

Hilbert irreducibility sequences and nonstandard arithmetic

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Let $\mathbb{Q}^*$ and $\mathbb{Z}^*$ be enlargements of $\mathbb{Q}$ and $\mathbb{Z}$ respectively. Our aim of this paper is to give a sufficient condition for $x \in \mathbb{Z}^* - \mathbb{Z}$ that $\mathbb{Q}(x)$ has no algebraic extension of degree not more than m within $\mathbb{Q}^*$. As its application to number theory, we give irreducibility sequences explicitly.

By an arithmetical prime divisor, we mean a prime number or the archimedean prime p_∞ . For each arithmetical prime p , we define p -adic absolute value of a rational number x ,

$$|x|_p = p^{-n}$$

$$|x|_{p_\infty} = |x|$$

where $x = rp^n$ and r has no p factor. For each finite set S , of arithmetical primes, we define

$$H_S(x) = \prod_{p \in S} \max(1, |x|_p)$$

$$H(x) = \prod_p \max(1, |x|_p) = \max(|m|, |n|)$$

where $x = m/n$ and $\text{g.c.d.}(m, n) = 1$.

THEOREM. Let x be a nonstandard rational number. Assume
 (1) there is a finite set S of standard prime divisors such that

$$\frac{\log(H_S(x)H_S(x^{-1}))}{\log H(x)} > 2 - \frac{1}{m} + \varepsilon$$

for some standard positive real ε ,

(2) for any nonzero standard rational number r and any natural number n with $2 \leq n \leq m$, there is no nonstandard rational $y \in \mathbb{Q}^* - \mathbb{Q}$ such that $rx = y^n$.

Then $\mathbb{Q}(x)$ has no algebraic extension of degree not more than m within $\mathbb{Q}^*$.

Let us give an application of the theorem to standard number theory. A sequence of integers $a_1, a_2, \dots, a_n, \dots$ is called a m -irreducibility sequence if for any polynomial $f(X, Y) \in \mathbb{Z}[X, Y]$ with $\deg_X(f) \leq m$, there are only finitely many a_n such that $f(X, a_n)$ is reducible. A sequence of integers is called a Hilbert irreducibility sequence (H.i.seq.) if it is a m -irreducibility sequence for all natural number m . In his papers [3] and [4], V.G.Sprindzuk proved that

$$a_n = [\exp \sqrt{\log \log n}] + n! 2^{n^2}$$

is a H.i.seq.. Our theorem can give a different type of H.i. seq. from those given by Sprindzuk. For example, we will show that $2^n p_n$, $2^n(n^3+1)$ and $n! 2^{n^2}$ are H.i.seq.s.

In nonstandard arithmetic, we have a beautiful characterization of a H.i.seq. due to Gilmor and Robinson.

PROPOSITION 1. a_n is a H.i.seq. if and only if for any nonstandard natural number $\omega \in \mathbb{N}^* - \mathbb{N}$, $\mathbb{Q}(a_\omega)$ is relatively

algebraically closed in $\mathbb{Q}^$.*

As for m -irreducibility we have the following sufficient condition for a sequence to be an m -irreducibility sequence

PROPOSITION 2. *If for any nonstandard natural number ω , $\mathbb{Q}(a_\omega)$ has no proper algebraic extension of degree not more than $m!$ within $\mathbb{Q}^*$, then a_n is an m -irreducibility sequence.*

Unfortunately the converse of Proposition 2 is not true but if $m!$ is replaced by m , then its converse holds.

PROPOSITION 3. *If a_n is an m -irreducibility sequence, then for any nonstandard natural number ω , $\mathbb{Q}(a_\omega)$ has no algebraic extension of degree not more than m within $\mathbb{Q}^*$.*

It is easily shown that Proposition 1 is a easy consequence of Proposition 2 and 3.

For the proofs of Theorem, Proposition 2 and 3, please refer to [5].

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3. V.G.Sprindzuk, Diophantine equations with unknown prime numbers, Trudy MIAN SSSR 158 (1981), 180-196.
4. V.G.Sprindzuk, Arithmetic specialization in polynomials, J. reine und angew. Math. 340 (1983) 26-52.
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