

On localizations of the characteristic classes of ℓ -adic sheaves of rank 1

By

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Abstract

The Grothendieck-Ogg-Shafarevich formula is generalized to any dimensional scheme by Abbes-Kato-Saito in [KS] and [AS]. In this paper, we introduce two methods of localization of the characteristic classes for sheaves of rank 1 and compare them. As a corollary of this comparison, we obtain a refinement of a formula proved by Abbes-Saito in [AS] without denominator for a smooth sheaf of rank 1 which is clean with respect to the boundary.

§ 1. Introduction

The Grothendieck-Ogg-Shafarevich formula is a formula calculating the Euler-Poincaré number of an ℓ -adic sheaf on a curve. A. Abbes, K. Kato and T. Saito generalized this formula to any dimensional case. To generalize this formula K. Kato and T. Saito define the Swan class which is produced by the wild ramification of an ℓ -adic sheaf using logarithmic blow-up and alteration in [KS]. This invariant realizes a deep insight of S. Bloch that the ramification of a higher dimensional arithmetic scheme produces a 0-cycle class on its boundary. K. Kato and T. Saito calculated the Euler-Poincaré number of an ℓ -adic sheaf on any dimensional scheme in terms of the Swan class in [KS]. A. Abbes and T. Saito refined this result using the characteristic class of an ℓ -adic sheaf in [AS]. They compared the characteristic class with two invariants produced by the wild ramification. One is the Swan class mentioned above. We call the comparison of the characteristic class with the Swan class the Abbes-Saito formula. The other is the 0-cycle class $c_{\mathcal{F}}$ defined by Kato in [K1] for a smooth Λ -sheaf of rank one which is clean with respect to the boundary where Λ is a finite commutative $\mathbb{Z}_\ell$ -algebra. We call the

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comparison of the characteristic class with the Kato 0-cycle class the Abbes-Kato-Saito formula.

We prove a localized version of the Abbes-Saito formula in [T] assuming the strong resolution of singularities, which is a refinement of their formula. We call this formula the localized Abbes-Saito formula. We use the localized characteristic class of an ℓ -adic sheaf to formulate it. The localized characteristic class is a lifting of the characteristic class to the étale cohomology group supported on the boundary locus and is defined in [AS, Section 5]. As an application of the localized Abbes-Saito formula, we proved the Kato-Saito conductor formula in characteristic $p > 0$ in [T]. This was the main motivation to consider the localized characteristic class and the main application of the localized Abbes-Saito formula. At the conference, I reported these results.

In this paper, we study two methods of localization of the characteristic classes for sheaves of rank 1. To refine the Kato-Saito conductor formula in characteristic $p > 0$ is the main purpose to consider these localizations. For this purpose, first we define a localization of the characteristic class as a cohomology class with support on the wild locus in Section 2 using logarithmic blow-up, and we call it the logarithmic localized characteristic class.

Recently T. Saito introduced a notion of non-degeneration of a Λ -sheaf in [S]. This notion is a natural generalization of the notion of cleanness of a Λ -sheaf of rank 1 defined by K. Kato to higher rank. He calculated the characteristic class of a smooth Λ -sheaf which is non-degenerate with respect to the boundary.

In Section 3, we define a further localization of the characteristic class as a cohomology class with support on the nonclean locus for a smooth Λ -sheaf of rank 1 inspired by an idea of T. Saito in [S]. We call it the nonclean localized characteristic class of an ℓ -adic sheaf. Our main theorem (Theorem 3.8) in this paper is the comparison of the logarithmic localized characteristic class with the nonclean localized characteristic class. As a corollary (Corollary 3.10), we obtain an equality of the logarithmic localized characteristic class of a Λ -sheaf of rank 1 which is clean with respect to the boundary and the Kato 0-cycle class in the étale cohomology group supported on the wild locus without denominator. This equality refines the Abbes-Kato-Saito formula mentioned above.

I would like to thank Professor T. Saito for introducing this subject to me, suggesting that there exists a cohomology class with support on the wild locus which refines the characteristic class, encouragements and so many advices and thank the referee for many comments on an earlier version of this paper.

In this paper, k denotes a field. Schemes over k are assumed to be separated and of finite type. For a divisor with simple normal crossings of a smooth scheme over k , we assume that the irreducible components and their intersections are also smooth over k .

The letter l denotes a prime number invertible in k and Λ denotes a finite commutative $\mathbb{Z}_l$ -algebra. For a scheme X over k , Let $\mathcal{K}_X$ denote $Rf^!\Lambda$ where $f : X \rightarrow \mathrm{Spec} k$ is the structure map. If f is smooth, the canonical class map $\Lambda(d)[2d] \rightarrow \mathcal{K}_X$ is an isomorphism by [AS] (1.8). When we say a scheme X is of dimension d , we understand that every irreducible component of X is of dimension d .

§ 2. Logarithmic localized characteristic class of a smooth Λ -sheaf of rank 1

In [T], we define a localization of the characteristic class of a smooth Λ -sheaf of any rank as a cohomology class with support on the wild locus which we call the logarithmic localized characteristic class. In this section, we introduce a more elementary definition of the logarithmic localized characteristic class of a smooth Λ -sheaf of rank 1.

Let X be a smooth scheme of dimension d over k , $U \subset X$ an open subscheme, the complement $X \setminus U = \bigcup_{i \in I} D_i$ a divisor with simple normal crossings and $j : U \rightarrow X$ the open immersion. We recall the definitions of the logarithmic blow-up and the logarithmic product from [AS, Section 2.2] and [KS, Section 1.1]. For $i \in I$, let $(X \times X)'_i \rightarrow X \times X$ be the blow-up at $D_i \times D_i$ and let $(X \times X)'_i \subset (X \times X)'_i$ be the complement of the proper transforms of $D_i \times X$ and $X \times D_i$. We define the log blow-up with respect to divisors $\{D_i\}_{i \in I}$

$$(X \times X)' \rightarrow X \times X$$

to be the fiber product $\coprod_{i \in I} (X \times X)'_i \rightarrow X \times X$ of $(X \times X)'_i (i \in I)$ over $X \times X$. We define the log product with respect to divisors $\{D_i\}_{i \in I}$

$$(X \times X)^\sim \subset (X \times X)'$$

to be the fiber product $\coprod_{i \in I} (X \times X)^\sim_i \rightarrow X \times X$ of $(X \times X)^\sim_i (i \in I)$ over $X \times X$. The log blow-up $(X \times X)'$ is a smooth scheme over k of dimension $2d$.

Lemma 2.1. *Let the notation be as above. We consider the following cartesian diagram*

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{i'} & (X \times X)^\sim \\ \downarrow & & \downarrow \\ X & \xrightarrow{\delta} & X \times X \end{array}$$

where the right vertical arrow $(X \times X)^\sim \rightarrow X \times X$ is the projection and $\delta : X \rightarrow X \times X$ is the diagonal closed immersion. Then the inverse image $\tilde{X}$ is the union of $(\mathbb{G}_m)^{\#J}$ -bundles $\{D_J \times (\mathbb{G}_m)^{\#J}\}_{J \subset I}$ of dimension d where D_J is the intersection of $\{D_i\}_{i \in J}$ in X .

Proof. Let $\tilde{X}_i$ denote the inverse image of the diagonal X by the projection $(X \times X)_i^\sim \longrightarrow X \times X$ for $i \in I$. By the definition of the log product, $\tilde{X}_i$ is the union of the diagonal $X \subset (X \times X)_i^\sim$ and $\mathbb{G}_{mD_i}$. Since $(X \times X)^\sim$ is the fiber product of $(X \times X)_i^\sim (i \in I)$ over $X \times X$, $\tilde{X}$ is the fiber product of the schemes $\{\tilde{X}_i\}_{i \in I}$ ($i \in I$) over X . Therefore the inverse image $\tilde{X}$ is the union of $(\mathbb{G}_m)^{\sharp J}$ -bundles $\{D_J \times (\mathbb{G}_m)^{\sharp J}\}_{J \subset I}$ of dimension d where D_J is the intersection of $\{D_i\}_{i \in J}$ in X . Hence the assertion follows. $\square$

We keep the above notation. For a subset $I^+ \subset I$, we put $D^+ = \bigcup_{i \in I^+} D_i \subset D$. Let $V \subset X$ denote the complement of D^+ in X . Let U' be the complement of $D' := \bigcup_{i \in I'} D_i$ in X where $I' = I \setminus I^+$. We have $U = U' \cap V$. Let $j' : U \longrightarrow V$ and $j_{U'} : U' \longrightarrow X$ be the open immersions. Let $(X \times X)_{U'}^\sim \subset (X \times X)'_{U'}$ denote the log product and log blow-up with respect to $\{D_i\}_{i \in I'}$ respectively.

We consider the cartesian diagram

$$\begin{array}{ccccc} (X \times X)_{U'}^\sim & \xleftarrow{\tilde{j}_1} & (V \times X)^\sim & \xleftarrow{\tilde{k}_1} & (V \times V)^\sim \\ f \downarrow & & \downarrow f_1 & & \downarrow \tilde{f} \\ X \times X & \xleftarrow{j_1} & V \times X & \xleftarrow{k_1} & V \times V \end{array}$$

where the horizontal arrows are the open immersions and the vertical arrows are the projections. Let $\delta_U : U \longrightarrow U \times U$ be the diagonal closed immersion, and $\tilde{\delta} : X \longrightarrow (X \times X)_{U'}^\sim$ and $\tilde{\delta}_V : V \longrightarrow (V \times V)^\sim$ be the logarithmic diagonal closed immersions induced by the universality of blow-up. We consider the cartesian diagrams

$$\begin{array}{ccccc} \tilde{X}_{U'} & \xrightarrow{\tilde{i}} & (X \times X)_{U'}^\sim & \xleftarrow{\tilde{g}} & (X \times X)_{U'}^\sim \setminus \tilde{X}_{U'} \\ \downarrow & & \downarrow f & & \downarrow \\ X & \xrightarrow{\delta_X} & X \times X & \xleftarrow{\quad} & X \times X \setminus \delta_X(X) \end{array}$$

and

$$\begin{array}{ccccc} \tilde{V} & \xrightarrow{\tilde{i}_V} & (V \times V)^\sim & \xleftarrow{\tilde{g}_V} & (V \times V)^\sim \setminus \tilde{V} \\ \downarrow & & \downarrow \tilde{f} & & \downarrow \\ V & \xrightarrow{\delta_V} & V \times V & \xleftarrow{\quad} & V \times V \setminus \delta_V(V) \end{array}$$

where $\tilde{g} : (X \times X)_{U'}^\sim \setminus \tilde{X}_{U'} \longrightarrow (X \times X)_{U'}^\sim$ and $\tilde{g}_V : (V \times V)^\sim \setminus \tilde{V} \longrightarrow (V \times V)^\sim$ are the open immersions.

Let $\mathcal{F}$ be a smooth Λ -sheaf of rank 1 on U which is tamely ramified along $V \setminus U$. We put $\mathcal{H}_0 := \mathcal{H}om(\mathrm{pr}_2^* \mathcal{F}, \mathrm{pr}_1^* \mathcal{F})$ and $\tilde{\mathcal{H}} := R\mathcal{H}om(\mathrm{pr}_2^* j_{!} \mathcal{F}, R\mathrm{pr}_1^! j_{!} \mathcal{F})$ on $U \times U$ and $X \times X$ respectively. Further, we put $\tilde{\mathcal{H}}_V := R\mathcal{H}om(\mathrm{pr}_2^* j'_{!} \mathcal{F}, R\mathrm{pr}_1^! j'_{!} \mathcal{F})$ on $V \times V$ and

$\tilde{\mathcal{H}} := (\tilde{j}_* \mathcal{H}_0)(d)[2d]$ on $(V \times V)^\sim$ respectively. There exists a unique map

$$(2.1) \quad \tilde{f}^* \bar{\mathcal{H}}_V \longrightarrow \tilde{\mathcal{H}}$$

inducing the canonical isomorphism $R\mathcal{H}om(\mathrm{pr}_2^* \mathcal{F}, R\mathrm{pr}_1^! \mathcal{F}) \longrightarrow \mathcal{H}_0(d)[2d]$ on $U \times U$ defined in [S, Proposition 3.1.1.1]. We put $\tilde{\mathcal{H}}_{U'} := \tilde{j}_{1!} R\tilde{k}_{1*} \tilde{\mathcal{H}}$ on $(X \times X)^\sim_{U'}$. We define a map

$$(2.2) \quad f^* \bar{\mathcal{H}} \longrightarrow \tilde{\mathcal{H}}_{U'}$$

to be the composition of the following maps

$$f^* \bar{\mathcal{H}} \simeq f^* j_{1!} Rk_{1*} \bar{\mathcal{H}}_V \simeq \tilde{j}_{1!} f_1^* Rk_{1*} \bar{\mathcal{H}}_V \longrightarrow \tilde{j}_{1!} R\tilde{k}_{1*} \tilde{f}^* \bar{\mathcal{H}}_V \longrightarrow \tilde{j}_{1!} R\tilde{k}_{1*} \tilde{\mathcal{H}} = \tilde{\mathcal{H}}_{U'}$$

where the first isomorphism is induced by the Kunneth formula, the second and third maps are induced by the base change maps $f^* j_{1!} \longrightarrow \tilde{j}_{1!} f_1^*$ and $f_1^* Rk_{1*} \longrightarrow R\tilde{k}_{1*} \tilde{f}^*$ respectively, and the fourth map is induced by applying the functor $\tilde{j}_{1!} R\tilde{k}_{1*}$ to the map (2.1).

By the definition of $\tilde{\mathcal{H}}_{U'}$, we have an isomorphism $\tilde{\delta}^* \tilde{\mathcal{H}}_{U'} \simeq j_{V!} \tilde{\delta}_V^* (\tilde{j}_* \mathcal{H}_0)(d)[2d]$. The base change map $\tilde{\delta}_V^* \tilde{j}_* \mathcal{H}_0 \longrightarrow j'_{*} \delta_U^* \mathcal{H}_0 \simeq j'_* \mathcal{E}nd(\mathcal{F})$ and the trace map $\mathcal{E}nd(\mathcal{F}) \longrightarrow \Lambda_U$ induce a trace map

$$(2.3) \quad \widetilde{\mathrm{Tr}} : \tilde{\delta}^* \tilde{\mathcal{H}}_{U'} \simeq j_{V!} \tilde{\delta}_V^* (\tilde{j}_* \mathcal{H}_0)(d)[2d] \longrightarrow j_{V!} \Lambda_V(d)[2d] = j_{V!} \mathcal{K}_V \longrightarrow \mathcal{K}_X.$$

The map (2.2) induces the pull-back

$$(2.4) \quad f^* : H_X^0(X \times X, \bar{\mathcal{H}}) \longrightarrow H_{\tilde{X}_{U'}}^0((X \times X)^\sim_{U'}, \tilde{\mathcal{H}}_{U'}).$$

The canonical map $\Lambda \longrightarrow R\tilde{g}_* \Lambda$ induces a map

$$(2.5) \quad H_{\tilde{X}_{U'}}^0((X \times X)^\sim_{U'}, \tilde{\mathcal{H}}_{U'}) \longrightarrow H_{\tilde{X}_{U'}}^0((X \times X)^\sim_{U'}, \tilde{\mathcal{H}}_{U'} \otimes R\tilde{g}_* \Lambda).$$

Lemma 2.2. *The canonical map*

$$(2.6) \quad H_{\tilde{X}_{U'} \setminus \tilde{V}}^0((X \times X)^\sim_{U'}, \tilde{\mathcal{H}}_{U'} \otimes R\tilde{g}_* \Lambda) \longrightarrow H_{\tilde{X}_{U'}}^0((X \times X)^\sim_{U'}, \tilde{\mathcal{H}}_{U'} \otimes R\tilde{g}_* \Lambda)$$

is an isomorphism.

Proof. By the localization sequence, it suffices to prove $H_V^i((V \times V)^\sim, \tilde{\mathcal{H}} \otimes R\tilde{g}_{V*} \Lambda) = 0$ for all i . Since $\mathcal{F}$ is a smooth sheaf of rank 1 which is tamely ramified along $V \setminus U$, $\tilde{\mathcal{H}}$ is a smooth Λ -sheaf of rank 1 on $(V \times V)^\sim$ by [AS, Proposition 4.2.2.1]. Therefore the canonical map $\tilde{\mathcal{H}} \otimes R\tilde{g}_{V*} \Lambda \longrightarrow R\tilde{g}_{V*} \tilde{g}_V^* \tilde{\mathcal{H}}$ is an isomorphism by the projection formula. By this isomorphism, we obtain isomorphisms $H_V^i((V \times V)^\sim, \tilde{\mathcal{H}} \otimes R\tilde{g}_{V*} \Lambda) \simeq$

$H^i(\tilde{V}, Ri_V^1(\tilde{\mathcal{H}} \otimes R\tilde{g}_{V*}\Lambda)) \simeq H^i(\tilde{V}, Ri_V^1 R\tilde{g}_{V*}\tilde{g}_V^*\tilde{\mathcal{H}}) = 0$ for all i . Hence the assertion follows. $\square$

We consider the pull-back by $\tilde{\delta}$

$$(2.7) \quad \tilde{\delta}^* : H_{\tilde{X}_{U'} \setminus \tilde{V}}^0((X \times X)_{U'}, \tilde{\mathcal{H}}_{U'} \otimes R\tilde{g}_*\Lambda) \longrightarrow H_{D+}^0(X, \tilde{\delta}^*\tilde{\mathcal{H}}_{U'} \otimes \tilde{\delta}^*R\tilde{g}_*\Lambda).$$

The trace map (2.3) induces a map

$$(2.8) \quad \widetilde{\text{Tr}} : H_{D+}^0(X, \tilde{\delta}^*\tilde{\mathcal{H}}_{U'} \otimes \tilde{\delta}^*R\tilde{g}_*\Lambda) \longrightarrow H_{D+}^0(X, \mathcal{K}_X \otimes \tilde{\delta}^*R\tilde{g}_*\Lambda).$$

Lemma 2.3. *The canonical map $\Lambda \longrightarrow \tilde{\delta}^*R\tilde{g}_*\Lambda$ induces an isomorphism*

$$(2.9) \quad H_{D+}^0(X, \mathcal{K}_X) \simeq H_{D+}^0(X, \mathcal{K}_X \otimes \tilde{\delta}^*R\tilde{g}_*\Lambda).$$

Proof. We consider the distinguished triangle

$$Ri^1\Lambda|_X(d)[2d] \longrightarrow \mathcal{K}_X \longrightarrow \mathcal{K}_X \otimes \tilde{\delta}^*R\tilde{g}_*\Lambda \longrightarrow$$

by the isomorphism $\Lambda(d)[2d] \simeq \mathcal{K}_X$. By this triangle, it is sufficient to prove that $H_{D+}^j(X, Ri^1\Lambda|_X)$ is zero for $j < 2d + 2$. For a subset $J \subset I'$, let $\tilde{X}_J = D_J \times (\mathbb{G}_m)^{\#J}$ denote the $(\mathbb{G}_m)^{\#J}$ -bundles over D_J . By Lemma 2.1 and the purity theorem, we obtain isomorphisms $R\hat{j}^1\Lambda = 0$ for $j < 2d$, $R^{2d}\hat{i}^1\Lambda \simeq \bigoplus_{J \subset I'} \Lambda_{\tilde{X}_J}(-d)$ and $R^{2d+1}\hat{i}^1\Lambda = 0$. By these isomorphisms and $D_J = \tilde{\delta}(X) \cap \tilde{X}_J$, we acquire $H_{D+}^j(X, Ri^1\Lambda|_X) = 0$ for $j < 2d$, $H_{D+}^{2d+i}(X, Ri^1\Lambda|_X) \simeq \bigoplus_{J \subset I'} H_{D+ \cap D_J}^i(D_J, \Lambda_{D_J}(-d))$ for $i = 0, 1$. Hence the assertion follows from the purity theorem. $\square$

Definition 2.4. We call the image of the element $\text{id}_{j_!\mathcal{F}} \in \text{End}_X(j_!\mathcal{F}) \simeq H_X^0(X \times X, \tilde{\mathcal{H}})$ under the composite of the maps (2.4)-(2.9) *the logarithmic localized characteristic class of $j_!\mathcal{F}$* and we denote it by $C_{D+}^{\log, 0}(j_!\mathcal{F}) \in H_{D+}^0(X, \mathcal{K}_X)$.

§ 3. Nonclean localization of the characteristic class and comparison with the logarithmic localized characteristic class

In this section, we will define a further localization of the characteristic class as a cohomology class with support on the nonclean locus which we call the nonclean localized characteristic class.

Let the notation be as in the previous section. Further, we assume that k is a perfect field. Let $R = \sum_{i \in I} r_i D_i$ be the Swan divisor of $\mathcal{F}$ with respect to the boundary D defined by K. Kato in [K1] and [K2]. We assume that the support of the divisor R is D^+ . We regard $R \subset X$ as a closed subscheme of $(X \times X)'$ by the log diagonal map $X \longrightarrow (X \times X)'$ and let $\pi : (X \times X)^{[R]} \longrightarrow (X \times X)'$ denote the blow-up at

R . Let $\Delta_i \subset (X \times X)'$ be the exceptional divisor above $D_i \times D_i$ for each $i \in I$. Let $(X \times X)^{(R)}$ be the complement of the union of the proper transforms of Δ_i for $i \in I^+$ in $\pi^{-1}((X \times X)^\sim)$. We consider the cartesian diagram

$$\begin{array}{ccc} X^{(R)} & \xrightarrow{i^{(R)}} & (X \times X)^{(R)} \\ \downarrow & & \downarrow f^{(R)} \\ X & \xrightarrow{\delta_X} & X \times X \end{array}$$

where $f^{(R)} : (X \times X)^{(R)} \rightarrow X \times X$ is the projection. Let $j^{(R)} : U \times U \rightarrow (X \times X)^{(R)}$ be the open immersion and $\delta^{(R)} : X \rightarrow (X \times X)^{(R)}$ the extended diagonal closed immersion induced by the universality of blow-up. (See [AS, Section 4.2] and [S, Section 2.3].)

Let E^+ be the complement of $(V \times V)^\sim$ in $(X \times X)^{(R)}$ which is a vector bundle over D^+ . Let $T \subset D^+$ be the nonclean locus of $\mathcal{F}$ which is a closed subscheme in X , and W the complement of T in X . (See [K2] for the notion of cleanness.) Then, the sheaf $\mathcal{F}$ is clean with respect to the boundary $W \setminus U$. Let R_W denote the restriction of R to the open subscheme $W \subset X$. We have the open subschemes $U \subset V \subset W \subset X$. We consider the cartesian diagram

$$\begin{array}{ccc} (X \times X)^{(R)} & \xleftarrow{\tilde{j}^{(R)}} & (V \times V)^\sim \\ \downarrow h & & \downarrow \tilde{k}_1 \\ (X \times X)_{U'}^\sim & \xleftarrow{\tilde{j}_1} & (V \times X)^\sim \end{array}$$

where $h : (X \times X)^{(R)} \rightarrow (X \times X)_{U'}^\sim$ is the projection and all the arrows except h are the open immersions.

We put $\mathcal{H}^{(R)} := j_*^{(R)} \mathcal{H}_0(d)[2d]$ on $(X \times X)^{(R)}$. The map of functors $\tilde{j}_!^{(R)} \rightarrow \tilde{j}_*^{(R)}$ induces a map

$$(3.1) \quad h^* \tilde{\mathcal{H}}_{U'} = h^* \tilde{j}_1^* R \tilde{k}_{1*} \tilde{\mathcal{H}} \simeq \tilde{j}_!^{(R)} \tilde{\mathcal{H}} \rightarrow \tilde{j}_*^{(R)} \tilde{j}_* \mathcal{H}_0(d)[2d] \simeq \mathcal{H}^{(R)}.$$

By the maps (2.2) and (3.1), we obtain the following map

$$(3.2) \quad f^{(R)*} \bar{\mathcal{H}} \rightarrow \mathcal{H}^{(R)}$$

defined in [S, Corollary 3.1.2.2]. This map induces the pull-back

$$(3.3) \quad f^{(R)*} : H_X^0(X \times X, \bar{\mathcal{H}}) \rightarrow H_{X^{(R)}}^0((X \times X)^{(R)}, \mathcal{H}^{(R)}).$$

There exists a unique section $e \in \Gamma(X, \delta^{(R)*} j_*^{(R)} \mathcal{H}_0)$ lifting the identity $\text{id}_{\mathcal{F}} \in \text{End}_U(\mathcal{F}) \simeq \Gamma(U, \delta_U^* \mathcal{H}_0)$. (See [S, Definition 2.3.1].) We consider the natural cup pairing

$$(3.4) \quad \cup : \Gamma(X, \delta^{(R)*} j_*^{(R)} \mathcal{H}_0) \times H_X^{2d}((X \times X)^{(R)}, \Lambda(d)) \rightarrow H_X^0((X \times X)^{(R)}, \mathcal{H}^{(R)}).$$

We write $[X]$ for the image of the cycle class $[X] \in CH_d(X)$ under the cycle class map $CH_d(X) \rightarrow H_X^{2d}((X \times X)^{(R)}, \Lambda(d))$. By the pairing (3.4) and the pull-back (3.3), we obtain an element $f^{(R)*} \text{id}_{j_! \mathcal{F}} - e \cup [X] \in H_{X^{(R)}}^0((X \times X)^{(R)}, \mathcal{H}^{(R)})$.

We have the localization sequence

$$(3.5) \quad \begin{aligned} H_{W^{(R_W)}}^{-1}((W \times W)^{(R_W)}, \mathcal{H}_W^{(R_W)}) &\rightarrow H_{T^{(R)}}^0((X \times X)^{(R)}, \mathcal{H}^{(R)}) \rightarrow H_{X^{(R)}}^0((X \times X)^{(R)}, \mathcal{H}^{(R)}) \\ &\rightarrow H_{W^{(R_W)}}^0((W \times W)^{(R_W)}, \mathcal{H}_W^{(R_W)}) \rightarrow H_{T^{(R)}}^1((X \times X)^{(R)}, \mathcal{H}^{(R)}) \rightarrow \dots \end{aligned}$$

where $T^{(R)}$ is the complement of $W^{(R_W)}$ in $X^{(R)}$ and $\mathcal{H}_W^{(R_W)}$ is the restriction of $\mathcal{H}^{(R)}$ to the open subscheme $(W \times W)^{(R_W)} \subset (X \times X)^{(R)}$.

Lemma 3.1. *The canonical map $H_{T^{(R)}}^0((X \times X)^{(R)}, \mathcal{H}^{(R)}) \rightarrow H_{X^{(R)}}^0((X \times X)^{(R)}, \mathcal{H}^{(R)})$ is injective.*

Proof. It suffices to prove $H_{W^{(R_W)}}^{-1}((W \times W)^{(R_W)}, \mathcal{H}_W^{(R_W)}) = 0$ by the localization sequence (3.5). We consider the exact sequence

$$H_{W^{(R_W)} \setminus \tilde{V}}^{-1}((W \times W)^{(R_W)}, \mathcal{H}_W^{(R_W)}) \rightarrow H_{W^{(R)}}^{-1}((W \times W)^{(R_W)}, \mathcal{H}_W^{(R_W)}) \rightarrow H_{\tilde{V}}^{-1}((V \times V)^\sim, \tilde{\mathcal{H}}).$$

Since we have $H_{W^{(R_W)} \setminus \tilde{V}}^{-1}((W \times W)^{(R_W)}, \mathcal{H}_W^{(R_W)}) = 0$ by [S, the proof of Theorem 3.1.3], it suffices to prove the vanishing $H_{\tilde{V}}^{-1}((V \times V)^\sim, \tilde{\mathcal{H}}) = 0$ by the exact sequence. Since $\tilde{\mathcal{H}}$ is a smooth sheaf on $(V \times V)^\sim$ by [AS, Proposition 4.2.2.1], the canonical map $\tilde{i}_V^* \tilde{\mathcal{H}} \otimes R\tilde{i}_V^! \Lambda \rightarrow R\tilde{i}_V^! \tilde{\mathcal{H}}$ is an isomorphism by the projection formula. By this isomorphism, we acquire isomorphisms $H_{\tilde{V}}^{-1}((V \times V)^\sim, \tilde{\mathcal{H}}) \simeq H^{-1}(\tilde{V}, R\tilde{i}_V^! \tilde{\mathcal{H}}) \simeq H^{-1}(\tilde{V}, \tilde{i}_V^* \tilde{\mathcal{H}} \otimes R\tilde{i}_V^! \Lambda) \simeq H^{2d-1}(\tilde{V}, \tilde{i}_V^* \tilde{j}_* \tilde{\mathcal{H}}_0(d) \otimes R\tilde{i}_V^! \Lambda)$. Thereby it is sufficient to prove that $H^{2d-1}(\tilde{V}, \tilde{i}_V^* \tilde{j}_* \tilde{\mathcal{H}}_0(d) \otimes R\tilde{i}_V^! \Lambda) = 0$. By Lemma 2.1 and the purity theorem, the sheaf $R^j \tilde{i}_V^! \Lambda$ is zero for $j < 2d$. Hence the assertion follows. $\square$

Corollary 3.2. *Let the notation be as above. Then there exists a unique element $(f^{(R)*} \text{id}_{j_! \mathcal{F}})'$ in $H_{T^{(R)}}^0((X \times X)^{(R)}, \mathcal{H}^{(R)})$ which goes to the element $f^{(R)*} \text{id}_{j_! \mathcal{F}} - e \cup [X] \in H_{X^{(R)}}^0((X \times X)^{(R)}, \mathcal{H}^{(R)})$ by the canonical map $H_{T^{(R)}}^0((X \times X)^{(R)}, \mathcal{H}^{(R)}) \rightarrow H_{X^{(R)}}^0((X \times X)^{(R)}, \mathcal{H}^{(R)})$.*

Proof. Since $\mathcal{F}$ is clean with respect to the boundary $W \setminus U$, the element $f^{(R)*} \text{id}_{j_! \mathcal{F}} - e \cup [X]$ goes to zero under the restriction map $H_{X^{(R)}}^0((X \times X)^{(R)}, \mathcal{H}^{(R)}) \rightarrow H_{W^{(R_W)}}^0((W \times W)^{(R_W)}, \mathcal{H}_W^{(R_W)})$ by [S, Corollary 3.1.2 and Theorem 3.1.3]. Therefore the assertion follows from Lemma 3.1 and the sequence (3.5). $\square$

The base change map $\delta^{(R)*} j_*^{(R)} \rightarrow j_* \delta_U^*$ and the trace map $\mathcal{E}nd(\mathcal{F}) \rightarrow \Lambda_U$ induce a trace map

$$(3.6) \quad \text{Tr}^{(R)} : \delta^{(R)*} \mathcal{H}^{(R)} \rightarrow j_* \delta_U^* \mathcal{H}_0(d)[2d] \simeq j_* \mathcal{E}nd(\mathcal{F})(d)[2d] \rightarrow \Lambda_X(d)[2d] \simeq \mathcal{K}_X.$$

Definition 3.3. We consider the following map

$$(f^{(R)*} \text{id}_{j_! \mathcal{F}})' \in H_{T^{(R)}}^0((X \times X)^{(R)}, \mathcal{H}^{(R)}) \xrightarrow{\delta^{(R)*}} H_T^0(X, \delta^{(R)*} \mathcal{H}^{(R)}) \xrightarrow{\text{Tr}^{(R)}} H_T^0(X, \mathcal{K}_X)$$

induced by the pull-back by $\delta^{(R)}$ and the trace map (3.6). We call the image of the element $(f^{(R)*} \text{id}_{j_! \mathcal{F}})'$ in Corollary 3.2 by this composition *the nonclean localized characteristic class of $j_! \mathcal{F}$* and we denote it by $C_T^{\text{ncl},0}(j_! \mathcal{F})$.

We will compare $C_T^{\text{ncl},0}(j_! \mathcal{F})$ with $C_{D^+}^{\log,0}(j_! \mathcal{F})$ in $H_{D^+}^0(X, \mathcal{K}_X)$ defined in Definition 2.4. (Theorem 3.8.)

Lemma 3.4. *The canonical maps*

$$(3.7) \quad H_{E^+}^0((X \times X)^{(R)}, \mathcal{H}^{(R)} \otimes h^* R\tilde{g}_* \Lambda) \longrightarrow H_{X^{(R)}}^0((X \times X)^{(R)}, \mathcal{H}^{(R)} \otimes h^* R\tilde{g}_* \Lambda),$$

$$(3.8) \quad H_{D^+}^0((X \times X)^{(R)}, \mathcal{H}^{(R)} \otimes h^* R\tilde{g}_* \Lambda) \longrightarrow H_X^0((X \times X)^{(R)}, \mathcal{H}^{(R)} \otimes h^* R\tilde{g}_* \Lambda),$$

$$(3.9) \quad H_{E^+}^0((X \times X)^{(R)}, h^* R\tilde{g}_* \Lambda) \longrightarrow H_{X^{(R)}}^0((X \times X)^{(R)}, h^* R\tilde{g}_* \Lambda)$$

and

$$(3.10) \quad H_{D^+}^{2d}((X \times X)^{(R)}, h^* R\tilde{g}_* \Lambda(d)) \longrightarrow H_X^{2d}((X \times X)^{(R)}, h^* R\tilde{g}_* \Lambda(d))$$

are isomorphisms.

Proof. We prove the assertions in the same way as Lemma 2.2. □

We consider the composite

$$\begin{aligned} H_{X^{(R)}}^0((X \times X)^{(R)}, \mathcal{H}^{(R)}) &\longrightarrow H_{X^{(R)}}^0((X \times X)^{(R)}, \mathcal{H}^{(R)} \otimes h^* R\tilde{g}_* \Lambda) \\ &\simeq H_{E^+}^0((X \times X)^{(R)}, \mathcal{H}^{(R)} \otimes h^* R\tilde{g}_* \Lambda) \end{aligned}$$

where the first map is induced by the canonical map $\Lambda \longrightarrow h^* R\tilde{g}_* \Lambda$ and the second isomorphism is (3.7). Let $(f^{(R)*} \text{id}_{j_! \mathcal{F}})^{\log} \in H_{E^+}^0((X \times X)^{(R)}, \mathcal{H}^{(R)} \otimes h^* R\tilde{g}_* \Lambda)$ denote the image of the element $f^{(R)*} \text{id}_{j_! \mathcal{F}} \in H_{X^{(R)}}^0((X \times X)^{(R)}, \mathcal{H}^{(R)})$ by this map.

The pull-back by $\delta^{(R)}$, the trace map (3.6) and the isomorphism (2.9) induce a map $\text{Tr}^{(R)} \cdot \delta^{(R)*} : H_{E^+}^0((X \times X)^{(R)}, \mathcal{H}^{(R)} \otimes h^* R\tilde{g}_* \Lambda) \longrightarrow H_{D^+}^0(X, \mathcal{K}_X \otimes \tilde{\delta}^* R\tilde{g}_* \Lambda) \simeq H_{D^+}^0(X, \mathcal{K}_X)$.

We write $\text{Tr}^{(R)} \delta^{(R)*} (f^{(R)*} \text{id}_{j_! \mathcal{F}})^{\log} \in H_{D^+}^0(X, \mathcal{K}_X)$ for the image of the element

$$(f^{(R)*} \text{id}_{j_! \mathcal{F}})^{\log} \in H_{E^+}^0((X \times X)^{(R)}, \mathcal{H}^{(R)} \otimes h^* R\tilde{g}_* \Lambda)$$

under the map $\text{Tr}^{(R)} \cdot \delta^{(R)*}$.

Lemma 3.5. *Let the notation be as above. We have an equality*

$$C_{D^+}^{\log, 0}(j_! \mathcal{F}) = \mathrm{Tr}^{(R)} \delta^{(R)*} (f^{(R)*} \mathrm{id}_{j_! \mathcal{F}})^{\log}$$

in $H_{D^+}^0(X, \mathcal{K}_X)$.

Proof. We consider the commutative diagram

$$\begin{array}{ccc} f^{(R)*} \mathrm{id}_{j_! \mathcal{F}} \in H_{X^{(R)}}^0((X \times X)^{(R)}, \mathcal{H}^{(R)}) & \longrightarrow & H_{X^{(R)}}^0((X \times X)^{(R)}, \mathcal{H}^{(R)} \otimes h^* R\tilde{g}_* \Lambda) \\ \uparrow h^* & & \uparrow h^* \\ f^* \mathrm{id}_{j_! \mathcal{F}} \in H_{\tilde{X}_{U'}}^0((X \times X)_{\tilde{U}'}, \tilde{\mathcal{H}}_{U'}) & \longrightarrow & H_{\tilde{X}_{U'}}^0((X \times X)_{\tilde{U}'}, \tilde{\mathcal{H}}_{U'} \otimes R\tilde{g}_* \Lambda) \\ \downarrow h^* & & \downarrow \mathrm{id} \\ \xleftarrow[\simeq]{(3.7)} H_{X^{(R)} \setminus \tilde{V}}^0((X \times X)^{(R)}, \mathcal{H}^{(R)} \otimes h^* R\tilde{g}_* \Lambda) & \xrightarrow{\mathrm{Tr}^{(R)} \cdot \delta^{(R)*}} & H_{D^+}^0(X, \mathcal{K}_X \otimes \tilde{\delta}^* R\tilde{g}_* \Lambda) \simeq H_{D^+}^0(X, \mathcal{K}_X) \\ \uparrow h^* & & \uparrow \mathrm{id} \\ \xleftarrow[\simeq]{(2.6)} H_{\tilde{X}_{U'} \setminus \tilde{V}}^0((X \times X)_{\tilde{U}'}, \tilde{\mathcal{H}}_{U'} \otimes R\tilde{g}_* \Lambda) & \xrightarrow{\tilde{\mathrm{Tr}} \cdot \tilde{\delta}^*} & H_{D^+}^0(X, \mathcal{K}_X \otimes \tilde{\delta}^* R\tilde{g}_* \Lambda) \simeq H_{D^+}^0(X, \mathcal{K}_X). \end{array}$$

The right(resp. left) hand side of the equality is the image of $f^{(R)*} \mathrm{id}_{j_! \mathcal{F}} \in H_{X^{(R)}}^0((X \times X)^{(R)}, \mathcal{H}^{(R)})$ (resp. $f^* \mathrm{id}_{j_! \mathcal{F}} \in H_{\tilde{X}_{U'}}^0((X \times X)_{\tilde{U}'}, \tilde{\mathcal{H}}_{U'})$) by the composite of the maps in the upper(resp. lower) lines in the commutative diagrams. Therefore the assertion follows from the commutative diagram. $\square$

We study the relationship between the elements $(f^{(R)*} \mathrm{id}_{j_! \mathcal{F}})'$ and $(f^{(R)*} \mathrm{id}_{j_! \mathcal{F}})^{\log}$ to compare the two localizations.

Lemma 3.6. *We consider the composition*

$$H_X^0((X \times X)^{(R)}, \mathcal{H}^{(R)}) \longrightarrow H_X^0((X \times X)^{(R)}, \mathcal{H}^{(R)} \otimes h^* R\tilde{g}_* \Lambda) \simeq H_{D^+}^0((X \times X)^{(R)}, \mathcal{H}^{(R)} \otimes h^* R\tilde{g}_* \Lambda)$$

where the first map is induced by the canonical map $\Lambda \longrightarrow h^* R\tilde{g}_* \Lambda$ and the second isomorphism is (3.8). Let $(e \cup [X])^{\log} \in H_{D^+}^0((X \times X)^{(R)}, \mathcal{H}^{(R)} \otimes h^* R\tilde{g}_* \Lambda)$ be the image of $e \cup [X] \in H_X^0((X \times X)^{(R)}, \mathcal{H}^{(R)})$ by this composition. Then, we have an equality

$$(f^{(R)*} \mathrm{id}_{j_! \mathcal{F}})' = (f^{(R)*} \mathrm{id}_{j_! \mathcal{F}})^{\log} - (e \cup [X])^{\log}$$

in $H_{E^+}^0((X \times X)^{(R)}, \mathcal{H}^{(R)} \otimes h^* R\tilde{g}_* \Lambda)$.

Proof. Since we have $T^{(R)} \subset E^+$, we obtain the following commutative diagram

$$\begin{array}{ccc} (f^{(R)*} \mathrm{id}_{j_! \mathcal{F}})' \in H_{T^{(R)}}^0((X \times X)^{(R)}, \mathcal{H}^{(R)}) & \longrightarrow & H_{E^+}^0((X \times X)^{(R)}, \mathcal{H}^{(R)} \otimes h^* R\tilde{g}_* \Lambda) \\ \downarrow \text{can.} & & \downarrow \text{(3.7)} \\ f^{(R)*} \mathrm{id}_{j_! \mathcal{F}} - e \cup [X] \in H_{X^{(R)}}^0((X \times X)^{(R)}, \mathcal{H}^{(R)}) & \xrightarrow{\log} & H_{X^{(R)}}^0((X \times X)^{(R)}, \mathcal{H}^{(R)} \otimes h^* R\tilde{g}_* \Lambda) \end{array}$$

where the horizontal arrows are induced by the canonical map $\Lambda \longrightarrow h^* R\tilde{g}_* \Lambda$. By this commutative diagram, the isomorphism (3.7) and Corollary 3.2, the assertion is proved. $\square$

By Lemmas 3.5 and 3.6, and Definition 3.3, it suffices to calculate the element $(e \cup [X])^{\log} \in H_{E^+}^0((X \times X)^{(R)}, \mathcal{H}^{(R)} \otimes h^* R\tilde{g}_* \Lambda)$ to prove our main theorem (Theorem 3.8). For the calculation, we will write it in terms of a cycle class with support on the vector bundle E^+ in the following Lemma 3.7.

We consider the composite

$$(3.11) \quad H_X^{2d}((X \times X)^{(R)}, \Lambda(d)) \longrightarrow H_X^{2d}((X \times X)^{(R)}, h^* R\tilde{g}_* \Lambda(d)) \simeq H_{D^+}^{2d}((X \times X)^{(R)}, h^* R\tilde{g}_* \Lambda(d))$$

where the first map is induced by the canonical map $\Lambda \longrightarrow h^* R\tilde{g}_* \Lambda$ and the second isomorphism is (3.10). Let $[X]^{\log} \in H_{D^+}^{2d}((X \times X)^{(R)}, h^* R\tilde{g}_* \Lambda(d))$ denote the image of the element $[X] \in H_X^{2d}((X \times X)^{(R)}, \Lambda(d))$ under the map (3.11). By the canonical cup pairing

$$(3.12) \quad \cup : \Gamma(X, \delta^{(R)*} j_*^{(R)} \mathcal{H}_0) \times H_X^{2d}((X \times X)^{(R)}, h^* R\tilde{g}_* \Lambda(d)) \longrightarrow H_X^0((X \times X)^{(R)}, \mathcal{H}^{(R)} \otimes h^* R\tilde{g}_* \Lambda)$$

and the isomorphisms (3.8), (3.10), we obtain an element $e \cup [X]^{\log}$ in $H_{D^+}^0((X \times X)^{(R)}, \mathcal{H}^{(R)} \otimes h^* R\tilde{g}_* \Lambda)$. By the definitions of the two pairings (3.4) and (3.12), we obtain an equality

$$(3.13) \quad (e \cup [X])^{\log} = e \cup [X]^{\log}$$

in $H_{D^+}^0((X \times X)^{(R)}, \mathcal{H}^{(R)} \otimes h^* R\tilde{g}_* \Lambda)$.

Lemma 3.7. *Let the notation be as above.*

1. *We consider the cartesian diagram*

$$\begin{array}{ccccc} h^{-1}(X) & \longrightarrow & (X \times X)^{(R)} & \longleftarrow & (V \times V)^\sim \\ \downarrow & & \downarrow h & & \downarrow \text{id} \\ X & \xrightarrow{\tilde{\delta}} & (X \times X)^\sim_{U'} & \longleftarrow & (V \times V)^\sim. \end{array}$$

Let $h^! : CH_d(X) \longrightarrow CH_d(h^{-1}(X))$ be the Gysin map. Then there exists a unique element $[X - h^! X] \in CH_d(E^+)$ which goes to $[X] - h^![X] \in CH_d(h^{-1}(X))$ under the canonical map $CH_d(E^+) \longrightarrow CH_d(h^{-1}(X))$.

2. *Further, this element satisfies an equality*

$$\delta^{(R)!} [X - h^! X] = (-1)^d \cdot c_d(\Omega_{X/k}^1(\log D)(R) - \Omega_{X/k}^1(\log D'))_{D^+}^X \cap [X]$$

in $CH_0(D^+)$ where $c_d(\cdot)_{D^+}^X \cap [X]$ is the localized Chern class introduced in [KS, Section 3.4].

3. We also write $\text{cl}([X - h^!X])$ for the image of the cycle class $\text{cl}([X - h^!X])$ by the canonical map $H_{E^+}^{2d}((X \times X)^{(R)}, \Lambda(d)) \longrightarrow H_{E^+}^{2d}((X \times X)^{(R)}, h^*R\tilde{g}_*\Lambda(d))$ where $\text{cl} : CH_d(E^+) \longrightarrow H_{E^+}^{2d}((X \times X)^{(R)}, \Lambda(d))$ is the cycle class map. Then we have an equality

$$\text{cl}([X - h^!X]) = [X]^{\log}$$

in $H_{E^+}^{2d}((X \times X)^{(R)}, h^*R\tilde{g}_*\Lambda(d))$.

Proof. First we prove 1,2. We consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\delta^{(R)}} & (X \times X)^{(R)} \\ \downarrow id & & \downarrow h \\ X & \xrightarrow{\tilde{\delta}} & (X \times X)_{U'} \end{array}$$

where the horizontal arrows are the diagonal closed immersions and $h : (X \times X)^{(R)} \longrightarrow (X \times X)_{U'}$ is the projection. The conormal sheaves $N_{X/(X \times X)^{(R)}}$ and $N_{X/(X \times X)_{U'}}$ are naturally identified with $\Omega_{X/k}^1(\log D)(R)$ and $\Omega_{X/k}^1(\log D')$ respectively, and we have $h^{-1}(X) \setminus V = E^+$. Therefore it is sufficient to apply Lemma 3.4.9 in [KS] to the diagram by taking $(V \times V) \subset (X \times X)^{(R)}$ as the open subscheme $U \subset Y$ in Lemma 3.4.9 in [KS].

We prove 3. By the following commutative diagram

$$\begin{array}{ccc} h^*[X] \in H_{X^{(R)}}^{2d}((X \times X)^{(R)}, \Lambda(d)) & \xrightarrow{\log} & H_{X^{(R)}}^{2d}((X \times X)^{(R)}, h^*R\tilde{g}_*\Lambda(d)) \ni (h^*[X])^{\log} \\ \uparrow h^* & & \uparrow h^* \\ [X] \in H_X^{2d}((X \times X)_{U'}, \Lambda(d)) & \longrightarrow & H_X^{2d}((X \times X)_{U'}, R\tilde{g}_*\Lambda(d)) = 0, \end{array}$$

we acquire $(h^*[X])^{\log} = 0$. We consider the commutative diagram

$$\begin{array}{ccc} \text{cl}([X - h^!X]) \in H_{E^+}^{2d}((X \times X)^{(R)}, \Lambda(d)) & \longrightarrow & H_{E^+}^{2d}((X \times X)^{(R)}, h^*R\tilde{g}_*\Lambda(d)) \\ \downarrow (1) & & \simeq \downarrow (3.9) \\ [X] - h^*[X] \in H_{X^{(R)}}^{2d}((X \times X)^{(R)}, \Lambda(d)) & \xrightarrow{\log} & H_{X^{(R)}}^{2d}((X \times X)^{(R)}, h^*R\tilde{g}_*\Lambda(d)). \end{array}$$

The element $\text{cl}([X - h^!X]) \in H_{E^+}^{2d}((X \times X)^{(R)}, \Lambda(d))$ goes to $[X] - h^*[X] \in H_{X^{(R)}}^{2d}((X \times X)^{(R)}, \Lambda(d))$ under the map (1) by Lemma 3.7. Hence the assertion follows from this commutative diagram and $(h^*[X])^{\log} = 0$. $\square$

We are ready to prove the main result in this paper.

Theorem 3.8. *Let X be a smooth scheme of dimension d over a perfect field k and $U \subset X$ be the complement of a divisor with simple normal crossings. Let $\mathcal{F}$ be*

a smooth sheaf of rank 1, and T and D^+ the nonclean locus and the wild locus of $\mathcal{F}$ respectively. Then we have an equality

$$C_T^{\text{ncl},0}(j_!\mathcal{F}) + (-1)^d \cdot c_d(\Omega_{X/k}^1(\log D)(R) - \Omega_{X/k}^1(\log D'))_{D^+}^X \cap [X] = C_{D^+}^{\log,0}(j_!\mathcal{F})$$

in $H_{D^+}^0(X, \mathcal{K}_X)$.

Proof. By Lemmas 3.5 and 3.6, and the commutative diagram

$$\begin{array}{ccc} (f^{(R)*} \text{id}_{j_!\mathcal{F}})' \in H_{T^{(R)}}^0((X \times X)^{(R)}, \mathcal{H}^{(R)}) & \longrightarrow & H_{E^+}^0((X \times X)^{(R)}, \mathcal{H}^{(R)} \otimes h^* R\tilde{g}_* \Lambda) \\ \downarrow \text{Tr}^{(R)} \cdot \delta^{(R)*} & & \downarrow \text{Tr}^{(R)} \cdot \delta^{(R)*} \\ C_T^{\text{ncl},0}(j_!\mathcal{F}) \in H_T^0(X, \mathcal{K}_X) & \xrightarrow{\text{can.}} & H_{D^+}^0(X, \mathcal{K}_X) \simeq H_{D^+}^0(X, \mathcal{K}_X \otimes \tilde{\delta}^* R\tilde{g}_* \Lambda), \end{array}$$

we obtain an equality

$$C_T^{\text{ncl},0}(j_!\mathcal{F}) = C_{D^+}^{\log,0}(j_!\mathcal{F}) - \text{Tr}^{(R)} \cdot \delta^{(R)*}((e \cup [X])^{\log})$$

in $H_{D^+}^0(X, \mathcal{K}_X)$. Since we have

$$\text{Tr}^{(R)} \delta^{(R)*}(e \cup [X]^{\log}) = \text{Tr}^{(R)}(e) \cup \delta^{(R)*}[X]^{\log} = \delta^{(R)*}[X]^{\log}$$

in $H_{D^+}^0(X, \mathcal{K}_X)$, we acquire equalities

$$\begin{aligned} \text{Tr}^{(R)} \cdot \delta^{(R)*}((e \cup [X])^{\log}) &= \text{Tr}^{(R)} \cdot \delta^{(R)*}(e \cup [X]^{\log}) = \delta^{(R)*} \text{cl}([X - h^! X]) \\ &= (-1)^d \cdot c_d(\Omega_{X/k}^1(\log D)(R) - \Omega_{X/k}^1(\log D'))_{D^+}^X \cap [X] \end{aligned}$$

by the equality (3.13), and Lemma 3.7. Therefore the assertion is proved. $\square$

We show that the logarithmic localized characteristic class refines the characteristic class. The localized Chern class $(-1)^d \cdot c_d(\Omega_{X/k}^1(\log D)(R) - \Omega_{X/k}^1(\log D'))_{D^+}^X \cap [X] \in H_{D^+}^0(X, \mathcal{K}_X)$ goes to the difference $(-1)^d \cdot c_d(\Omega_{X/k}^1(\log D)(R)) \cap [X] - (-1)^d \cdot c_d(\Omega_{X/k}^1(\log D')) \cap [X] \in H^0(X, \mathcal{K}_X)$ by the canonical map $H_{D^+}^0(X, \mathcal{K}_X) \longrightarrow H^0(X, \mathcal{K}_X)$. (See [KS, Section 3.4].)

Let $(\Delta_X, \Delta_X) \in H^0(X, \mathcal{K}_X)$ denote the self-intersection product. Since the conormal sheaf $N_{X/(X \times X)^{(R)}}$ (resp. $N_{X/(X \times X)_{U'}}$) is isomorphic to $\Omega_{X/k}^1(\log D)(R)$, (resp. $\Omega_{X/k}^1(\log D')$), we have an equality $(-1)^d \cdot c_d(\Omega_{X/k}^1(\log D)(R)) \cap [X] = (\Delta_X, \Delta_X)_{(X \times X)^{(R)}}$. (resp. $(-1)^d \cdot c_d(\Omega_{X/k}^1(\log D')) \cap [X] = (\Delta_X, \Delta_X)_{(X \times X)_{U'}}$.)

Corollary 3.9. *Let the notation be as in Theorem 3.8. Let $C(\cdot)$ denote the characteristic class defined in [AS, Definition 2.1.1]. Then, the image of the logarithmic localized characteristic class $C_{D^+}^{\log,0}(j_!\mathcal{F}) \in H_{D^+}^0(X, \mathcal{K}_X)$ under the canonical map $H_{D^+}^0(X, \mathcal{K}_X) \longrightarrow H^0(X, \mathcal{K}_X)$ is the difference $C(j_!\mathcal{F}) - C(j_{U'} \mathcal{F})$.*

Proof. By Definition 3.3, the nonclean localized characteristic class $C_T^{\text{ncl},0}(j_!\mathcal{F})$ goes to the difference $C(j_!\mathcal{F}) - (\Delta_X, \Delta_X)_{(X \times X)^{(R)}} \in H^0(X, \mathcal{K}_X)$ by the canonical map $H_T^0(X, \mathcal{K}_X) \longrightarrow H^0(X, \mathcal{K}_X)$. The localized Chern class $(-1)^d \cdot c_d(\Omega_{X/k}^1(\log D)(R) - \Omega_{X/k}^1(\log D'))_{D^+}^X \cap [X]$ goes to the difference

$$(\Delta_X, \Delta_X)_{(X \times X)^{(R)}} - (\Delta_X, \Delta_X)_{(X \times X)_{\widetilde{U}'}} ,$$

under the canonical map $H_{D^+}^0(X, \mathcal{K}_X) \longrightarrow H^0(X, \mathcal{K}_X)$. We have an equality

$$C(j_{U'!}\Lambda_{U'}) = (\Delta_X, \Delta_X)_{(X \times X)_{\widetilde{U}'}} ,$$

in $H^0(X, \mathcal{K}_X)$ by [AS, Corollary 2.2.5.1]. Hence the assertion follows from Theorem 3.8. $\square$

Let the notation be as above. We assume that $\mathcal{F}$ is a smooth Λ -sheaf of rank 1 which is clean with respect to the boundary D . Let $c_{\mathcal{F}} \in CH_0(D^+)$ be the Kato 0-cycle class of $\mathcal{F}$. (See [K2] or [AS, Definition 4.2.1].) We have an equality $c_{\mathcal{F}} = -(-1)^d c_d(\Omega_{X/k}^1(\log D)(R) - \Omega_{X/k}^1(\log D'))_{D^+}^X \cap [X]$. (See loc. cit.)

Corollary 3.10. *Let X be a smooth scheme of dimension d over a perfect field k and $U \subset X$ be the complement of a divisor with simple normal crossings. Let $\mathcal{F}$ be a smooth sheaf of rank 1 and D^+ the wild locus of $\mathcal{F}$. If $\mathcal{F}$ is clean with respect to the boundary, we have*

$$-c_{\mathcal{F}} = C_{D^+}^{\log,0}(j_!\mathcal{F}) - C_{D^+}^{\log,0}(j_!\Lambda_U)$$

in $H_{D^+}^0(X, \mathcal{K}_X)$.

Proof. Since we assume that $\mathcal{F}$ is clean with respect to the boundary D , we have $C_T^{\text{ncl},0}(j_!\mathcal{F}) = 0$. Thereby the equalities $(-1)^d c_d(\Omega_{X/k}^1(\log D)(R) - \Omega_{X/k}^1(\log D'))_{D^+}^X \cap [X] = C_{D^+}^{\log,0}(j_!\mathcal{F})$ and $(-1)^d c_d(\Omega_{X/k}^1(\log D) - \Omega_{X/k}^1(\log D'))_{D^+}^X \cap [X] = C_{D^+}^{\log,0}(j_!\Lambda_U)$ hold in $H_{D^+}^0(X, \mathcal{K}_X)$ by Theorem 3.8. Hence we obtain

$$C_{D^+}^{\log,0}(j_!\mathcal{F}) - C_{D^+}^{\log,0}(j_!\Lambda_U) = (-1)^d c_d(\Omega_{X/k}^1(\log D)(R) - \Omega_{X/k}^1(\log D'))_{D^+}^X \cap [X] = -c_{\mathcal{F}}.$$

$\square$

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