

Equivariant definable homotopy extensions

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Abstract

Let G be a definably compact definable group, X a definable G set and Y a definable closed G subset of X . We prove that a pair (X, Y) admits an equivariant definable homotopy extension.

1 Introduction

In this paper we consider equivariant definable homotopy extensions in an o-minimal expansion $\mathcal{N} = (R, +, \cdot, <, \dots)$ of a real closed field R . It is known that there exist uncountably many o-minimal expansions of the field $\mathbb{R}$ of real numbers([8]).

Definable set and definable maps are studied in [3], [4], and see also [9]. Everything is considered in $\mathcal{N} = (R, +, \cdot, <, \dots)$ and definable maps are assumed to be continuous unless otherwise stated.

2 Preliminaries

Let R be a real closed field.

A *structure* $\mathcal{N}$ is given by the following data.

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1. A set R is called the *universe* or *underlying set* of $\mathcal{N}$.
2. A collection of *functions* $\{f_i | i \in I\}$, where $f_i : R^{n_i} \rightarrow R$ for some $n_i \geq 1$.
3. A collection of *relations* $\{R_j | j \in J\}$, where $R_j \subset R^{m_j}$ for some $m_j \geq 1$.
4. A collection of distinguished elements $\{c_k | k \in K\} \subset R$, and each c_k is called a *constant*.

Any (or all) of the sets I, J, K may be empty.

We say that f (resp. L) is m -place function (resp. m -place relation) if $f : R^m \rightarrow R$ (resp. $L \subset R^m$).

A *term* is a finite string of symbols obtained by repeated applications of the following three rules:

1. Constants are terms.
2. Variables are terms.
3. If f is an m -place function of $\mathcal{N}$ and $t_1, \dots, t_m$ are terms, then the concatenated string $f(t_1, \dots, t_m)$ is a term.

A *formula* is a finite string of symbols $s_1 \dots s_k$, where each s_i is either a variable, a function, a relation, one of the logical symbols $=, \neg, \vee, \wedge, \exists, \forall$, one of the brackets $(,)$, or comma $,$. Arbitrary formulas are generated inductively by the following three rules:

1. For any two terms t_1 and t_2 , $t_1 = t_2$ and $t_1 < t_2$ are formulas.
2. If R is an m -place relation and $t_1, \dots, t_m$ are terms, then $R(t_1, \dots, t_m)$ is a formula.
3. If ϕ and ψ are formulas, then the negation $\neg\phi$, the disjunction $\phi \vee \psi$, and the conjunction $\phi \wedge \psi$ are formulas. If ϕ is a formula and v is a variable, then $(\exists v)\phi$ and $(\forall v)\phi$ are formulas.

A subset X of R^n is *definable* (in $\mathcal{N}$) if it is defined by a formula (with parameters). Namely, there exist a formula $\phi(x_1, \dots, x_n, y_1, \dots, y_m)$ and elements $b_1, \dots, b_m \in R$ such that $X = \{(a_1, \dots, a_n) \in R^n | \phi(a_1, \dots, a_n, b_1, \dots, b_m) \text{ is true in } \mathcal{N}\}$.

For any $-\infty \leq a < b \leq \infty$, an open interval $(a, b)_R$ means $\{x \in R \mid a < x < b\}$, for any $a, b \in R$ with $a < b$, a closed interval $[a, b]_R$ means $\{x \in R \mid a \leq x \leq b\}$. We call $\mathcal{N}$ *o-minimal* (*order-minimal*) if every definable subset of R is a finite union of points and open intervals.

A real closed field $(R, +, \cdot, <)$ is an o-minimal structure and every definable set is a semialgebraic set [10], and a definable map is a semialgebraic map [10]. In particular, the semialgebraic category is a special case of the definable one.

The topology of R is the interval topology and the topology of R^n is the product topology. Note that R^n is a Hausdorff space.

The field $\mathbb{R}$ of real numbers, $\mathbb{R}_{alg} = \{x \in \mathbb{R} \mid x \text{ is algebraic over } \mathbb{Q}\}$ are Archimedean real closed fields.

The Puiseux series $\mathbb{R}[X]^\wedge$, namely $\sum_{i=k}^\infty a_i X^{\frac{i}{q}}$, $k \in \mathbb{Z}, q \in \mathbb{N}, a_i \in \mathbb{R}$ is a non-Archimedean real closed field.

Fact 2.1. (1) *The characteristic of a real closed field is 0.*

(2) *For any cardinality $\kappa \geq \aleph_0$, there exist 2^κ many non-isomorphic real closed fields whose cardinality are κ .*

(3) *In a general real closed field, even for a C^∞ function, the intermediate value theorem, existence theorem of maximum and minimum, Rolle's theorem, the mean value theorem do not hold. Even for a C^∞ function f in one variable, the result that $f' > 0$ implies f is increasing does not hold.*

Definition 2.2. Let $X \subset R^n, Y \subset R^m$ be definable sets.

(1) A continuous map $f : X \rightarrow Y$ is a *definable map* if the graph of f ($\subset R^n \times R^m$) is definable.

(2) A definable map $f : X \rightarrow Y$ is a *definable homeomorphism* if there exists a definable map $f' : Y \rightarrow X$ such that $f \circ f' = id_Y, f' \circ f = id_X$.

Definition 2.3. A group G is a *definable group* if G is definable and the group operations $G \times G \rightarrow G, G \rightarrow G$ are definable.

As in the field $\mathbb{R}$, for any real closed field R , we can define the n -th general linear $G(n, R)$, the n -th orthogonal group $O(n)$.

Let G, G' be definable groups. A group homomorphism $f : G \rightarrow G'$ is a *definable group homomorphism* if f is definable. A definable group homomorphism $f : G \rightarrow GL(n, R)$ is called a *definable G representation*. A definable group homomorphism $f : G \rightarrow O(n)$ is called a *definable orthogonal*

G representation and R^n with the orthogonal action induced from an orthogonal G representation is called a *definable orthogonal G representation space*.

Definition 2.4. (1) A G invariant definable subset of a definable orthogonal G representation space is a *definable G set*.

Let X, Y be definable G sets.

(2) A definable map $f : X \rightarrow Y$ is a *definable G map* if for any $x \in X, g \in G, f(gx) = gf(x)$.

(3) A definable G map $f : X \rightarrow Y$ is a *definable G homeomorphism* if there exists a definable G map $h : Y \rightarrow X$ such that $f \circ h = id_Y, h \circ f = id_X$.

Definition 2.5. (1) A definable set $X \subset R^n$ is *definably compact* if for any definable map $f : (a, b)_R \rightarrow X$, there exist the limits $\lim_{x \rightarrow a+0} f(x), \lim_{x \rightarrow b-0} f(x)$ in X .

(2) A definable set $X \subset R^n$ is *definably connected* if there exist no definable open subsets U, V of X such that $X = U \cup V, U \cap V = \emptyset, U \neq \emptyset, V \neq \emptyset$.

A compact (resp. A connected) definable set is definably compact (resp. definably connected). But a definably compact (resp. a definably connected) definable set is not always compact (resp. connected). For example, if $R = \mathbb{R}_{alg}$, then $[0, 1]_{\mathbb{R}_{alg}} = \{x \in \mathbb{R}_{alg} | 0 \leq x \leq 1\}$ is definably compact and definably connected, but it is neither compact nor connected.

Theorem 2.6 ([7]). *For a definable set $X \subset R^n$, X is definably compact if and only if X is closed and bounded.*

The following is a definable version of the fact that the image of a compact (resp. a connected) set by a continuous map is compact (resp. connected).

Proposition 2.7. *Let $X \subset R^n, Y \subset R^m$ be definable set, $f : X \rightarrow Y$ a definable map. If X is definably compact (resp. definably connected), then $f(X)$ is definably compact (resp. definably connected).*

Theorem 2.8. (1) *(The intermediate value theorem) For a definable function f on a definably connected set X , if $a, b \in X, f(a) \neq f(b)$ then f takes all values between $f(a)$ and $f(b)$.*

(2) *(Existence theorem of maximum and minimum) Every definable function on a definably compact set attains maximum and minimum.*

(3) (*Rolle's theorem*) Let $f : [a, b]_R \rightarrow R$ be a definable function such that f is differentiable on $(a, b)_R$ and $f(a) = f(b)$. Then there exists c between a and b with $f'(c) = 0$.

(4) (*The mean value theorem*) Let $f : [a, b]_R \rightarrow R$ be a definable function which is differentiable on $(a, b)_R$. Then there exists c between a and b with $f'(c) = \frac{f(b)-f(a)}{b-a}$.

(5) Let $f : (a, b)_R \rightarrow R$ be a differentiable definable function. If $f' > 0$ on $(a, b)_R$, then f is increasing.

Example 2.9. (1) Let $\mathcal{N}$ be $(\mathbb{R}_{alg}, +, \cdot, <)$. Then $f : \mathbb{R}_{alg} \rightarrow \mathbb{R}_{alg}, f(x) = 2^x$ is not defined ([11]).

(2) Let $\mathcal{N}$ be $(\mathbb{R}, +, \cdot, <)$. Then $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = 2^x$ is defined but not definable, and $h : \mathbb{R} \rightarrow \mathbb{R}, h(x) = \sin x$ is defined but not definable.

3 Equivariant definable homotopy extensions

Let X, Y be definable set and $f : X \rightarrow Y$ a definable map. We say that f is *definably proper* if for any definably compact subset C of Y , $f^{-1}(C)$ is a definably compact subset of X .

Let $A \subset R^n, S \subset R^m$ be definable sets, and let $f : S \rightarrow A$ be a definable map. We say that f is *definably trivial* if there exist a definable set $F \subset R^N$ for some $N \in \mathbb{N}$, and a definable map $h : S \rightarrow F$ such that $(f, h) : S \rightarrow A \times F$ is a definable homeomorphism. In this case, each fiber $f^{-1}(a)$ of f over a is definably homeomorphic to F .

In o-minimal expansions of real closed fields, the following five theorems are known.

Theorem 3.1. (1) (*Monotonicity theorem* (e.g. 3.1.2, 3.1.6. [3])). Let $f : (a, b)_R \rightarrow R$ be a function with the definable graph. Then there exist finitely many points $a = a_0 < a_1 < \dots < a_k = b$ such that on each subinterval $(a_j, a_{j+1})_R$, the function is either constant, or strictly monotone and continuous. Moreover for any $c \in (a, b)_R$, the limits $\lim_{x \rightarrow c+0} f(x), \lim_{x \rightarrow c-0} f(x)$ exist in $R \cup \{\infty\} \cup \{-\infty\}$.

(2) (*Cell decomposition theorem* (e.g. 3.2.11. [3])). For any definable subsets $A_1, \dots, A_k$ of R^n , there exists a cell decomposition of R^n partitioning each $A_1, \dots, A_k$.

Let A be a definable subset of R^n and $f : A \rightarrow R$ a function with the definable graph. Then there exists a cell decomposition $\mathcal{D}$ of R^n partitioning A such that each $B \subset A, B \in \mathcal{D}$, $f|_B : B \rightarrow R$ is continuous.

(3) (Triangulation theorem (e.g. 8.2.9. [3])). Let $S \subset R^n$ be a definable set and let $S_1, S_2, \dots, S_k$ be definable subsets of S . Then S has a triangulation in R^n compatible with $S_1, \dots, S_k$.

(4) (Piecewise trivialization theorem (e.g. 8.2.9. [3])). Let $f : S \rightarrow A$ be a definable map between definable sets S and A . Then there is a finite partition $A_1, \dots, A_k$ of A into definable sets A_i such that each $f|_{f^{-1}(A_i)} : f^{-1}(A_i) \rightarrow A_i$ is definably trivial.

(5) (Existence of definable quotients (e.g. 10.2.18 [3])). Let G be a definably compact definable group and X a definable G set. Then the orbit space X/G exists as a definable set and the orbit map $\pi : X \rightarrow X/G$ is surjective, definable and definably proper.

Question 3.2. Let X, Y be definable sets and A a definable subset of X .

(1) (Extensions of definable maps) Let $f : A \rightarrow Y$ be a definable map. When does $f : A \rightarrow Y$ extend a definable map $F : X \rightarrow Y$?

(2) (Definable homotopy extensions) Let $f : X \rightarrow Y$ be a definable map and a definable homotopy $F : A \times [0, 1]_R \rightarrow Y$ such that $F(x, 0) = f(x)$ for any $x \in A$. When does a definable homotopy $H : X \times [0, 1]_R \rightarrow Y$ exist such that $H(x, 0) = f(x)$ for any $x \in X$ and $H|_{A \times [0, 1]_R} = F$?

Theorem 3.3 (Definable Tietze extension theorem [1]). Let X, Y be definable sets, A a definable closed subset of X and $f : A \rightarrow R$ a definable function. Then there exists a definable function $F : X \rightarrow R$ such that $F|_A = f$.

Theorem 3.4 (Definable homotopy extension theorem [2]). Let X, Y be definable sets and A a definable closed subset of X . For any definable map $f : X \rightarrow Y$ and for any definable homotopy $F : A \times [0, 1]_R \rightarrow Y$ such that $F(x, 0) = f(x)$ for any $x \in A$, there exists a definable homotopy $H : X \times [0, 1]_R \rightarrow Y$ such that $H(x, 0) = f(x)$ for any $x \in X$ and $H|_{A \times [0, 1]_R} = F$.

To consider Question 3.2, we need to construct an obstruction theory in the definable category.

The following question is an equivariant version of Question 3.2.

Question 3.5. Let G be a definable group, X, Y a definable G sets and A a definable G subset of X .

(1) (*Extensions of definable G maps*) Let $f : A \rightarrow Y$ be a definable G map. When does $f : A \rightarrow Y$ extend a definable G map $F : X \rightarrow Y$?

(2) (*Equivariant definable homotopy extensions*) Let $f : X \rightarrow Y$ be a definable G map and an equivariant definable homotopy $F : A \times [0, 1]_R \rightarrow Y$ such that $F(x, 0) = f(x)$ for any $x \in A$. When does an equivariant definable homotopy $H : X \times [0, 1]_R \rightarrow Y$ exist such that $H(x, 0) = f(x)$ for any $x \in X$ and $H|_{A \times [0, 1]_R} = F$?

We have the following result.

Theorem 3.6 ([6]). *Let G be a definably compact definable group, X a definable G set and A a definable closed G subset of X . For any definable G map $f : X \rightarrow Y$ and for any equivariant definable homotopy $F : A \times [0, 1]_R \rightarrow Y$ such that $F(x, 0) = f(x)$ for any $x \in A$, there exists an equivariant definable homotopy $H : X \times [0, 1]_R \rightarrow Y$ such that $H(x, 0) = f(x)$ for any $x \in X$ and $H|_{A \times [0, 1]_R} = F$.*

Theorem 3.6 is proved in the case where $R = \mathbb{R}$ ([5]).

To prove Theorem 3.6, we need the following results.

Theorem 3.7 ([6]). *Let G be a definably compact definable group and Y a definable closed G subset of a definable G set X . Then there exists a G invariant definable open neighborhood U of Y in X such that Y is a definable strong G deformation retract of both U and of the closure $\text{cl } U$ of U in X .*

Proposition 3.8 ([6]). *Let G be a definably compact definable group and A, B disjoint definable closed G subsets of a definable G set X . Then there exists a G invariant definable map $f : X \rightarrow [0, 1]_R$ with $A = f^{-1}(0)$ and $B = f^{-1}(1)$.*

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