

# American Options with Uncertainty of the Stock Prices: The Discrete-Time Model

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## 1. Introduction

A discrete-time mathematical model for American put option with uncertainty is presented, and the randomness and fuzziness are evaluated by both probabilistic expectation and  $\lambda$ -weighted possibilistic mean values.

## 2. Fuzzy stochastic processes

First we give some mathematical notations regarding fuzzy numbers. Let  $(\Omega, \mathcal{M}, P)$  be a probability space, where  $\mathcal{M}$  is a  $\sigma$ -field and  $P$  is a non-atomic probability measure.  $\mathbb{R}$  denotes the set of all real numbers, and let  $\mathcal{C}(\mathbb{R})$  be the set of all non-empty bounded closed intervals. A ‘fuzzy number’ is denoted by its membership function  $\tilde{a} : \mathbb{R} \mapsto [0, 1]$  which is normal, upper-semicontinuous, fuzzy convex and has a compact support. Refer to Zadeh [12] regarding fuzzy set theory.  $\mathcal{R}$  denotes the set of all fuzzy numbers. In this paper, we identify fuzzy numbers with its corresponding membership functions. The  $\alpha$ -cut of a fuzzy number  $\tilde{a} (\in \mathcal{R})$  is given by

$$\tilde{a}_\alpha := \{x \in \mathbb{R} \mid \tilde{a}(x) \geq \alpha\} \quad (\alpha \in (0, 1]) \quad \text{and} \quad \tilde{a}_0 := \text{cl}\{x \in \mathbb{R} \mid \tilde{a}(x) > 0\},$$

where  $\text{cl}$  denotes the closure of an interval. In this paper, we write the closed intervals by

$$\tilde{a}_\alpha := [\tilde{a}_\alpha^-, \tilde{a}_\alpha^+] \quad \text{for } \alpha \in [0, 1].$$

Hence we introduce a partial order  $\succeq$ , so called the ‘fuzzy max order’, on fuzzy numbers  $\mathcal{R}$ : Let  $\tilde{a}, \tilde{b} \in \mathcal{R}$  be fuzzy numbers.

$$\tilde{a} \succeq \tilde{b} \quad \text{means that} \quad \tilde{a}_\alpha^- \geq \tilde{b}_\alpha^- \quad \text{and} \quad \tilde{a}_\alpha^+ \geq \tilde{b}_\alpha^+ \quad \text{for all } \alpha \in [0, 1].$$

Then  $(\mathcal{R}, \succeq)$  becomes a lattice. For fuzzy numbers  $\tilde{a}, \tilde{b} \in \mathcal{R}$ , we define the maximum  $\tilde{a} \vee \tilde{b}$  with respect to the fuzzy max order  $\succeq$  by the fuzzy number whose  $\alpha$ -cuts are

$$(\tilde{a} \vee \tilde{b})_\alpha = [\max\{\tilde{a}_\alpha^-, \tilde{b}_\alpha^-\}, \max\{\tilde{a}_\alpha^+, \tilde{b}_\alpha^+\}], \quad \alpha \in [0, 1]. \quad (2.1)$$

An addition, a subtraction and a scalar multiplication for fuzzy numbers are defined as follows: For  $\tilde{a}, \tilde{b} \in \mathcal{R}$  and  $\lambda \geq 0$ , the addition and subtraction  $\tilde{a} \pm \tilde{b}$  of  $\tilde{a}$  and  $\tilde{b}$  and the scalar multiplication  $\lambda \tilde{a}$  of  $\lambda$  and  $\tilde{a}$  are fuzzy numbers given by

$$(\tilde{a} + \tilde{b})_\alpha := [\tilde{a}_\alpha^- + \tilde{b}_\alpha^-, \tilde{a}_\alpha^+ + \tilde{b}_\alpha^+], \quad (\tilde{a} - \tilde{b})_\alpha := [\tilde{a}_\alpha^- - \tilde{b}_\alpha^+, \tilde{a}_\alpha^+ - \tilde{b}_\alpha^-]$$

$$\text{and } (\lambda \tilde{a})_\alpha := [\lambda \tilde{a}_\alpha^-, \lambda \tilde{a}_\alpha^+] \quad \text{for } \alpha \in [0, 1].$$

A fuzzy-number-valued map  $\tilde{X} : \Omega \mapsto \mathcal{R}$  is called a ‘fuzzy random variable’ if the maps  $\omega \mapsto \tilde{X}_\alpha^-(\omega)$  and  $\omega \mapsto \tilde{X}_\alpha^+(\omega)$  are measurable for all  $\alpha \in [0, 1]$ , where  $\tilde{X}_\alpha(\omega) = [\tilde{X}_\alpha^-(\omega), \tilde{X}_\alpha^+(\omega)] = \{x \in \mathbb{R} \mid \tilde{X}(\omega)(x) \geq \alpha\}$  (see [10]). Next we need to introduce expectations of fuzzy random variables in order to describe an optimal stopping model in the next section. A fuzzy random variable  $\tilde{X}$  is called integrably bounded if both  $\omega \mapsto \tilde{X}_\alpha^-(\omega)$  and  $\omega \mapsto \tilde{X}_\alpha^+(\omega)$  are integrable for all  $\alpha \in [0, 1]$ . Let  $\tilde{X}$  be an integrably bounded fuzzy random variable. The expectation  $E(\tilde{X})$  of the fuzzy random variable  $\tilde{X}$  is defined by a fuzzy number (see [7])

$$E(\tilde{X})(x) := \sup_{\alpha \in [0, 1]} \min\{\alpha, 1_{E(\tilde{X})_\alpha}(x)\}, \quad x \in \mathbb{R}, \quad (2.2)$$

where closed intervals  $E(\tilde{X})_\alpha := \left[ \int_\Omega \tilde{X}_\alpha^-(\omega) dP(\omega), \int_\Omega \tilde{X}_\alpha^+(\omega) dP(\omega) \right]$  ( $\alpha \in [0, 1]$ ).

In the rest of this section, we introduce stopping times for fuzzy stochastic processes. Let  $T$  ( $T > 0$ ) be an ‘expiration date’ and let  $\mathbb{T} := \{0, 1, 2, \dots, T\}$  be the time space. Let a ‘fuzzy stochastic process’  $\{\tilde{X}_t\}_{t=0}^T$  be a sequence of integrably bounded fuzzy random variables such that  $E(\max_{t \in \mathbb{T}} \tilde{X}_{t,0}^+) < \infty$ , where  $\tilde{X}_{t,0}^+(\omega)$  is the right-end of the 0-cut of the fuzzy number  $\tilde{X}_t(\omega)$ . For  $t \in \mathbb{T}$ ,  $\mathcal{M}_t$  denotes the smallest  $\sigma$ -field on  $\Omega$  generated by all random variables  $\tilde{X}_{s,\alpha}^-$  and  $\tilde{X}_{s,\alpha}^+$  ( $s = 0, 1, 2, \dots, t; \alpha \in [0, 1]$ ). We call  $(\tilde{X}_t, \mathcal{M}_t)_{t=0}^\infty$  a fuzzy stochastic process. A map  $\tau : \Omega \mapsto \mathbb{T}$  is called a ‘stopping time’ if

$$\{\omega \in \Omega \mid \tau(\omega) = t\} \in \mathcal{M}_t \quad \text{for all } t = 0, 1, 2, \dots, T.$$

Then, the following lemma is trivial from the definitions ([11]).

**Lemma 2.1.** *Let  $\tau$  be a stopping time. We define*

$$\tilde{X}_\tau(\omega) := \tilde{X}_t(\omega) \quad \text{if } \tau(\omega) = t \quad \text{for } t = 0, 1, 2, \dots, T \text{ and } \omega \in \Omega.$$

*Then,  $\tilde{X}_\tau$  is a fuzzy random variable.*

### 3. American put option with uncertainty of stock prices

In this section, we formulate American put option with uncertainty of stock prices by fuzzy random variables. Let  $\mathbb{T} := \{0, 1, 2, \dots, T\}$  be the time space with an expiration date  $T$  ( $T > 0$ ) similarly to the previous section, and take a probability space  $\Omega := \mathbb{R}^{T+1}$ . Let  $r$  ( $r > 0$ ) be an interest rate of a bond price, which is riskless asset, and put a discount

rate  $\beta = 1/(1+r)$ . Define a 'stock price process'  $\{S_t\}_{t=0}^T$  as follows: An initial stock price  $S_0$  is a positive constant and stock prices are given by

$$S_t := S_0 \prod_{s=1}^t (1 + Y_s) \quad \text{for } t = 1, 2, \dots, T, \quad (3.1)$$

where  $\{Y_t\}_{t=1}^T$  is a uniform integrable sequence of independent, identically distributed real random variables on  $[r-1, r+1]$  such that  $E(Y_t) = r$  for all  $t = 1, 2, \dots, T$ . The  $\sigma$ -fields  $\{\mathcal{M}_t\}_{t=0}^T$  are defined as follows:  $\mathcal{M}_0$  is the completion of  $\{\emptyset, \Omega\}$  and  $\mathcal{M}_t$  ( $t = 1, 2, \dots, T$ ) denote the complete  $\sigma$ -fields generated by  $\{Y_1, Y_2, \dots, Y_t\}$ .

We consider a finance model where the stock price process  $\{S_t\}_{t=0}^T$  takes fuzzy values. Now we give fuzzy values by triangular fuzzy numbers for simplicity. Let  $\{a_t\}_{t=0}^T$  be an  $\mathcal{M}_t$ -adapted stochastic process such that  $0 < a_t(\omega) \leq S_t(\omega)$  for  $\omega \in \Omega$ . A 'stock price process with fuzzy values' are represented by a sequence of fuzzy random variables  $\{\tilde{S}_t\}_{t=0}^T$ :

$$\tilde{S}_t(\omega)(x) := L((x - S_t(\omega))/a_t(\omega)) \quad (3.2)$$

for  $t \in \mathbb{T}$ ,  $\omega \in \Omega$  and  $x \in \mathbb{R}$ , where  $L(x) := \max\{1 - |x|, 0\}$  ( $x \in \mathbb{R}$ ) is the triangle shape function. Hence,  $a_t(\omega)$  is a spread of triangular fuzzy numbers  $\tilde{S}_t(\omega)$  and corresponds to the amount of fuzziness in the process. Then,  $a_t(\omega)$  should be an increasing function of the stock price  $S_t(\omega)$  (see Assumption S in the next section).

Let  $K$  ( $K > 0$ ) be a 'strike price'. The 'price process'  $\{\tilde{P}_t\}_{t=0}^T$  of American put option under uncertainty is represented by

$$\tilde{P}_t(\omega) := \beta^t(1_{\{K\}} - \tilde{S}_t(\omega)) \vee 1_{\{0\}} \quad \text{for } t = 0, 1, 2, \dots, T, \quad (3.3)$$

where  $\vee$  is given by (2.1), and  $1_{\{K\}}$  and  $1_{\{0\}}$  denote the crisp number  $K$  and zero respectively. An 'exercise time' in American put option is given by a stopping time  $\tau$  with values in  $\mathbb{T}$ . For an exercise time  $\tau$ , we define

$$\tilde{P}_\tau(\omega) := \tilde{P}_t(\omega) \quad \text{if } \tau(\omega) = t \quad \text{for } t = 0, 1, 2, \dots, T, \text{ and } \omega \in \Omega. \quad (3.4)$$

Then, from Lemma 2.1,  $\tilde{P}_\tau$  is a fuzzy random variable. The expectation of the fuzzy random variable  $\tilde{P}_\tau$  is a fuzzy number (see (2.2))

$$E(\tilde{P}_\tau)(x) := \sup_{\alpha \in [0,1]} \min\{\alpha, 1_{E(\tilde{P}_\tau)_\alpha}(x)\}, \quad x \in \mathbb{R}, \quad (3.5)$$

where  $E(\tilde{P}_\tau)_\alpha = \left[ \int_\Omega \tilde{P}_{\tau,\alpha}^-(\omega) dP(\omega), \int_\Omega \tilde{P}_{\tau,\alpha}^+(\omega) dP(\omega) \right]$ . In American put option, we must maximize the expected values (3.5) of the price process by stopping times  $\tau$ , and we need to evaluate the fuzzy numbers (3.5) since the fuzzy max order (2.1) on  $\mathcal{R}$  is a partial order and not a linear order. In this paper, we consider the following estimation regarding the price process  $\{\tilde{P}_t\}_{t=0}^T$  of American put option. Let  $g : \mathcal{C}(\mathbb{R}) \mapsto \mathbb{R}$  be a map such that

$$g([x, y]) := \lambda x + (1 - \lambda)y, \quad [x, y] \in \mathcal{C}(\mathbb{R}), \quad (3.6)$$

where  $\lambda$  is a constant satisfying  $0 \leq \lambda \leq 1$ . This scalarization is used for the evaluation of fuzzy numbers, and  $\lambda$  is called a ‘pessimistic-optimistic index’ and means the pessimistic degree in decision making. We call  $g$  a ‘ $\lambda$ -weighting function’ and we evaluate fuzzy numbers  $\tilde{a}$  by “ $\lambda$ -weighted possibilistic mean value’

$$\int_0^1 2\alpha g(\tilde{a}_\alpha) d\alpha, \quad (3.7)$$

where  $\tilde{a}_\alpha$  is the  $\alpha$ -cut of fuzzy numbers  $\tilde{a}$ . (see Carlsson and Fullér [1], Goetschel and Voxman [4]) When we apply a  $\lambda$ -weighting function  $g$  to (3.5), its evaluation follows

$$\int_0^1 2\alpha g(E(\tilde{P}_\tau)_\alpha) d\alpha. \quad (3.8)$$

Now we analyze (3.8) by  $\alpha$ -cuts technique of fuzzy numbers. The  $\alpha$ -cuts of fuzzy random variables (3.2) are

$$\tilde{S}_{t,\alpha}(\omega) = [S_t(\omega) - (1 - \alpha)a_t(\omega), S_t(\omega) + (1 - \alpha)a_t(\omega)], \quad \omega \in \Omega, \quad (3.9)$$

and so

$$\tilde{S}_{t,\alpha}^\pm(\omega) = S_t(\omega) \pm (1 - \alpha)a_t(\omega), \quad \omega \in \Omega \quad (3.10)$$

for  $t \in \mathbb{T}$  and  $\alpha \in [0, 1]$ . Therefore, the  $\alpha$ -cuts of (3.3) are

$$\tilde{P}_{t,\alpha}(\omega) = [\tilde{P}_{t,\alpha}^-(\omega), \tilde{P}_{t,\alpha}^+(\omega)] := [\beta^t \max\{K - \tilde{S}_{t,\alpha}^+(\omega), 0\}, \beta^t \max\{K - \tilde{S}_{t,\alpha}^-(\omega), 0\}], \quad (3.11)$$

and we obtain  $E(\max_{t \in \mathbb{T}} \sup_{\alpha \in [0,1]} \tilde{P}_{t,\alpha}^+) \leq K < \infty$  since  $\tilde{S}_{t,\alpha}^-(\omega) \geq 0$ , where  $E(\cdot)$  is the expectation with respect to some risk-neutral equivalent martingale measure ([2],[6]). For a stopping time  $\tau$ , the expectation of the fuzzy random variable  $\tilde{P}_\tau$  is a fuzzy number whose  $\alpha$ -cut is a closed interval

$$E(\tilde{P}_\tau)_\alpha = E(\tilde{P}_{\tau,\alpha}) = [E(\tilde{P}_{\tau,\alpha}^-), E(\tilde{P}_{\tau,\alpha}^+)] \quad \text{for } \alpha \in [0, 1], \quad (3.12)$$

where  $\tilde{P}_{\tau(\omega),\alpha}(\omega) = [\tilde{P}_{\tau(\omega),\alpha}^-(\omega), \tilde{P}_{\tau(\omega),\alpha}^+(\omega)]$  is the  $\alpha$ -cut of fuzzy number  $\tilde{P}_\tau(\omega)$ . Using the  $\lambda$ -weighting function  $g$ , from (3.7) the evaluation of the fuzzy random variable  $\tilde{P}_\tau$  is given by the integral

$$\int_0^1 2\alpha g(E(\tilde{P}_{\tau,\alpha})) d\alpha. \quad (3.13)$$

Put the value by  $\mathbf{P}(\tau)$ . Then, from (2.2), the terms (3.8) and (3.13) coincide:

$$\mathbf{P}(\tau) = \int_0^1 2\alpha g(E(\tilde{P}_{\tau,\alpha})) d\alpha = \int_0^1 2\alpha g(E(\tilde{P}_\tau)_\alpha) d\alpha. \quad (3.14)$$

Therefore  $\mathbf{P}(\tau)$  means an evaluation of the expected price of American put option when  $\tau$  is an exercise time. Further, we have the following equality.

**Lemma 3.1.** For a stopping time  $\tau$  ( $\tau \leq T$ ), it holds that

$$P(\tau) = \int_0^1 2\alpha g(E(\tilde{P}_{\tau,\alpha})) d\alpha = \int_0^1 2\alpha E(g(\tilde{P}_{\tau,\alpha})) d\alpha = E\left(\int_0^1 2\alpha g(\tilde{P}_{\tau,\alpha}(\cdot)) d\alpha\right). \quad (3.15)$$

We put the ‘optimal expected price’ by

$$V := \sup_{\tau: \tau \leq T} P(\tau) = \sup_{\tau: \tau \leq T} \int_0^1 2\alpha g(E(\tilde{P}_{\tau,\alpha})) d\alpha. \quad (3.16)$$

In the next section, this paper discusses the following optimal stopping problem regarding American put option with fuzziness.

**Problem P.** Find a stopping time  $\tau^*$  ( $\tau^* \leq T$ ) and the optimal expected price  $V$  such that

$$P(\tau^*) = V, \quad (3.17)$$

where  $V$  is given by (3.16).

Then,  $\tau^*$  is called an ‘optimal exercise time’.

#### 4. The optimal expected price and the optimal exercise time

In this section, we discuss the optimal fuzzy price  $V$  and the optimal exercise time  $\tau^*$ , by using dynamic programming approach. Now we introduce an assumption.

**Assumption S.** The stochastic process  $\{a_t\}_{t=0}^T$  is represented by

$$a_t(\omega) := cS_t(\omega), \quad t = 0, 1, 2, \dots, T, \quad \omega \in \Omega,$$

where  $c$  is a constant satisfying  $0 < c < 1$ .

Assumption S is reasonable since  $a_t(\omega)$  means a size of fuzziness and it should depend on the volatility and the stock price  $S_t(\omega)$  because one of the most difficulties is estimation of the actual volatility ([8, Sect.7.5.1]). In this model, we represent by  $c$  the fuzziness of the volatility, and we call  $c$  a ‘fuzzy factor’ of the process. From now on, we suppose that Assumption S holds. For a stopping time  $\tau$  ( $\tau \leq T$ ), we define a random variable

$$\Pi_\tau(\omega) := \int_0^1 2\alpha g(\tilde{P}_{\tau,\alpha}(\omega)) d\alpha, \quad \omega \in \Omega. \quad (4.1)$$

From Lemma 3.1,  $P(\tau) = E(\Pi_\tau)$  is the evaluated price of American put option when  $\tau$  is an exercise time. Then we have the following representation about (4.1).

**Lemma 4.1.** For a stopping time  $\tau$  ( $\tau \leq T$ ), it holds that

$$\Pi_\tau(\omega) = \beta^{\tau(\omega)} f^P(S_\tau(\omega)), \quad \omega \in \Omega, \quad (4.2)$$

where  $f^P$  is a function on  $(0, \infty)$  such that

$$f^P(y) := \begin{cases} K - y - \frac{1}{3}cy(2\lambda - 1) + \lambda\varphi^1(y) & \text{if } 0 < y < K \\ (1 - \lambda)\varphi^2(y) & \text{if } y \geq K, \end{cases} \quad (4.3)$$

and

$$\varphi^1(y) := \frac{1}{(cy)^2}((-K + y + cy) \max\{0, -K + y + cy\}^2 - \frac{2}{3} \max\{0, -K + y + cy\}^3), \quad y > 0, \quad (4.4)$$

$$\varphi^2(y) := \frac{1}{(cy)^2}((K - y + cy) \max\{0, K - y + cy\}^2 - \frac{2}{3} \max\{0, K - y + cy\}^3), \quad y > 0. \quad (4.5)$$

Now we give an optimal stopping time for Problem P and we discuss an iterative method to obtain the optimal expected price  $V$  in (3.16). To analyze the optimal fuzzy price  $V$ , we put

$$V_t^P(y) = \sup_{\tau: t \leq \tau \leq T} E(\beta^{-t} \Pi_\tau | S_t = y) \quad (4.6)$$

for  $t = 0, 1, 2, \dots, T$  and an initial stock price  $y$  ( $y > 0$ ). Then we note that  $V = V_0^P(y)$ .

**Theorem 4.1** (Optimality equation).

- (i) The optimal expected price  $V = V_0^P(y)$  with an initial stock price  $y$  ( $y > 0$ ) is given by the following backward recursive equations (4.7) and (4.8):

$$V_t^P(y) = \max\{\beta E(V_{t+1}^P(y(1 + Y_1))), f^P(y)\}, \quad t = 0, 1, \dots, T - 1, \quad y > 0, \quad (4.7)$$

$$V_T^P(y) = f^P(y), \quad y > 0. \quad (4.8)$$

- (ii) Define a stopping time

$$\tau^P(\omega) := \inf\{t \in \mathbb{T} \mid V_0^P(S_t(\omega)) = f^P(S_t(\omega))\}, \quad \omega \in \Omega, \quad (4.9)$$

where the infimum of the empty set is understood to be  $T$ . Then,  $\tau^P$  is an optimal exercise time for Problem P, and the optimal value of American put option is

$$V = V_0^P(y) = P(\tau^P) \quad (4.10)$$

for an initial stock price  $y > 0$ .

## 5. A numerical example

Now we give a numerical example to illustrate our idea in Sections 3 and 4.

**Example 5.1.** We consider CRR type American put option model (see Ross [8, Sect.7.4]). Put an expiration date  $T = 10$ , an interest rate of a bond  $r = 0.05$ , a fuzzy factor  $c = 0.05$ , an initial stock price  $y = 30$  and a strike price  $K = 35$ . Assume that

$\{Y_t\}_{t=1}^T$  is a uniform sequence of independent, identically distributed real random variables such that

$$Y_t := \begin{cases} e^\sigma - 1 & \text{with probability } p \\ e^{-\sigma} - 1 & \text{with probability } (1 - p) \end{cases}$$

for all  $t = 1, 2, \dots, T$ , where  $\sigma = 0.25$  and  $p = (1 + r - e^{-\sigma})/(e^\sigma - e^{-\sigma})$ . Then we have  $E(Y_t) = r$ . The corresponding optimal exercise time is given by

$$\tau^P(\omega) = \inf\{t \in \mathbb{T} \mid V_0^P(S_t(\omega)) = f^P(S_t(\omega))\}.$$

In the following Table, the optimal expected price  $V = V_0^P(y)$  at initial stock price  $y = 30$  changes with the pessimistic-optimistic index  $\lambda$  of the  $\lambda$ -weighting function  $g$ .

Table. The optimal expected price  $V = V_0^P(y)$  at initial stock prices  $y = 30$ .

| $\lambda$ | 1/3     | 1/2     | 2/3     |
|-----------|---------|---------|---------|
| $V$       | 7.48169 | 7.39649 | 7.31130 |

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