

REVERSE INEQUALITIES ASSOCIATED WITH TSALLIS RELATIVE OPERATOR ENTROPY VIA GENERALIZED KANTOROVICH CONSTANT

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§1. Introduction

A capital letter means an operator on a Hilbert space H . An operator X is said to be *strictly positive* (denoted by $X > 0$) if X is positive definite and invertible. For two strictly positive operators A, B and $p \in [0, 1]$, p -power mean $A \sharp_p B$ is defined by

$$A \sharp_p B = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^p A^{\frac{1}{2}}$$

and we remark that $A \sharp_p B = A^{1-p} B^p$ if A commutes with B .

Very recently, Tsallis relative operator entropy $T_p(A|B)$ in Yanagi-Kuriyama-Furuichi [17] is defined by

$$(1.1) \quad T_p(A|B) = \frac{A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^p A^{\frac{1}{2}} - A}{p} \quad \text{for } p \in (0, 1]$$

and $T_p(A|B)$ can be written by using the notion of $A \sharp_p B$ as follows:

$$(1.1') \quad T_p(A|B) = \frac{A \sharp_p B - A}{p} \quad \text{for } p \in (0, 1].$$

The relative operator entropy $\hat{S}(A|B)$ in [3] is defined by

$$(1.2) \quad \hat{S}(A|B) = A^{\frac{1}{2}} (\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}$$

as an extension of [15].

On the other hand, the generalized Kantorovich constant $K(p)$ is defined by

$$(1.3) \quad K(p) = \frac{(h^p - h)}{(p-1)(h-1)} \left(\frac{(p-1)}{p} \frac{(h^p - 1)}{(h^p - h)} \right)^p$$

for any real number p and $h > 1$. Also $S(p)$ is defined by

$$(1.4) \quad S(p) = \frac{h^{\frac{p}{h^p-1}}}{e \log h^{\frac{p}{h^p-1}}}$$

for any real number p . In particular $S(1) = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}}$ is said to be the *Specht ratio* and $S(1) > 1$ is well known.

Theorem A. *Let A be strictly positive operator satisfying $MI \geq A \geq mI > 0$, where $M > m > 0$. Put $h = \frac{M}{m} > 1$. Then the following inequalities hold:*

$$(1.5) \quad (Ax, x)^p \geq (A^p x, x) \geq K(p)(Ax, x)^p \quad \text{for any } 1 \geq p > 0.$$

$$(1.6) \quad S(1)\Delta_x(A) \geq (Ax, x) \geq \Delta_x(A).$$

$$(1.7) \quad K(p) \in (0, 1] \text{ for } p \in [0, 1].$$

$$(1.8) \quad K(0) = K(1) = 1.$$

$$(1.9) \quad S(1) = e^{K'(1)} = e^{-K'(0)}.$$

where the determinant $\Delta_x(A)$ for strictly positive operator A at a unit vector x is defined by $\Delta_x(A) = \exp\langle (\log A)x, x \rangle$ and (1.6) is shown in [4].

(1.8) and (1.9) of Theorem A are shown in [8, Proposition 1] and (1.7) is shown in [9].

§2 Two reverse inequalities involving Tsallis relative operator entropy $T_p(A|B)$ via generalized Kantorovich constant $K(p)$

At first we shall state the following two reverse inequalities involving Tsallis relative operator entropy $T_p(A|B)$ via generalized Kantorovich constant $K(p)$.

Theorem 2.1. *Let A and B be strictly positive operators such that $M_1 I \geq A \geq m_1 I > 0$ and $M_2 I \geq B \geq m_2 I > 0$. Put $m = \frac{m_2}{M_1}$, $M = \frac{M_2}{m_1}$, $h = \frac{M}{m} = \frac{M_1 M_2}{m_1 m_2} > 1$ and $p \in (0, 1]$. Let Φ be normalized positive linear map on $B(H)$. Then the following inequalities hold:*

$$(2.1) \quad \left(\frac{1 - K(p)}{p} \right) \Phi(A) \sharp_p \Phi(B) + \Phi(T_p(A|B)) \geq T_p(\Phi(A)|\Phi(B)) \geq \Phi(T_p(A|B))$$

and

$$(2.2) \quad F(p)\Phi(A) + \Phi(T_p(A|B)) \geq T_p(\Phi(A)|\Phi(B)) \geq \Phi(T_p(A|B))$$

where $K(p)$ is the generalized Kantorovich constant defined in (1.3) and

$$F(p) = \frac{m^p}{p} \left(\frac{h^p - h}{h - 1} \right) \left(1 - K(p)^{\frac{1}{p-1}} \right) \geq 0.$$

Remark 2.1. We remark that the second inequality of (2.1) of Theorem 2.1 is shown in [6] along [3] and the first one of (2.1) is a reverse one of the second one and also the second inequality of (2.2) is as the same as the second one in (2.1) and the first one of (2.2) is a reverse one of the second one. We shall give simple proofs of (2.1) and (2.2) including its reverse inequality, respectively, via generalized Kantorovich constant $K(p)$ in (1.3).

We state the following result to prove Theorem 2.1.

Proposition 2.2. *Let $h > 1$ and let $g(p)$ be defined by:*

$$g(p) = \frac{h^p - h}{h - 1} + (1 - p) \left(\frac{h^p - 1}{p(h - 1)} \right)^{\frac{p}{p-1}} \quad \text{for } p \in [0, 1].$$

Then the following results hold:

- (i) $g(0) = \lim_{p \rightarrow 0} g(p) = 0.$
- (ii) $g(p) = \frac{h^p - h}{h - 1} \left(1 - K(p)^{\frac{1}{p-1}} \right) \geq 0 \quad \text{for all } p \in [0, 1].$
- (iii) $g'(0) = \lim_{p \rightarrow 0} g'(p) = \log S(1).$
- (iv) $\lim_{p \rightarrow 0} \frac{g(p)}{p} = \log S(1).$

Also we need the following result to prove Theorem 2.1.

Theorem B. *Let A and B be strictly positive operators on a Hilbert space H such that $M_1 I \geq A \geq m_1 I > 0$ and $M_2 I \geq B \geq m_2 I > 0$. Put $m = \frac{m_2}{M_1}$, $M = \frac{M_2}{m_1}$ and $h = \frac{M}{m} = \frac{M_1 M_2}{m_1 m_2} > 1$. Let $p \in (0, 1)$ and also let Φ be normalized positive linear map on $B(H)$. Then the following inequalities hold:*

- (i) $\Phi(A) \sharp_p \Phi(B) \geq \Phi(A \sharp_p B) \geq K(p) \Phi(A) \sharp_p \Phi(B)$
- (ii) $\Phi(A) \sharp_p \Phi(B) \geq \Phi(A \sharp_p B) \geq \Phi(A) \sharp_p \Phi(B) - f(p) \Phi(A)$

where $f(p) = m^p \left[\frac{h^p - h}{h - 1} + (1 - p) \left(\frac{h^p - 1}{p(h - 1)} \right)^{\frac{p}{p-1}} \right]$ and $K(p)$ is defined in (1.3).

The right hand side inequalities of (i) and (ii) of Theorem B follow by [14, Corollary 3.5] and the left hand side of (i) is well known [13].

§3 Two results by Furuichi-Yanagi-Kuriyama which are useful to prove our results in §6

Throughout this section, we deal with $n \times n$ matrix. A matrix X is said to be strictly positive definite matrix (denoted by $X > 0$) if X is positive definite and invertible. Let A and B be positive definite matrices. Tsallis relative entropy $D_p(A||B)$ in Furuichi-Yanagi-Kuriyama [5] is defined by

$$(3.1) \quad D_p(A||B) = \frac{\text{Tr}[A] - \text{Tr}[A^{1-p} B^p]}{p} \quad \text{for } p \in (0, 1].$$

Umegaki relative entropy $S(A, B)$ in [16] is defined by

$$(3.2) \quad S(A, B) = \text{Tr}[A(\log A - \log B)] \quad \text{for } A, B > 0.$$

Theorem C. (Generalized Peierls-Bogoliubov inequality [5]) *Let $A, B > 0$ and also let $p \in (0, 1]$. Then the following inequality holds:*

$$(3.3) \quad D_p(A||B) \geq \frac{\text{Tr}[A] - (\text{Tr}[A])^{1-p}(\text{Tr}[B])^p}{p}.$$

Theorem D [5]. *Let $A, B > 0$. The following inequality holds:*

$$(3.4) \quad -\text{Tr}[\text{T}_p(A|B)] \geq D_p(A||B) \quad \text{for } p \in (0, 1].$$

We remark that (3.4) implies $-\text{Tr}[\hat{S}(A|B)] \geq S(A, B)$ which is well known in [11],[12],[2] and [5].

§4 A result which unifies Theorem C and Theorem D in §3

Also throughout this section, we deal with $n \times n$ matrix. In this section, we shall state the following Proposition E which unifies Theorem C and Theorem D in §3.

Proposition E. *Let $A, B > 0$ and also let $p \in (0, 1]$. Then the following inequalities hold:*

$$(4.1) \quad \begin{aligned} \text{Tr}[(1-p)A + pB] &\geq (\text{Tr}[A])^{1-p}(\text{Tr}[B])^p \\ &\geq \text{Tr}[A^{1-p}B^p] \\ &\geq \text{Tr}[A \sharp_p B]. \end{aligned}$$

Proposition F. *Let $A, B > 0$ and also let $p \in (0, 1]$. Then the following inequalities hold:*

$$(4.5) \quad \begin{aligned} -\text{Tr}[\text{T}_p(A|B)] &\geq D_p(A||B) \\ &\geq \frac{\text{Tr}[A] - (\text{Tr}[A])^{1-p}(\text{Tr}[B])^p}{p} \\ &\geq \text{Tr}[A - B]. \end{aligned}$$

Needless to say, the first inequality of (4.5) of Proposition F is just (3.4) of Theorem D and the second one of (4.5) is just (3.3) of Theorem C, and also Proposition E is nothing but another expression form of Proposition F.

Proposition F yields the following result by putting $p \rightarrow 0$.

Proposition G. *Let $A, B > 0$. Then the following inequalities hold:*

$$(4.6) \quad -\text{Tr}[\hat{S}(A|B)] \geq S(A, B)$$

$$\begin{aligned}
&\geq \text{Tr}[A(\log \text{Tr}[A] - \log \text{Tr}[B])] \\
&\geq \text{Tr}[A - B].
\end{aligned}$$

§5 Related counterexamples to several questions caused by the results in §4

Also throughout this section, we deal with $n \times n$ matrix too. We shall give related counterexamples to several questions caused by the results in §4

Remark 5.1. The following matrix inequality (AG) is quite well known as the matrix version of (4.2) and there are a lot of references (for example, [13],[7]):

$$(AG) \quad (1-p)A + pB \geq A \sharp_p B \text{ holds for } A, B > 0 \text{ and } p \in (0, 1].$$

Suggested by the matrix inequality (AG), the second inequality and the third one on trace-inequality (4.1) of Proposition E, we might be apt to suppose that the following matrix inequalities as more exact precise estimation than (AG) : let $A, B > 0$ and $p \in (0, 1]$,

$$(AG-1?) \quad (1-p)A + pB \geq B^{\frac{p}{2}} A^{1-p} B^{\frac{p}{2}} \geq A \sharp_p B$$

and

$$(AG-2?) \quad (1-p)A + pB \geq A^{\frac{1-p}{2}} B^p A^{\frac{1-p}{2}} \geq A \sharp_p B.$$

But we have the following common counterexample to (AG-1?) and (AG-2?).

Remark 5.2. (i). *If A and B are positive definite matrices and $p \in (0, 1]$, then the following inequality holds:*

$$(5.2) \quad D_p(A||B) \geq \text{Tr}[A - B].$$

We remark that (5.2) is shown in the proof of [5, (1) of Proposition 2.4] and the second inequality and the third one of (4.5) of Proposition F yield the inequality (5.2), that is, the second inequality and the third one of (4.5) of Proposition F are somewhat more precise estimation than (5.2).

(ii). Also we recall the following result [1, Problem IX.8.12]:

If A and B are strictly positive matrices, then the following inequality holds:

$$(5.3) \quad \text{Tr}[A(\log A - \log B)] \geq \text{Tr}[A - B].$$

We remark that the second inequality and the third one of (4.6) of Proposition G imply (5.3) since $S(A, B) = \text{Tr}[A(\log A - \log B)]$, that is, the second inequality and the third one of (4.6) are somewhat more precise estimation than (5.3).

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Suggested by (5.3), we might be apt to expect that the following matrix inequality:

$$(5.3-1?) \quad A^{\frac{1}{2}}(\log A - \log B)A^{\frac{1}{2}} \geq A - B.$$

so that it turns out that (5.3-1?) does not hold.

§6. Two trace reverse inequalities associated with $-\text{Tr}[T_p(A|B)]$ and $D_p(A||B)$ via generalized Kantorovich constant $K(p)$

As an application of Theorem 2.1, we shall show the following two trace reverse inequalities associated with $-\text{Tr}[T_p(A|B)]$ and $D_p(A||B)$ via generalized Kantorovich constant $K(p)$.

Theorem 6.1.

Let A and B be strictly positive definite matrices such that $M_1 I \geq A \geq m_1 I > 0$ and $M_2 I \geq B \geq m_2 I > 0$. Put $m = \frac{m_2}{M_1}$, $M = \frac{M_2}{m_1}$ and $h = \frac{M}{m} = \frac{M_1 M_2}{m_1 m_2} > 1$ and $p \in (0, 1]$. Then the following inequalities hold:

$$(6.1) \quad \begin{aligned} & \left(\frac{1 - K(p)}{p} \right) (\text{Tr}[A])^{1-p} (\text{Tr}[B])^p + D_p(A||B) \\ & \geq -\text{Tr}[T_p(A|B)] \\ & \geq D_p(A||B) \end{aligned}$$

$$(6.2) \quad \begin{aligned} & F(p)(\text{Tr}[A]) + D_p(A||B) \\ & \geq -\text{Tr}[T_p(A|B)] \\ & \geq D_p(A||B) \end{aligned}$$

where $K(p)$ is the generalized Kantorovich constant defined in (1.3) and

$$F(p) = \frac{m^p}{p} \left(\frac{h^p - h}{h - 1} \right) \left(1 - K(p)^{\frac{1}{p-1}} \right) \geq 0.$$

Corollary 6.2. [10] Let A and B be strictly positive definite matrices such that $M_1 I \geq A \geq m_1 I > 0$ and $M_2 I \geq B \geq m_2 I > 0$. Put $h = \frac{M_1 M_2}{m_1 m_2} > 1$. Then the following inequality hold:

$$(6.5) \quad \begin{aligned} & \log S(1) \text{Tr}[A] + S(A, B) \\ & \geq -\text{Tr}[\hat{S}(A|B)] \\ & \geq S(A, B) \end{aligned}$$

where $S(1)$ is the Specht ratio defined in (1.4) and the first inequality is the reverse one of the second inequality.

The complete paper with proofs will appear elsewhere.

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