \\ &\leq M r^{1-\beta} \log(4r/y_n). \end{aligned}$$

Then we have by Hölder's inequality

$$\begin{aligned} J_1 &= r^{1-\beta} \int_{\{y \in D_0: |x_0 - y| \leq 2r\}} g(y) y_n^{-\gamma/p} \log(4r/y_n) dy \\ &\leq r^{1-\beta} \left(\int_{\{y \in D_0: |x_0 - y| \leq 2r\}} \{\log(4r/y_n)\}^{p'} y_n^{-\gamma p'/p} dy \right)^{1/p'} \left(\int_{D_0} g(y)^p dy \right)^{1/p} \\ &\leq M r^{1-\beta-\gamma/p+n/p'} \left(\int_{D_0} g(y)^p dy \right)^{1/p} \end{aligned}$$

and

$$\begin{aligned} J_2 &= r^{1-\beta+n} \int_{\{y \in D_0: |x_0 - y| > 2r\}} g(y) y_n^{-\gamma/p} |x_0 - y|^{-n} dy \\ &\leq r^{1-\beta+n} \left(\int_{\{y \in D_0: |x_0 - y| > 2r\}} y_n^{-\gamma p'/p} |x_0 - y|^{-p'n} dy \right)^{1/p'} \left(\int_{D_0} g(y)^p dy \right)^{1/p} \\ &\leq M r^{1-\beta-\gamma/p+n/p'} \left(\int_{D_0} g(y)^p dy \right)^{1/p}. \end{aligned}$$

Therefore we have

$$|B_+| \leq M r^{1-\beta-\gamma/p+n/p'} \left(\int_{D_0} g(y)^p dy \right)^{1/p},$$

so that it follows from the definition of $C_{k\beta, \gamma, p}$ that

$$r^{n+\gamma-(1-\beta)p} \leq M C_{k\beta, \gamma, p}(B_+; D_0),$$

as required.

To obtain the converse inequality, note that for $x \in B_+$

$$\begin{aligned} &\int_{B_+} x_n^{1-\beta} |x - y|^{1-n} |\bar{x} - y|^{-1} y_n^{-\gamma/p} dy \\ &\geq \int_{B_+ \cap B(x, x_n/2)} x_n^{1-\beta} |x - y|^{1-n} |\bar{x} - y|^{-1} y_n^{-\gamma/p} dy \\ &\geq M x_n^{1-\beta-1-\gamma/p} \int_{B_+ \cap B(x, x_n/2)} |x - y|^{1-n} dy \\ &\geq M x_n^{1-\beta-\gamma/p} \geq M r^{1-\beta-\gamma/p}, \end{aligned}$$

since $1 - \beta < \gamma/p$. Hence it follows from the definition of $C_{k_{\beta,\gamma,p}}$ that

$$C_{k_{\beta,\gamma,p}}(B_+; D_0) \leq M r^{-(1-\beta)p+\gamma} \int_{B_+} dy = M r^{n+\gamma-(1-\beta)p}.$$

Thus the proof is completed.

For a nondecreasing function φ on $\mathbf{R}^1$ such that $0 < \varphi(2t) \leq M\varphi(t)$ for $t > 0$ with a positive constant M , we set

$$T_\varphi = \{x = (x', x_n); 0 < x_n < \varphi(|x'|)\}.$$

PROPOSITION 4 (cf. Aikawa [1, Proposition 5.1]). *Let $0 < \beta \leq 1$ and $p(1 - \beta) - 1 < \gamma < p - 1$. Assume further that*

$$\lim_{r \rightarrow \infty} \frac{\varphi(r)}{r} = 0. \quad (11)$$

Then T_φ is $(k_{\beta,\gamma,p})$ -thin at infinity if and only if

$$\int_1^\infty \left(\frac{\varphi(t)}{t} \right)^{p(-1+\beta)+\gamma+1} \frac{dt}{t} < \infty. \quad (12)$$

For example, $\varphi(r) = r[\log(1 + r)]^{-\delta}$ satisfies (12), when $\delta\{p(-1 + \beta) + \gamma + 1\} > 1$.

5 Limits of monotone functions

Finally we consider the limits at infinity for monotone BLD functions. A continuous function u is called monotone on $\mathbf{D}$ in the sense of Lebesgue (see [5]) if for every relatively compact open subset G of $\mathbf{D}$,

$$\max_{G \cup \partial G} u = \max_{\partial G} u \quad \text{and} \quad \min_{G \cup \partial G} u = \min_{\partial G} u.$$

For examples and fundamental properties of monotone functions, see [12] and [16]. Among them the following result is only needed for monotone functions.

LEMMA 1. *If u is a monotone BLD function on $B(x, 2r)$ and $p > n - 1$, then*

$$|u(z) - u(x)|^p \leq M r^{p-n} \int_{B(x, 2r)} |\nabla u(y)|^p dy \quad (13)$$

for every $z \in B(x, r)$.

THEOREM 5. Let $p > n - 1$, $-1 < \gamma < p - 1$ and $n + \gamma - p \geq 0$. If u is a monotone function on D satisfying (1), then there exist a number A such that

$$\lim_{|x| \rightarrow \infty, x \in D} x_n^{(n+\gamma-p)/p} [u(x) - A] = 0$$

in case $n + \gamma - p > 0$ and

$$\lim_{|x| \rightarrow \infty, x \in D} (\max\{\log(1/x_n), \log|x|\})^{-1/p'} [u(x) - A] = 0$$

in case $n + \gamma - p = 0$.

PROOF. For $x \in D$, let $r = |x|$, $C(x) = (0, \dots, 0, r)$ and $\rho_D(x)$ denote the distance of $x \in D$ from the boundary ∂D , that is, $\rho_D(x) = x_n$. We take a finite covering $\{B_j\}$, $B_j = B(X_j, 4^{-1}\rho_D(X_j))$, such that

- (i) $X_1 = x$ and $X_{N+1} = C(x)$;
- (ii) $r/2 < |z| < 2r$ for $z \in A(r) = \cup_j 2B_j$, where $2B_j = B(X_j, 2^{-1}\rho_D(X_j))$;
- (iii) $B_j \cap B_{j+1} \neq \emptyset$ for each j ;
- (iv) $\sum_j \chi_{2B_j}$ is bounded, where χ_A denotes the characteristic function of A .

By the monotonicity of u , we see that

$$|u(y) - u(X_j)| \leq M \rho_D(X_j)^{(p-n)/p} \int_{2B_j} |\nabla u(z)|^p dz$$

for $y \in B_j$. First suppose $n + \gamma - p > 0$. Using Theorem 1, we can find a number A and $C_1(x)$ such that $C_1(x) \in B_{N+1}$ and

$$\lim_{|x| \rightarrow \infty} |x|^{(n+\gamma-p)/p} [u(C_1(x)) - A] = 0.$$

Then we have by Hölder's inequality

$$\begin{aligned} |u(x) - A| &\leq |u(X_1) - u(X_2)| + |u(X_2) - u(X_3)| + \dots + |u(X_N) - u(X_{N+1})| \\ &\quad + |u(X_{N+1}) - u(C_1(x))| + |u(C_1(x)) - A| \\ &\leq M \sum_j \rho_D(X_j)^{(p-n-\gamma)/p} \left(\int_{2B_j} |\nabla u(z)|^p \rho_D(z)^\gamma dz \right)^{1/p} + |u(C_1(x)) - A| \\ &\leq M \left(\sum_j \rho_D(X_j)^{p'(p-n-\gamma)/p} \right)^{1/p'} \left(\int_{A(r)} |\nabla u(z)|^p \rho_D(z)^\gamma dz \right)^{1/p} \\ &\quad + |u(C_1(x)) - A| \\ &\leq M x_n^{(p-n-\gamma)/p} \left(\int_{D-B(0,r/2)} |\nabla u(z)|^p \rho_D(z)^\gamma dz \right)^{1/p} + |u(C_1(x)) - A|, \end{aligned}$$

which proves

$$\lim_{|x| \rightarrow \infty} x_n^{(n+\gamma-p)/p} [u(x) - A] = 0,$$

as required.

The case $n + \gamma - p = 0$ can be treated similarly.

References

- [1] H. Aikawa, On the behavior at infinity of non-negative superharmonic functions in a half space, *Hiroshima Math. J.* **11** (1981), 425-441.
- [2] J. Deny and J. L. Lions, Les espaces du type de Beppo Levi, *Ann. Inst. Fourier* **5** (1955), 305-370.
- [3] M. Essén and H.L.Jackson, On the covering properties of certain exceptional sets in a half space, *Hiroshima Math. J.* **10** (1980), 233-262.
- [4] T. Kurokawa and Y. Mizuta, On the order at infinity of Riesz potentials, *Hiroshima Math. J.* **9** (1979), 533-545.
- [5] H. Lebesgue, Sur le problème de Dirichlet, *Rend. Circ. Mat. Palermo* **24** (1907), 371-402.
- [6] J. Lelong-Ferrand, Étude au voisinage de la frontière des fonctions surharmoniques positives dans un demi-espace, *Ann. Sci. École Norm. Sup.* **66** (1949), 125-159.
- [7] N. G. Meyers, A theory of capacities for potentials in Lebesgue classes, *Math. Scand.* **8** (1970), 255-292.
- [8] I. Miyamoto and H. Yoshida, Two criterions of Wiener type for minimally thin sets and rarefied sets in a cone, *J. Math. Soc. Japan* **54** (2002), 487-512.
- [9] Y. Mizuta, On the behavior at infinity of Green potentials in a half space, *Hiroshima Math. J.* **10** (1980), 607-613.
- [10] Y. Mizuta, On semi-fine limits of potentials, *Analysis* **2** (1982), 115-139.
- [11] Y. Mizuta, Continuity properties of potentials and Beppo-Levi-Deny functions, *Hiroshima Math. J.* **23** (1993), 79-153.
- [12] Y. Mizuta, Potential theory in Euclidean spaces, Gakkōtoshō, Tokyo, 1996.
- [13] Y. Mizuta, Integral representations, differentiability properties and limits at infinity for Beppo Levi functions, *Potential Analysis* **6** (1997), 237-276.
- [14] Y. Mizuta and T. Shimomura, Minimally fine limits at infinity for p -precise functions, to appear in *J. Math. Soc. Japan*.

- [15] M. Ohtsuka, Extremal length and precise functions in 3-space, Lecture Notes at Hiroshima University, 1972.
- [16] M. Vuorinen, Conformal geometry and quasiregular mappings, Lecture Notes in Math. **1319**, Springer-Verlag, 1988.
- [17] W. P. Ziemer, Extremal length as a capacity, Michigan Math. J. **17** (1970), 117-128.

*The Division of Mathematical and Information Sciences
Faculty of Integrated Arts and Sciences
Hiroshima University
Higashi-Hiroshima 739-8521, Japan
E-mail : mizuta@mis.hiroshima-u.ac.jp
and
Department of Mathematics
Graduate School of Education
Hiroshima University
Higashi-Hiroshima 739-8524, Japan
E-mail : tshimo@hiroshima-u.ac.jp*